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SOME TYPES OF η-RICCI SOLITONS ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. In this paper, we study some types of η-Ricci solitons on Lorentzian para-Sasakian manifolds and we give an example of η-Ricci solitons on a 3-dimensional Lorentzian para-Sasakian manifold. We obtain the conditions for η-Ricci solitons on φ-conformally flat, φ-conharmonically flat and φ-projectively flat Lorentzian para-Sasakian manifolds. The existence of η-Ricci solitons implies that (M, g) is an η-Einstein manifold. In these cases there is no Ricci soliton on M with the potential vector field ξ.
Keywords: η-Ricci solitons, Lorentzian para-Sasakian structure, conformal curvature, conharmonic curvature and projective curvature.

1. Introduction

In 1982, Hamilton [12] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij.}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0,$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively. Ricci solitons have

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been studied in many contexts: on Kähler manifolds [10], on contact and Lorentzian manifolds [1, 7, 15, 21], on Sasakian [14], α-Sasakian [15], on Kenmotsu [2] etc. In paracontact geometry, Ricci solitons firstly appeared in the paper of G. Calvaruso and D. Perrone [8]. Ricci solitons on 3-dimensional normal paracontact manifolds were studied by C. L. Bejan and M. Crasmareanu [3].

A more general notion is that of η-Ricci soliton introduced by J. T. Cho and M. Kimura [9], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [7]. Recently, η-Ricci solitons on para-Kenmotsu manifolds were studied by A. M. Blaga [4] and η-Ricci solitons on Lorentzian para-Sasakian manifolds were also studied by A. M. Blaga [5].

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞ and ∇ be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor R (see [20]), the Weyl conformal curvature tensor C (see [23]), the conharmonic curvature tensor H (see [16]) and the projective curvature tensor P (see [23]) of (M, g) are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.2}$$

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(Y,Z)QX - g(X,Z)QY], \tag{1.4}$$

respectively, where Q is the Ricci operator, defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, $r = tr(S)$ is the scalar curvature and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

This paper is organized as follows: Section 2 consists of the basic definitions of the Lorentzian para-Sasakian manifold. In Section 3, we define Ricci and η-Ricci soliton on $(M, \varphi, \xi, \eta, g)$ and also give an example of η-Ricci solitons on a 3-dimensional Lorentzian para-Sasakian manifold. In Section 4, we obtain the conditions for η-Ricci solitons on φ-conformally flat, φ-conharmonically flat and φ-projectively flat Lorentzian para-Sasakian manifolds. The existence of η-Ricci solitons implies that (M, g) is an η-Einstein manifold. In these cases there is no Ricci soliton on M with the potential vector field ξ.

2. Lorentzian para-Sasakian manifolds

The notion of a Lorentzian para-Sasakian manifold was introduced by K. Matsumoto [17].

An n-dimensional differential manifold M^n is a Lorentzian para-Sasakian (LP-Sasakian) manifold if it admits a $(1,1)$-tensor field φ, contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g, which satisfy

$$\varphi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which imply

$$(a)\ \ \varphi\xi = 0, \quad (b)\ \ \eta(\varphi X) = 0, \quad (c)\quad rank(\varphi) = n-1, \tag{2.2}$$

Then M^n admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

and M^n is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). In this case, we have

$$(a)\quad g(X,\xi) = \eta(X), \qquad (b)\quad \nabla_X \xi = \varphi X, \tag{2.4}$$

$$(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

If we put

$$\Omega(X,Y) = g(X,\varphi Y) = g(\varphi X, Y) = \Omega(Y,X), \tag{2.6}$$

for any vector fields X and Y, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$-tensor field.

Also, since the vector field is closed in an LP-Sasakian manifold, we have

$$(\nabla_X \eta)(Y) = \Omega(X,Y) = g(\varphi X, Y) = (\nabla_Y \eta)(X), \qquad \nabla_\xi \eta = 0, \tag{2.7}$$

for any vector fields X and Y.

Also, in an LP-Sasakian manifold $(M^n, \varphi, \xi, \eta, g)$, for any $X, Y, Z \in \chi(M^n)$, the following relations hold:

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{2.8}$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.9}$$

$$\eta(R(X,Y)\xi) = 0, \tag{2.10}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.11}$$

$$L_\xi\varphi = 0, \quad L_\xi\eta = 0, \quad L_\xi g = 2g(\varphi\cdot,\cdot), \tag{2.12}$$

where R is the Riemann curvature tensor field, L is the Lie derivatives and ∇ is the Levi-Civita connection associated to g.

3. Ricci and η-Ricci Solitons on (M,φ,ξ,η,g)

Let (M,φ,ξ,η,g) be paracontact metric manifolds. Consider the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = 0, \tag{3.1}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we have:

$$2S(X,Y) = -g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any $X,Y\in\chi(M)$, or equivalent:

$$S(X,Y) = -g(\varphi X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y), \tag{3.3}$$

for any $X,Y\in\chi(M)$. The data (g,ξ,λ,μ) satisfying the equation (3.1) is said to be an η-Ricci soliton on M [9]; in particular, if $\mu=0$, (g,ξ,λ) is a Ricci soliton [13] and it is called shrinking, steady or expanding accordingly as λ is negative, zero or positive, respectively [11]. In [18] and [19] the the authors proved that on a Lorentzian para-Sasakian manifold (M,φ,ξ,η,g), the Ricci tensor field satisfies

$$S(X,\xi) = (dim(M)-1)\eta(X), \tag{3.4}$$

$$S(\varphi X,\varphi Y) = S(X,Y) + (dim(M)-1)\eta(X)\eta(Y). \tag{3.5}$$

Again putting $X=\varphi X$ and $Y=\varphi Y$ in the equation (3.3), we get

$$S(\varphi X,\varphi Y) = -g(X,\varphi Y) - \lambda g(\varphi X,\varphi Y), \tag{3.6}$$

for any $X,Y\in\chi(M)$. From (3.3) and (3.4), we obtain

$$\mu - \lambda = n-1. \tag{3.7}$$

Putting $X = Y = e_i$ in (3.3) and summing over $i = 1, 2, ..., n$, we have

$$r = \sum_{i=1}^{n} S(e_i, e_i) = -\psi - \lambda n - \mu, \tag{3.8}$$

where $\psi = tr\varphi$.

Example 3.1. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent frame field on M given by [22]

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

where a is a non-zero constant such that $a \neq 1$. Let g be the Lorentzian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,$$

$$g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1.$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$, for any $U \in \chi(M)$ and φ be the $(1, 1)$-tensor field defined by

$$\varphi E_1 = -E_1, \quad \varphi E_2 = -E_2 \quad and \quad \varphi E_3 = 0.$$

Then, using the linearity of φ and g, we have $\eta(E_3) = -1, \varphi^2 U = U + \eta(U)E_3$ and $g(\varphi U, \varphi W) = g(U, W) + \eta(U)\eta(W)$, for any $U, W \in \chi(M)$. Thus for $E_3 = \xi, (\varphi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad -g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1,$$

$$\nabla_{E_2} E_1 = ae^z E_2, \quad \nabla_{E_2} E_2 = -ae^z E_1 - E_3, \quad \nabla_{E_2} E_3 = -E_2,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

It can be easily seen that for $E_3 = \xi, (\varphi, \xi, \eta, g)$ is a Lorentzian para-Sasakian structure on M. Consequently, $(M, \varphi, \xi, \eta, g)$ is a Lorentzian para-Sasakian manifold.

Also, the Riemannian curvature tensor R is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$R(E_1,E_2)E_2 = (1-a^2e^{2z})E_1,\ R(E_1,E_3)E_3 = -E_1,\ R(E_2,E_1)E_1 = (1-a^2e^{2z})E_2,$$

$$R(E_2,E_3)E_3 = -E_2,\ R(E_3,E_1)E_1 = E_3,\ R(E_3,E_2)E_2 = E_3 + ae^z E_1.$$

Then, the Ricci tensor S is given by

$$S(E_1,E_1) = S(E_2,E_2) = -a^2e^{2z}, \quad S(E_3,E_3) = -2.$$

From (3.3), we obtain $S(E_1,E_1) = 1-\lambda$ and $S(E_{3,}E_3) = \lambda - \mu$, therefore $\lambda = 1+a^2e^{2z}$, and $\mu = 3 + a^2e^{2z}$. The data (g,ξ,λ,μ) for $\lambda = 1 + a^2e^{2z}$, and $\mu = 3 + a^2e^{2z}$ defines an η-Ricci soliton on the Lorentzian para-Sasakian manifold M.

4. Main results

In this section, we consider an η-Ricci soliton on φ-conformally flat, φ-conharmonically flat and φ-projectively flat Lorentzian para-Sasakian manifolds.

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \varphi(T_p(M^n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n)) \oplus L(\xi_p).$$

Let us consider the following particular cases:

(1) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to L(\xi_p)$, i.e., the projection of the image of C in $\varphi(T_p(M^n))$ is zero.

(2) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n))$, i.e., the projection of the image of C in $L(\xi_p)$ is zero.

$$C(X,Y)\xi = 0. \tag{4.1}$$

(3) $C : \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \to L(\xi_p)$, i.e., when C is restricted to $\varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n))$, the projection of the image of C in $\varphi(T_p(M^n))$ is zero. This condition is equivalent to

$$\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0, \tag{4.2}$$

(see[6]).

Here the cases (1), (2) and (3) are conformally symmetric, ξ-conformally flat and φ-conformally flat, respectively. The cases (1) and (2) were considered in [24] and [25], respectively. The case (3) was considered in [6] for M a K-contact manifold.

Now we will study the condition (4.2) for η-Ricci solitons on Lorentzian para-Sasakian manifolds.

Definition 4.1. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition (4.2) is called φ-conformally flat.

Suppose that $(M^n, g), n > 3$, is a φ-conformally flat Lorentzian para-Sasakian manifold. It is easy to see that $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(C(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. So by the use of (1.2), φ-conformally flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &\quad -g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) \\ &\quad -g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)] - \frac{r}{(n-1)(n-2)} \\ &\quad [g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{4.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n; then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.3) and summing over $i = 1, ..., n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) &= \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &\quad -g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) \\ &\quad -g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)] - \frac{r}{(n-1)(n-2)} \\ &\quad \sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)]. \end{aligned} \tag{4.4}$$

It can be easy to verify that

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{4.5}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + n - 1, \tag{4.6}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \tag{4.7}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n + 1, \tag{4.8}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z). \tag{4.9}$$

So applying (4.5) – (4.9) into (4.4), we obtain

$$S(\varphi Y, \varphi Z) = \left(\frac{r}{n-1} - 1\right) g(\varphi Y, \varphi Z). \tag{4.10}$$

Using (3.6) and (3.8) in (4.10), we get

$$(n-1) g(Y, \varphi Z) = (\psi + \mu + \lambda + n - 1) g(\varphi Y, \varphi Z), \tag{4.11}$$

for any $Y, Z \in \chi(M^n)$ and for $Y \mapsto \varphi Y$, we get

$$(n-1) g(\varphi Y, \varphi Z) = (\psi + \mu + \lambda + n - 1) g(Y, \varphi Z). \tag{4.12}$$

Adding the previous two equations, we have

$$(\psi + \mu + \lambda + 2n - 2)[g(Y, \varphi Z) - g(\varphi Y, \varphi Z)] = 0, \tag{4.13}$$

for any $Y, Z \in \chi(M^n)$ and follows

$$\psi + \mu + \lambda + 2n - 2 = 0. \tag{4.14}$$

Now using (3.7) in (4.24), we get

$$\lambda = \frac{3 - \psi - 3n}{2} \quad \textit{and} \quad \mu = \frac{1 - \psi - n}{2}. \tag{4.15}$$

Hence, we can state the following:

Theorem 4.1. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conformally flat, then*

$$\lambda = \frac{3 - \psi - 3n}{2} \quad \textit{and} \quad \mu = \frac{1 - \psi - n}{2}.$$

Corollary 4.1. *If (φ, ξ, η, g) is a φ-conformally flat Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, then there is no Ricci soliton with a potential vector field ξ.*

From (3.3), (3.7) and (4.11), we obtain

$$S(X,Y) = -\left(\frac{\psi + n\lambda + \mu + n - 1}{n-1}\right) g(X,Y) - \left(\frac{\psi + \lambda + \mu n + n - 1}{n-1}\right) \eta(X)\eta(Y). \tag{4.16}$$

Hence, we can state the following:

Proposition 4.1. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conformally flat, then (M^n, g) is an η-Einstein manifold.*

Let H be the conharmonic curvature tensor of M^n.

Definition 4.2. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\varphi^2 H(\varphi X, \varphi Y)\varphi Z = 0,$$

is called φ-conharmonically flat.

Now our aim is to find the characterization of η-Ricci solitons on Lorentzian para-Sasakian manifolds satisfying the above condition.

Assume that $(M^n, g), n > 3$, is a φ-conharmonically flat Lorentzian para-Sasakian manifold. It can be easily seen that $\varphi^2 H(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(H(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.3), φ-conharmonically flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-2}[& g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ & - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) \\ & - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)]. \end{aligned} \tag{4.17}$$

In a manner similar to the method in the proof of Theorem (4.1), choosing $\{e_1, e_2, ..., e_{n-1}, \xi\}$ the local orthonormal basis of vector fields in M^n, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.17) and summing over $i = 1, ..., n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \frac{1}{n-2}\sum_{i=1}^{n-1}[& g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ & - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) \\ & - g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)]. \end{aligned} \tag{4.18}$$

So applying (4.5) – (4.9) into (4.18), we get

$$S(\varphi Y, \varphi Z) = -(r+1)g(\varphi Y, \varphi Z). \tag{4.19}$$

Using (3.6) and (3.8) in the above equation, we get

$$g(Y, \varphi Z) = (-\psi - \lambda n - \lambda - \mu + 1)g(\varphi Y, \varphi Z), \tag{4.20}$$

for any $Y, Z \in \chi(M^n)$ and for $Y \mapsto \varphi Y$, we get

$$g(\varphi Y, \varphi Z) = (-\psi - \lambda n - \lambda - \mu + 1)g(Y, \varphi Z). \tag{4.21}$$

Adding the previous two equations, we have

$$(-\psi - \lambda n - \lambda - \mu + 2)[g(Y, \varphi Z) - g(\varphi Y, \varphi Z)] = 0, \tag{4.22}$$

for any $Y, Z \in \chi(M^n)$ and follows

$$[\psi + \lambda(n+1) + \mu - 2] = 0. \tag{4.23}$$

In view of (3.7) and (4.23), we obtain

$$\lambda = \frac{-(\psi + n - 3)}{(n+2)} \quad and \quad \mu = \frac{-\psi + n^2 + 1}{(n+2)}. \tag{4.24}$$

Hence, we can state the following:

Theorem 4.2. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conharmonically flat, then*

$$\lambda = \frac{-(\psi + n - 3)}{(n+2)} \quad and \quad \mu = \frac{-\psi + n^2 + 1}{(n+2)}.$$

Corollary 4.2. *If (φ, ξ, η, g) is a φ-conharmonically flat Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, then there is no Ricci soliton with the potential vector field ξ.*

From (3.3), (3.7) and (4.20), we obtain

$$\begin{aligned} S(X, Y) \;&=\; (\psi + n\lambda + \mu - 1)g(X, Y) \\ &\quad +(\psi + n\mu + \lambda - 1)\eta(X)\eta(Y). \end{aligned} \tag{4.25}$$

Hence, we can state the following:

Proposition 4.2. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conharmonically flat, then (M^n, g) is η-Einstein manifold.*

Let P be the projective curvature tensor of M^n.

Definition 4.3. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0,$$

is called φ-projectively flat.

Assume that $(M^n, g), n > 3$, is a φ-projectively flat Lorentzian para-Sasakian manifold. It can be easily seen that $\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.4), φ-projectively flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-1}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &\quad - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W). \end{aligned} \tag{4.26}$$

Similar to the proof of Theorem (4.1), we can suppose that $\{e_1, e_2, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M^n, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.26) and summing over $i = 1, ..., n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) &= \frac{1}{n-1}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &\quad - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \end{aligned} \tag{4.27}$$

So applying (4.5) – (4.9) into (4.27), we get

$$nS(\varphi Y, \varphi Z) = rg(\varphi Y, \varphi Z), \tag{4.28}$$

In view of (3.6), (3.8) and (4.28), we obtain

$$ng(Y, \varphi Z) = (\psi + \mu)g(\varphi Y, \varphi Z), \tag{4.29}$$

for any $Y, Z \in \chi(M^n)$ and for $Y \mapsto \varphi Y$, we get

$$ng(\varphi Y, \varphi Z) = (\psi + \mu)g(Y, \varphi Z). \tag{4.30}$$

Adding the previous two equations, we have

$$(\psi + \mu + n)[g(Y, \varphi Z) - g(\varphi Y, \varphi Z)] = 0, \tag{4.31}$$

for any $Y, Z \in \chi(M^n)$ and follows

$$\psi + \mu + n = 0. \tag{4.32}$$

In view of (3.7) and (4.32), we obtain

$$\lambda = -\psi - 2n + 1 \quad \textit{and} \quad \mu = -(\psi + n). \tag{4.33}$$

Hence, we can state the following:

Theorem 4.3. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-projectively flat, then*

$$\lambda = -\psi - 2n + 1 \quad \textit{and} \quad \mu = -(\psi + n).$$

Corollary 4.3. *If (φ, ξ, η, g) is a φ-projectively flat Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, then there is no Ricci soliton with the potential vector field ξ.*

From (3.3), (3.7) and (4.29), we obtain

$$\begin{aligned} S(X, Y) \quad = \quad & \left(\frac{\psi + \mu - n\lambda}{n}\right) g(X, Y) \\ & + \left(\frac{\psi + \mu - \mu n}{n}\right) \eta(X)\eta(Y). \end{aligned} \tag{4.34}$$

Hence, we can state the following:

Proposition 4.3. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-projectively flat, then (M^n, g) is an η-Einstein manifold.*

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