SIMPSON’S TYPE INEQUALITY FOR $F$-CONVEX FUNCTION

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Abstract. In this paper, we obtain Simpson’s type inequality for the function whose second derivatives absolute values are $F$-convex. Then, we give some special cases of the mappings $F$.

Keywords: Simpson’s inequality, $F$-convex mapping

1. Introduction

The well-known [2] in the literatüre as Simpson’s inequality is described by the following theorem:

**Theorem 1.1.** Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $\|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:

$$\left| \frac{1}{3} \left( f(a) + f(b) \right) + 2f \left( \frac{a + b}{2} \right) \right| - \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b - a)^4.$$

For many years, many types of convexity have been defined, such as quasi-convex [1], pseudo-convex [5], strongly convex [6], $\varepsilon$-convex [4], $s$-convex [3], $h$-convex [9] and etc. Recently, a new convexity that depends on a certain function satisfying some axioms was defined by Samet in the paper [7] which generalizes different types of convexity, including $\varepsilon$-convex functions, $\alpha$-convex functions, $h$-convex functions and many others.

Let us recall the family $F$ of mappings $F : \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfying the following axioms:

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(A1) If \( u_i \in L^1(0,1), \, i = 1, 2, 3 \), then for every \( \lambda \in [0, 1] \), we have
\[
\int_0^1 F(u_1(t), u_2(t), u_3(t), \lambda)dt = F \left( \int_0^1 u_1(t)dt, \int_0^1 u_2(t)dt, \int_0^1 u_3(t)dt, \lambda \right).
\]

(A2) For every \( u \in L^1(0,1) \), \( w \in L^\infty(0,1) \) and \((z_1, z_2) \in \mathbb{R}^2\), we have
\[
\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)dt = T_{F,w} \left( \int_0^1 w(t)u(t)dt, z_1, z_2 \right),
\]
where \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a function that depends on \((F, w)\), and it is nondecreasing with respect to the first variable.

(A3) For any \((w, u_1, u_2, u_3) \in \mathbb{R}^4\), \( u_4 \in [0, 1] \), we have
\[
wF(u_1, u_2, u_3, u_4) = F(wu_1, wu_2, wu_3, u_4) + L_w
\]
where \( L_w \in \mathbb{R} \) is a constant that depends only on \( w \).

**Definition 1.1.** Let \( f : [a, b] \to \mathbb{R}, \, (a, b) \in \mathbb{R}^2, \, a < b \), be a given function. We say that \( f \) is a convex function with respect to some \( F \in \mathcal{F} \) (or \( F \)-convex function) iff
\[
F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].
\]

One can obtain many types of convexity with the special cases of \( F \). Some of them are listed below:

**Remark 1.1.** 1) If we choose the functions \( F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R} \) by
\[
F(u_1, u_2, u_3, u_4) = u_1 - u_4u_2 - (1 - u_4)u_3 - \varepsilon
\]
and \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \frac{1}{w} \int w(t)dt \right) u_2 - \left( \frac{1}{w} \int (1-t)w(t)dt \right) u_3 - \varepsilon,
\]
then it is clear that \( F \in \mathcal{F} \) for
\[
L_w = (1 - w)\varepsilon
\]
and
\[
F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,
\]
that is, \( f \) is an \( \varepsilon \)-convex function. Particularly, if we take \( \varepsilon = 0 \), then \( f \) is a convex function.
2) Let \( h : J \to [0, \infty) \) be a given function which is not identical to 0, where \( J \) is an interval in \( \mathbb{R} \) such that \((0, 1) \subseteq J \). If we choose the functions \( F : \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R} \) by
\[
F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1 - u_4)u_3
\]
and \( T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \int_0^1 h(t)w(t)dt \right) u_2 - \left( \int_0^1 h(1 - t)w(t)dt \right) u_3,
\]
then it is clear that \( F \in \mathcal{F} \) if and
\[
F(f(tx + (1 - t)y), f(x), f(y), t) = f(tx + (1 - t)y) - h(t)f(x) - h(1 - t)f(y) \leq 0,
\]
that is, \( f \) is an \( h \)-convex function.

The following lemma obtained by Sarikaya et. al. in the paper [8] which motivates our main result.

**Lemma 1.1.** Let \( f : I^0 \subseteq \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I^0 \), \( a, b \in I^0 \) with \( a < b \). If \( f'' \in L^1[a, b] \), then the following equality holds:
\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx = (b-a)^2 \int_0^1 k(t)f''(tb+(1-t)a)dt,
\]
where
\[
k(t) = \begin{cases} \frac{1}{2} \left( \frac{1}{2} - t \right), & t \in \left[ 0, \frac{1}{2} \right) \\ \left( 1 - t \right) \left( \frac{1}{2} - \frac{1}{3} \right), & t \in \left[ \frac{1}{2}, 1 \right]. \end{cases}
\]

**2. A Simpson type inequality for \( F \)-convex function**

In this part, we obtain Theorem related to Simpson’s type inequality for functions whose second derivatives absolute values are \( F \)-convex. Then, we give special cases of this.

**Theorem 2.1.** Let \( I \subseteq \mathbb{R} \) be an interval, \( f : I^0 \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \), \((a, b) \in I^0 \times I^0 \), \( a < b \). If \( \left| f'' \right| \) is \( F \)-convex on \([a, b]\), for some \( F \in \mathcal{F} \) and the function \( t \in [0, 1] \to L_{w(t)} \) belongs to \( L^1[0, 1] \), then we have the following inequality
\[
T_{F,w} \left( \frac{1}{(b-a)^2} \left| f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x)dx \right., \left| f''(b) \right|, \left| f''(a) \right| \right)
+ \int_0^1 L_{w(t)}dt \leq 0,
\]
where \( w(t) = |k(t)| \).
Proof. Since $|f''|$ is $F$-convex, we have
\[ F(|f''(tb + (1-t)a)|, |f''(b)|, |f''(a)|, t) \leq 0, \quad t \in [0,1]. \]
Multiplying this inequality by $w(t) = |k(t)|$ and using axiom (A3), we get
\[ F(w(t)|f''(tb + (1-t)a)|, w(t)|f''(b)|, w(t)|f''(a)|, t) + L_{w(t)} \leq 0, \]
for $t \in [0,1]$. Integrating over $[0,1]$ with respect to the variable $t$ and using axiom (A2), we obtain
\[ T_{F,w} \left( \int_0^1 w(t)|f''(tb + (1-t)a)| dt, |f''(b)|, |f''(a)| \right) + \int_0^1 L_{w(t)} dt \leq 0 \]
for $t \in [0,1]$. On the other hand, using Lemma 1.1, we have
\[ \frac{1}{(b-a)^2} \left| \frac{1}{6} [f(a) + 4f \left( \frac{a+b}{2} \right) + f(b)] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{0} \int |k(t)||f''(tb + (1-t)a)| dt. \]
Since $T_{F,w}$ is nondecreasing with respect to the first variable, we get
\[ T_{F,w} \left( \frac{1}{(b-a)^2} \left| \frac{1}{6} [f(a) + 4f \left( \frac{a+b}{2} \right) + f(b)] - \frac{1}{b-a} \int_a^b f(x) dx \right|, |f''(b)|, |f''(a)| \right) \]
\[ + \int_0^1 L_{w(t)} dt \leq 0. \]
This completes the proof. \[ \square \]

Corollary 2.1. Under assumptions of Theorem 2.1, if we choose $F(u_1, u_2, u_3, u_4) = u_1 - u_4 u_2 - (1 - u_4) u_3 - \varepsilon$, then the function $|f''|$ is $\varepsilon$-convex on $[a,b]$, $\varepsilon \geq 0$ and we have the inequality
\[ \frac{1}{6} [f(a) + 4f \left( \frac{a+b}{2} \right) + f(b)] - \frac{1}{b-a} \int_a^b f(x) dx \]
\[ \leq \frac{(b-a)^2}{162} (|f''(a)| + |f''(b)|) + \frac{1}{81} (b-a)^2 \varepsilon. \]

Proof. Using (1.3) with $w(t) = |k(t)|$, we obtain
\[ \int_0^1 L_{w(t)} dt = \int_0^1 \frac{1}{(1 - |k(t)|)} dt = \varepsilon \left( \frac{1}{\int_0^1 (1 - |k(t)|) dt} + \int_0^1 (1 - |k(t)|) dt \right) = \frac{80}{81} \varepsilon. \]
From (1.2) with \( w(t) = |k(t)| \), we get

\[
T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \int_0^1 t |k(t)| \, dt \right) u_2 - \left( \int_0^1 (1 - t) |k(t)| \, dt \right) u_3 - \varepsilon 
\]

for \( u_1, u_2, u_3 \in \mathbb{R} \). Hence,

\[
0 \geq T_{F,w} \left( \frac{1}{(b-a)^2} \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|, |f''(b)|, |f''(a)| \right) 
+ \int_0^1 L_w \, dt 
\]

\[
= \frac{1}{(b-a)^2} \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| 
- \frac{1}{162} [f''(a)] + [f''(b)] - \varepsilon + \frac{80}{81} \varepsilon 
\]

that is

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| 
\leq \frac{(b-a)^2}{162} [f''(a)] + [f''(b)] + \frac{1}{81} (b-a)^2 \varepsilon. 
\]

This completes the proof. \( \square \)

**Remark 2.1.** Taking \( \varepsilon = 0 \) in Corollary 2.1, then the function \( |f''| \) is convex and we have the inequality

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{162} [f''(a)] + [f''(b)] 
\]

which is given by Sarikaya et al. in [8].

**Corollary 2.2.** Under the assumptions of Theorem 2.1, if we choose \( F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3 \), then the function \( |f''| \) is \( h \)-convex on \([a, b]\) and we have
the inequality

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|
\leq (b - a)^2 \left( \int_0^1 h(t) |k(t)| \, dt \right) [\|f''(a)\| + |f''(b)|]
\]

Proof. From (1.5) with \(w(t) = |k(t)|\), we have

\[
T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \int_0^1 h(t) |k(t)| \, dt \right) u_2 - \left( \int_0^1 h(1-t) |k(t)| \, dt \right) u_3
\]

\[
= u_1 - \left( \int_0^1 h(t) |k(t)| \, dt \right) u_2 - \left( \int_0^1 h(t) |k(1-t)| \, dt \right) u_3
\]

\[
= u_1 - \left( \int_0^1 h(t) |k(t)| \, dt \right) (u_2 + u_3)
\]

for \(u_1, u_2, u_3 \in \mathbb{R}\). Then, by Theorem 2.1,

\[
T_{F,w} \left( \frac{1}{(b-a)^2} \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|, |f''(b)|, |f''(a)| \right)
\]

\[
= \frac{1}{(b-a)^2} \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
- \left( \int_0^1 h(t) |k(t)| \, dt \right) [\|f''(a)\| + |f''(b)|] \leq 0
\]

that is,

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq (b - a)^2 \left( \int_0^1 h(t) |k(t)| \, dt \right) [\|f''(a)\| + |f''(b)|]
\]

which completes the proof. □
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REFERENCES


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