SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR 
h–CONVEX FUNCTIONS ON THE CO-ORDINATES VIA FRACTIONAL 
INTEGRALS

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Abstract. By making use of the identity obtained by Sarıkaya, some new Hermite-Hadamard type inequalities for h-convex functions on the co-ordinates via fractional integrals are established. Our results have some relationships with the results of Sarıkaya ([20]).

Keywords: Hermite-Hadamard inequality; h-convex functions; co-ordinated convex function; Riemann-Liouville integral.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \), then

\[
 f\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

holds, the double inequality is well known in the literature as Hermite-Hadamard inequality (see [7]).

Both inequalities hold in the reversed direction if \( f \) is concave. For recent results, generalizations and new inequalities related to the Hermite-Hadamard inequality see ([5],[15],[18],[22]).

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function \( f : [a, b] \to \mathbb{R} \).

Let us now consider a bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A mapping \( f : \Delta \to \mathbb{R} \) is said to be convex on \( \Delta \) if the following inequality:

\[
f(t(x + (1-t)z), t(y + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)
\]

is satisfied for any \( t \in [0,1] \).
A function \( f : \Delta \rightarrow \mathbb{R} \) is said to be on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v) \) are convex where defined for all \( x \in [a, b] \) and \( y \in [c, d] \) (see [6]).

A formal definition for coordinated convex function may be stated as follows:

**Definition 1.1.** A function \( f : \Delta \rightarrow \mathbb{R} \) will be called coordinated convex on \( \Delta \), for all \( t, k \in [0, 1] \) and \( (x, u), (y, w) \in \Delta \), if the following inequality holds:

\[
f(t x + (1 - t) y, ku + (1 - k) w) \leq tk f(x, u) + k (1 - t) f(y, u) + t (1 - k) f(x, w) + (1 - k) (1 - t) f(y, w).
\]

Clearly, every convex mapping is convex on the co-ordinates, but the converse is not generally true ([6]). Some interesting and important inequalities for convex functions on the co-ordinates can be found in ([10, 11, 13, 14, 21]).

In [1], Alomari and Darus established the following definition of s-convex function in the second sense on co-ordinates.

**Definition 1.2.** Consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \([0, \infty)^2\) with \( a < b \) and \( c < d \). The mapping \( f : \Delta \rightarrow \mathbb{R} \) is s-convex on \( \Delta \) if

\[
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w)
\]

holds for all \((x, y), (z, w) \in \Delta\) with \( \lambda \in [0, 1] \), and for some fixed \( s \in (0, 1] \).

A function \( f : \Delta \rightarrow \mathbb{R} \) is s-convex on \( \Delta \) is called co-ordinated s-convex on \( \Delta \) if the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v) \) are s-convex for all \( x \in [a, b] \) and \( y \in [c, d] \) with some fixed \( s \in (0, 1] \).

In [12], Latif and Alomari gave the notion of h-convexity of a function \( f \) on a rectangle from the plane \( \mathbb{R}^2 \) and h-convexity on the co-ordinates on a rectangle from the plane \( \mathbb{R}^2 \) as follows:

**Definition 1.3.** Let us consider a bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). Let \( h : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a positive function. A mapping \( f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R} \) is said to be h-convex on \( \Delta \), if \( f \) is non-negative and if the following inequality:

\[
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq h(\lambda) f(x, y) + h(1 - \lambda) f(z, w)
\]

holds, for all \((x, y), (z, w) \in \Delta\) with \( \lambda \in (0, 1) \). Let us denote this class of functions by \( \text{SX}(h, \Delta) \). The function \( f \) is said to be h-concave if the inequality reversed. We denote this class of functions by \( \text{SV}(h, \Delta) \).

A function \( f : \Delta \rightarrow \mathbb{R} \) is said to be h-convex on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v) \) are h-convex where defined for all \( x \in [a, b] \) and \( y \in [c, d] \). A formal definition of h-convex functions may also be stated as follows:
Definition 1.4. [12] A function \( f : \Delta \to \mathbb{R} \) is said to be \( h \)-convex on the co-ordinates on \( \Delta \), if the following inequality:

\[
\begin{align*}
&f \left( (tx + (1 - t)y, ku + (1 - k)w) \right) \\
&\leq h(t) h(k) f(x, u) + h(k) h(1 - t) f(y, u) \\
&+ h(t) h(1 - k) f(x, w) + h(1 - t) h(1 - k) f(y, w)
\end{align*}
\]

holds for all \( t, k \in [0, 1] \) and \((x, u), (x, w), (y, u), (y, w) \in \Delta\).

Obviously, if \( h(x) = x \), then all the non-negative convex (concave) functions on \( \Delta \) belong to the class \( SX(h, \Delta) \) (SV \((h, \Delta)\)) and if \( h(x) = x^s \), where \( s \in (0, 1) \), then the class of \( s \)-convex on \( \Delta \) belong to the class \( SX(h, \Delta) \). Similarly we can say that if \( h(x) = x \), then the class of non-negative convex (concave) functions on the co-ordinates on \( \Delta \) is contained in the class of \( h \)-convex (concave) functions on the co-ordinates on \( \Delta \) and if \( h(x) = x^s \), where \( s \in (0, 1) \), then the class of \( s \)-convex functions on the co-ordinates on \( \Delta \) is contained in the class of \( h \)-convex functions on the co-ordinates on \( \Delta \).

In the following we will give some necessary definitions which are used further in this paper. For more details, one can consult \((8),(9),(17)\).

Definition 1.5. Let \( f \in L_1[a,b] \). The Riemann-Liouville integrals \( J^\alpha_a f \) and \( J^\alpha_b f \) order \( \alpha > 0 \) and \( \alpha \geq 0 \) are defined by

\[
J^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \ dt, \ x > a
\]

and

\[
J^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \ dt, \ x < b
\]

respectively. Here \( \Gamma(\alpha) \) is the Gamma function and \( J^\alpha_a f(x) = J^\alpha_b f(x) = f(x) \).

Definition 1.6. Let \( f \in L_1([a,b] \times [c,d]) \). The Riemann-Liouville integrals \( J^\alpha_{a,c} f \), \( J^\alpha_{a,d} f \), \( J^\beta_{b,c} f \) and \( J^\beta_{b,d} f \) of order \( \alpha, \beta > 0 \) with \( a, c \geq 0 \) are defined by

\[
J^\alpha_{a,c} f(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^c \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t, s) \ ds \ dt, \ x > a, y > c
\]

and

\[
J^\alpha_{a,d} f(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^d \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t, s) \ ds \ dt, \ x > a, y < d
\]
\[ \int_{b}^{a} f(x, y) \]
\[ = \frac{1}{\Gamma(a) \Gamma(\beta)} \int_{x}^{b} \int_{y}^{c} (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s) ds dt, \; x < b, \; y > c \]
\[ \int_{d}^{a} f(x, y) \]
\[ = \frac{1}{\Gamma(a) \Gamma(\beta)} \int_{x}^{b} \int_{y}^{d} (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s) ds dt, \; x < b, \; y < d \]

respectively. Here, \( \Gamma \) is the Gamma function,
\[ \int_{a}^{b} f(x, y) = \int_{a}^{b} f(x, y) = \int_{b}^{a} f(x, y) = \int_{b}^{a} f(x, y) = f(x, y) \]
and
\[ \int_{a}^{1} f(x, y) = \int_{a}^{b} f(t,s) ds dt. \]

For some recent results connected with fractional integral inequalities see ([2, 3, 4, 16, 19, 23]).

In [20], Sarıkaya established the following inequalities of Hadamard’s type for coordinated convex mapping on a rectangle from the plane \( \mathbb{R}^2 \):

**Theorem 1.1.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be coordinated convex on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \) and \( f \in L_1(\Delta) \). Then one has the inequalities:

\[ (1.2) \]
\[ f\left( \frac{a + b}{2}, \frac{c + d}{2} \right) \]
\[ \leq \frac{\Gamma(a + 1) \Gamma(\beta + 1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \]
\[ \times \left[ \int_{a}^{b} f(b, d) + \int_{a}^{d} f(a, c) + \int_{b}^{d} f(a, d) + \int_{a}^{b} f(c, c) \right] \]
\[ \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \]

**Theorem 1.2.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \frac{\partial^2 f}{\partial x \partial y} \) is a convex function on the co-ordinates on
\[\Delta\text{, then one has the inequalities:}\]

\[(1.3)\]
\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right|
\]
\[
+ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta}
\]
\[
x \left[ f_{\alpha, \beta}, f(b, d) + f_{\alpha, \beta}, f(b, c) + f_{\alpha, \beta}, f(a, d) + f_{\alpha, \beta}, f(a, c) \right] - A
\]
\[
\leq \frac{(b-a)(d-c)}{4(\alpha + 1)(\beta + 1)}
\]
\[
x \left( \left| \frac{\partial^2 f}{\partial k \partial t}(a, c) \right| + \left| \frac{\partial^2 f}{\partial k \partial t}(a, d) \right| + \left| \frac{\partial^2 f}{\partial k \partial t}(b, c) \right| + \left| \frac{\partial^2 f}{\partial k \partial t}(b, d) \right| \right)
\]

where

\[A = \frac{\Gamma(\beta + 1)}{4(d-c)^\beta} \left[ f_{\beta, \gamma}^{\alpha, \beta}, f(a, d) + f_{\beta, \gamma}^{\alpha, \beta}, f(b, d) + f_{\beta, \gamma}^{\alpha, \beta}, f(a, c) + f_{\beta, \gamma}^{\alpha, \beta}, f(b, c) \right]
\]
\[
+ \frac{\Gamma(\alpha + 1)}{4(b-a)^\alpha} \left[ f_{\alpha, \beta}^{\alpha, \beta}, f(b, c) + f_{\alpha, \beta}^{\alpha, \beta}, f(b, d) + f_{\alpha, \beta}^{\alpha, \beta}, f(a, c) + f_{\alpha, \beta}^{\alpha, \beta}, f(a, d) \right].
\]

**Theorem 1.3.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \left| \frac{\partial^q f}{\partial x^p} \right| q > 1 \) is a convex function on the co-ordinates on \( \Delta \), then one has the inequalities:

\[(1.4)\]
\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right|
\]
\[
+ \left\{ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \right\}
\]
\[
x \left[ f_{\alpha, \beta}^{\alpha, \beta}, f(b, d) + f_{\alpha, \beta}^{\alpha, \beta}, f(b, c) + f_{\alpha, \beta}^{\alpha, \beta}, f(a, d) + f_{\alpha, \beta}^{\alpha, \beta}, f(a, c) \right] - A
\]
\[
\leq \frac{(b-a)(d-c)}{[(\alpha + 1)(\beta \gamma + 1)]^\frac{1}{2}} \left( \frac{1}{4} \right)^\frac{1}{2}
\]
\[
x \left( \left| \frac{\partial^2 f}{\partial k \partial t}(a, c) \right| q + \left| \frac{\partial^2 f}{\partial k \partial t}(a, d) \right| q + \left| \frac{\partial^2 f}{\partial k \partial t}(b, c) \right| q + \left| \frac{\partial^2 f}{\partial k \partial t}(b, d) \right| q \right)^\frac{1}{2}
Lemma 1.1. \( f: \Delta \subset R^2 \rightarrow R \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( R^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \frac{\partial f}{\partial xk} \in L(\Delta) \), then the following equality holds:

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy = \frac{1}{4} \int_0^1 \int_0^1 t^a k^b \frac{\partial^2 f}{\partial t \partial k} (ta + (1 - t)b, kc + (1 - k)d) \, dk \, dt
\]

where

\[
A = \frac{\Gamma(\beta + 1)}{4(d - c)^\beta} \left[ j_c^\beta f(a, d) + j_{c-}^\beta f(b, d) + j_a^\beta f(c, a) + j_{a-}^\beta f(b, c) \right] + \frac{\Gamma(\alpha + 1)}{4(b - a)^\alpha} \left[ j_a^\alpha f(b, c) + j_{a-}^\alpha f(b, d) + j_c^\alpha f(a, c) + j_{c-}^\alpha f(a, d) \right]
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

In order to prove our main results we need the following lemma (see [20]).

Lemma 1.1. Let \( f: \Delta \subset R^2 \rightarrow R \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( R^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \frac{\partial f}{\partial xk} \in L(\Delta) \), then the following equality holds:

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy = \frac{1}{4} \int_0^1 \int_0^1 t^a k^b \frac{\partial^2 f}{\partial t \partial k} (ta + (1 - t)b, kc + (1 - k)d) \, dk \, dt
\]
Hermite-Hadamard Type Inequalities for $h$–convex Function

2. Main Results

Theorem 2.1. Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be $h$-convex function on the co-ordinates on $\Delta$ in $\mathbb{R}^2$ and $f \in L_2(\Delta)$. Then one has the inequalities:

(2.1) \[
\begin{align*}
&f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \\
&\leq \left[h\left(\frac{1}{2}\right)\right]^2 \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(b-a)\alpha (d-c)\beta} \\
&\times \left[\int_{a+c}^{b+d} f(x) + \int_{a+c}^{b+d} f(x) + \int_{b+c}^{d+a} f(x) + \int_{b+c}^{d+a} f(x)\right] \\
&\leq \left[h\left(\frac{1}{2}\right)\right]^2 \alpha \beta \left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right] \\
&\times \left[\int_0^1 \int_0^1 t^{a-1} k^{\beta-1} \right. \\
&\left.\times \left[h(t)h(k) + h(t)h(1-k) + h(1-t)h(k) + h(1-t)h(1-k)\right] dtdk\right].
\end{align*}
\]

Proof. According to (1.1) with $x = t_1 a + (1-t_1) b, y = (1-t_1) a + t_1 b, u = k_1 c + (1-k_1) d, w = (1-k_1) c + k_1 d$ and $t = k = \frac{1}{2}$, we find that

(2.2) \[
\begin{align*}
&f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \\
&\leq \left[h\left(\frac{1}{2}\right)\right]^2 \\
&\times \left[f(t_1 a + (1-t_1) b, k_1 c + (1-k_1) d) + f(t_1 a + (1-t_1) b, (1-k_1) c + k_1 d) \\
+f((1-t_1) a + t_1 b, k_1 c + (1-k_1) d) + f((1-t_1) a + t_1 b, (1-k_1) c + k_1 d)\right].
\end{align*}
\]

Thus, multiplying both sides of (2.2) by $t_1^{a-1} k_1^{\beta-1}$, then by integrating with respect to $(t_1, k_1)$ on $[0, 1] \times [0, 1]$, we obtain
\[
\frac{1}{\alpha \beta} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right)
\leq \left[ h \left( \frac{1}{2} \right) \right]^2 
\times \left[ \int_0^1 \int_0^{1-t} t^{-1} k_1^{\beta-1} \right.
\times \left[ f \left( t_1 a + (1 - t_1) b, k_1 c + (1 - k_1) d \right) 
+ f \left( t_1 a + (1 - t_1) b, (1 - k_1) c + k_1 d \right) 
+ f \left((1 - t_1) a + t_1 b, k_1 c + (1 - k_1) d \right) 
\right. 
\left. + f \left((1 - t_1) a + t_1 b, (1 - k_1) c + k_1 d \right) \right] \, dk_1 \, dt_1
\]

On the right side of the above inequality, using the changes of the variable

\[
\begin{align*}
&\{ x = t_1 a + (1 - t_1) b, y = k_1 c + (1 - k_1) d \}, \\
&\{ x = t_1 a + (1 - t_1) b, y = (1 - k_1) c + k_1 d \}, \\
&\{ x = (1 - t_1) a + t_1 b, y = k_1 c + (1 - k_1) d \}, \\
&\{ x = (1 - t_1) a + t_1 b, y = (1 - k_1) c + k_1 d \}
\end{align*}
\]

respectively, we get

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right)
\leq \left[ h \left( \frac{1}{2} \right) \right]^2 
\times \left\{ \int_a^b \int_c^d (b - x)^{\alpha-1} (d - y)^{\beta-1} f(x, y) \, dy \, dx 
+ \int_a^b \int_c^d (b - x)^{\alpha-1} (y - c)^{\beta-1} f(x, y) \, dy \, dx 
+ \int_a^b \int_c^d (x - a)^{\alpha-1} (d - y)^{\beta-1} f(x, y) \, dy \, dx 
+ \int_a^b \int_c^d (x - a)^{\alpha-1} (y - c)^{\beta-1} f(x, y) \, dy \, dx \right\}
\]

from which the first inequality is proved.

For the proof of the second inequality (2.1), we first note that if \( f \) is a \( h \)-convex function on \( \Delta \), then, by using (1.1) with \( x = a, y = b, u = c, w = d \), it yields

\[
\begin{align*}
f \left( t a + (1 - t) b, k c + (1 - k) d \right) 
\leq h \left( t \right) h \left( k \right) f(a, c) + h \left( t \right) h \left( 1 - k \right) f(a, d) \\
+ h \left( 1 - t \right) h \left( k \right) f(b, c) + h \left( 1 - t \right) h \left( 1 - k \right) f(b, d)
\end{align*}
\]
Then, multiplying both sides of (2.3) by $(t, k)$ over $[0 \leq t \leq 1] \times [0 \leq k \leq 1]$, we get

$$f ((1 - t)a + tb, kc + (1 - k)d)$$

$$\leq h (1 - t) h (k) f (a, c) + h (1 - t) h (1 - k) f (a, d)$$

$$+ h (t) h (k) f (b, c) + h (t) h (1 - k) f (b, d)$$

$$f (ta + (1 - t)b, (1 - k)c + kd)$$

$$\leq h (t) h (1 - k) f (a, c) + h (t) h (k) f (a, d)$$

$$+ h (1 - t) h (1 - k) f (b, c) + h (t) h (k) f (b, d).$$

By adding these inequalities we have

(2.3)

$$f (ta + (1 - t)b, kc + (1 - k)d) + f ((1 - t)a + tb, kc + (1 - k)d)$$

$$+ f (ta + (1 - t)b, (1 - k)c + kd) + f ((1 - t)a + tb, (1 - k)c + kd)$$

$$\leq [f (a, c) + f (b, c) + f (a, d) + f (b, d)]$$

$$\times [h (t) h (k) + h (t) h (1 - k) + h (1 - t) h (k) + h (1 - t) h (1 - k)].$$

Then, multiplying both sides of (2.3) by $t^{a-1}k^{b-1}$ and integrating with respect to $(t, k)$ over $[0, 1] \times [0, 1]$, we get

$$\int_0^1 \int_0^1 t^{a-1}k^{b-1}$$

$$\times [f (ta + (1 - t)b, kc + (1 - k)d) + f ((1 - t)a + tb, kc + (1 - k)d)$$

$$+ f (ta + (1 - t)b, (1 - k)c + kd) + f ((1 - t)a + tb, (1 - k)c + kd)] dkdtdt$$

$$\leq [f (a, c) + f (b, c) + f (a, d) + f (b, d)]$$

$$\times \left\{ \int_0^1 \int_0^1 t^{a-1}k^{b-1}\right.$$
Here, using the change of the variable we have
\[
\left[ h \left( \frac{1}{2} \right) \right]^2 \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{(b - a)^\alpha (d - c)^\beta} \\
\times \left[ J_{a\alpha,\beta}^{\alpha,\beta} f (b, d) + J_{a\alpha,\beta}^{\alpha,\beta} f (b, c) + J_{b\beta,\alpha}^{\beta,\alpha} f (a, d) + J_{b\beta,\alpha}^{\beta,\alpha} f (a, c) \right] \\
\leq \left[ h \left( \frac{1}{2} \right) \right]^2 \alpha \beta \left[ f (a, c) + f (a, d) + f (b, c) + f (b, d) \right] \\
\times \left[ \int_0^1 \int_0^1 t^{\alpha-1} k^{\beta-1} \\
\times [ h (t) h (k) + h (t) h (1 - k) + h (1 - t) h (k) + h (1 - t) h (1 - k) ] dkdt \right].
\]

The proof is completed. \(\square\)

Remark 2.1. If we take \(h(\alpha) = \alpha\) in Theorem 2.1, then the inequality (2.1) becomes the inequality (1.2) of Theorem 1.1.

Corollary 2.1. If we take \(h (\alpha) = \alpha^s\) in Theorem 2.1, we have the following inequality:
\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
\leq \left( \frac{1}{2} \right)^{2s} \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{(b - a)^\alpha (d - c)^\beta} \\
\times \left[ J_{a\alpha,\beta}^{\alpha,\beta} f (b, d) + J_{a\alpha,\beta}^{\alpha,\beta} f (b, c) + J_{b\beta,\alpha}^{\beta,\alpha} f (a, d) + J_{b\beta,\alpha}^{\beta,\alpha} f (a, c) \right] \\
\leq \left( \frac{1}{2} \right)^{2s} \alpha \beta \left[ f (a, c) + f (a, d) + f (b, c) + f (b, d) \right] \\
\times \left( \frac{1}{\alpha + s} + B (\alpha, s + 1) \left( \frac{1}{\beta + s} + B (\beta, s + 1) \right) \right)
\]
\[\text{where } B \text{ is the Beta function,} \]
\[B (x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt.\]

Theorem 2.2. Let \(f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}\) be a partial differentiable mapping on \(\Delta\) in \(\mathbb{R}^2\). If
Hermite-Hadamard Type Inequalities for $h$-convex Function

$\frac{\partial^2 f}{\partial t \partial k}$ is a $h$-convex function on the co-ordinates on $\Delta$, then one has the inequalities:

\begin{align}
\left| f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
+ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4 (b-a)^\alpha (d-c)^\beta} \\
\times [ J^{\alpha,\beta}_{a,c} f(b,d) + J^{\alpha,\beta}_{b,c} f(b,c) + J^{\alpha,\beta}_{a,c} f(a,d) + J^{\alpha,\beta}_{b,c} f(a,c) ] - A \\
\leq \frac{(b-a)(d-c)}{4} \times \left\{ \left| \frac{\partial^2 f}{\partial t \partial k} (a,c) \right| \int_0^1 \int_0^1 \left( t^\alpha + (1-t)^\alpha \right) \left( k^\beta + (1-k)^\beta \right) h(t) h(k) \, dk \, dt \\
+ \left| \frac{\partial^2 f}{\partial t \partial k} (b,c) \right| \int_0^1 \int_0^1 \left( t^\alpha + (1-t)^\alpha \right) \left( k^\beta + (1-k)^\beta \right) h(t) h(1-k) \, dk \, dt \\
+ \left| \frac{\partial^2 f}{\partial t \partial k} (a,d) \right| \int_0^1 \int_0^1 \left( t^\alpha + (1-t)^\alpha \right) \left( k^\beta + (1-k)^\beta \right) h(1-t) h(k) \, dk \, dt \\
+ \left| \frac{\partial^2 f}{\partial t \partial k} (b,d) \right| \int_0^1 \int_0^1 \left( t^\alpha + (1-t)^\alpha \right) \left( k^\beta + (1-k)^\beta \right) h(1-t) h(1-k) \, dk \, dt \right\}
\end{align}

where

\begin{align}
A &= \frac{\Gamma(\beta + 1)}{4 (d-c)^\beta} \left[ J^{\beta}_{a,c} f(b,d) + J^{\beta}_{c,b} f(b,c) + J^{\beta}_{a,c} f(a,d) + J^{\beta}_{c,b} f(a,c) \right] \\
+ \frac{\Gamma(\alpha + 1)}{4 (b-a)^\alpha} \left[ J^{\alpha}_{a,d} f(b,c) + J^{\alpha}_{d,a} f(b,d) + J^{\alpha}_{a,d} f(a,c) + J^{\alpha}_{d,a} f(a,d) \right].
\end{align}
Proof. From Lemma 1.1, we have

\[
\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \times \left[ \int_{a_{\alpha, \beta}}^{c_{\alpha, \beta}} f(b,d) + \int_{a_{\alpha, \beta}}^{b_{\alpha, \beta}} f(b,c) + \int_{a_{\alpha, \beta}}^{c_{\alpha, \beta}} f(a,d) + \int_{b_{\alpha, \beta}}^{c_{\alpha, \beta}} f(a,c) \right] - \Delta
\]

\[
\leq \frac{(b-a)(d-c)}{4} \left\{ \int_0^1 \int_0^1 (t^\alpha (1-t)^\beta)(k^\beta + (1-k)^\beta) \frac{\partial^2 f}{\partial t \partial k}(ta + (1-t)b, kc + (1-k)d) \, dk \, dt \right. \\
+ \int_0^1 \int_0^1 (1-t)^\alpha k^\beta \frac{\partial^2 f}{\partial t \partial k}(ta + (1-t)b, kc + (1-k)d) \, dk \, dt \\
+ \int_0^1 \int_0^1 t^\alpha (1-k)^\beta \frac{\partial^2 f}{\partial t \partial k}(ta + (1-t)b, kc + (1-k)d) \, dk \, dt \\
+ \int_0^1 \int_0^1 (1-t)^\alpha (1-k)^\beta \frac{\partial^2 f}{\partial t \partial k}(ta + (1-t)b, kc + (1-k)d) \, dk \, dt \right\}.
\]

Since \( \frac{\partial^2 f}{\partial t \partial k} \) is h-convex function on the co-ordinates on \( \Delta \), then one has:

\[
\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \times \left[ \int_{a_{\alpha, \beta}}^{c_{\alpha, \beta}} f(b,d) + \int_{a_{\alpha, \beta}}^{b_{\alpha, \beta}} f(b,c) + \int_{a_{\alpha, \beta}}^{c_{\alpha, \beta}} f(a,d) + \int_{b_{\alpha, \beta}}^{c_{\alpha, \beta}} f(a,c) \right] - \Delta
\]

\[
\leq \frac{(b-a)(d-c)}{4} \times \left\{ \int_0^1 \int_0^1 (t^\alpha (1-t)^\beta) (k^\beta + (1-k)^\beta) h(t)h(k) \, dk \, dt \\
+ \int_0^1 \int_0^1 (1-t)^\alpha k^\beta h(t)h(k) \, dk \, dt \\
+ \int_0^1 \int_0^1 t^\alpha (1-k)^\beta h(t)h(k) \, dk \, dt \\
+ \int_0^1 \int_0^1 (1-t)^\alpha (1-k)^\beta h(t)h(k) \, dk \, dt \right\}.
\]
The proof is completed. □

**Remark 2.2.** If we take $h(\alpha) = \alpha$ in Theorem 2.2, then the inequality (2.4) becomes the inequality (1.3) of Theorem 1.2.

**Corollary 2.2.** If we take $h(\alpha) = \alpha^s$ in Theorem 2.2, we have

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \left\{ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b - a)^\alpha (d - c)^\beta} \left[ \int_{a,c}^{b,d} f(b, d) + \int_{a,d}^{b,c} f(b, c) + \int_{a,c}^{b,d} f(a, d) + \int_{b,c}^{a,d} f(a, c) \right] - A \right\}
\]

\[\leq \frac{(b - a)(d - c)}{4} \times \left[ \frac{\partial^2 f}{\partial t \partial k} (a, c) \right] + \left[ \frac{\partial^2 f}{\partial t \partial k} (b, c) \right] + \left[ \frac{\partial^2 f}{\partial t \partial k} (a, d) \right] + \left[ \frac{\partial^2 f}{\partial t \partial k} (b, d) \right] \times \left( \frac{1}{\alpha + s + 1} + B(s + 1, \alpha + 1) \right) \left( \frac{1}{\beta + s + 1} + B(s + 1, \beta + 1) \right)
\]

where

\[A = \frac{\Gamma(\beta + 1)}{4(d - c)^\beta} \left[ \int_{a,c}^{b,d} f(b, d) + \int_{a,d}^{b,c} f(b, c) + \int_{a,c}^{b,d} f(a, d) + \int_{b,c}^{a,d} f(a, c) \right] + \frac{\Gamma(\alpha + 1)}{4(b - a)^\alpha} \left[ \int_{a,c}^{b,d} f(b, c) + \int_{a,d}^{b,c} f(b, d) + \int_{a,c}^{b,d} f(a, c) + \int_{b,c}^{a,d} f(a, d) \right]
\]

and $B$ is the Beta function,

\[B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.
\]

**Theorem 2.3.** Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta$ in $\mathbb{R}^2$. If $\left| \frac{\partial^2 f}{\partial t \partial k} \right|, q > 1$, is an h-convex function on the co-ordinates on $\Delta$, then one has the inequalities:
Proof. From Lemma 1.1, we have

\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right|
\]

\[
+ \left\{ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \times \left[ I_{a^{\alpha}, c^{\beta}} f(b, d) + I_{a^{\alpha}, d^{\beta}} f(b, c) + I_{b^{\beta}, c^{\alpha}} f(a, c) + I_{b^{\beta}, d^{\alpha}} f(a, d) \right] \right\} - A
\]

\[
\leq \frac{(b-a)(d-c)}{[(\alpha + 1)(\beta p + 1)]^\frac{1}{q}}
\]

\[
\times \left[ \left( \frac{\partial^2 f}{\partial k \partial t}(a, c) \right)^q \int_0^1 \int_0^1 h(t) h(k) dk dt \right]
\]

\[
+ \left( \frac{\partial^2 f}{\partial k \partial t}(a, d) \right)^q \int_0^1 \int_0^1 h(t) h(1-k) dk dt
\]

\[
+ \left( \frac{\partial^2 f}{\partial k \partial t}(b, c) \right)^q \int_0^1 \int_0^1 h(1-t) h(k) dk dt
\]

\[
+ \left( \frac{\partial^2 f}{\partial k \partial t}(b, d) \right)^q \int_0^1 \int_0^1 h(1-t)(1-k) dk dt \right] \right)^\frac{1}{q},
\]

where

\[
A = \frac{\Gamma(\beta + 1)}{4(d-c)^\beta} \left[ I_{c^{\beta}} f(a, d) + I_{c^{\beta}} f(b, d) + I_{a^{\alpha}, c^{\beta}} f(a, c) + I_{a^{\alpha}, d^{\beta}} f(a, d) \right]
\]

\[
+ \frac{\Gamma(\alpha + 1)}{4(b-a)^\alpha} \left[ I_{a^{\alpha}, c^{\beta}} f(b, c) + I_{a^{\alpha}, d^{\beta}} f(b, d) + I_{b^{\beta}, c^{\alpha}} f(a, c) + I_{b^{\beta}, d^{\alpha}} f(a, d) \right]
\]

and \(\frac{1}{p} + \frac{1}{q} = 1\).

Proof. From Lemma 1.1, we have

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
\]

\[
+ \left\{ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \times \left[ I_{a^{\alpha}, c^{\beta}} f(b, d) + I_{a^{\alpha}, d^{\beta}} f(b, c) + I_{b^{\beta}, c^{\alpha}} f(a, c) + I_{b^{\beta}, d^{\alpha}} f(a, d) \right] \right\} - A
\]
By using the well known H"{o}lder’s inequality for double integrals, we get

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| \\
+ \left\{ \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \times \left[ \int_a^b f^\alpha \left( \frac{\partial^2 f}{\partial t \partial k} \right)^\alpha (ta + (1-t)b, kc + (1-k)d) \, dk \, dt \right] \right\}^{1/\gamma}\]

\[
\leq \frac{(b-a)(d-c)}{4} \left\{ \left( \int_0^1 \int_0^1 t^\alpha k^\beta \, dk \, dt \right)^{1/\gamma} + \left( \int_0^1 \int_0^1 (1-t)^{\alpha} k^{\beta} \, dk \, dt \right)^{1/\gamma} \left[ \int_a^b f^\alpha \left( \frac{\partial^2 f}{\partial t \partial k} \right)^\alpha (ta + (1-t)b, kc + (1-k)d) \, dk \, dt \right] \right\}^{1/\gamma}.
\]

Since \( \frac{\partial^2 f}{\partial t \partial k} \) is h-convex function on the co-ordinates on \( \Delta \), then one has:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| \\
+ \left\{ \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \times \left[ \int_a^b f^\alpha \left( \frac{\partial^2 f}{\partial t \partial k} \right)^\alpha (ta + (1-t)b, kc + (1-k)d) \, dk \, dt \right] \right\}.
\]
\[ \leq \frac{(b-a)(d-c)}{[(ap+1)(\beta p+1)]^\frac{1}{2}} \]
\[ \quad \times \left( \left\| \frac{\partial^2 f}{\partial k \partial t}(a,c) \right\|^q \int_0^1 \int_0^1 h(t) h(k) \, dk \, dt \right. 
\[ \quad + \frac{\partial f}{\partial k}(a,d) \int_0^1 \int_0^1 h(t) h(1-k) \, dk \, dt 
\[ \quad + \frac{\partial^2 f}{\partial k \partial t}(b,c) \int_0^1 \int_0^1 h(1-t) h(k) \, dk \, dt 
\[ \quad + \left. \frac{\partial^2 f}{\partial k \partial t}(b,d) \int_0^1 \int_0^1 h(1-t) h(1-k) \, dk \, dt \right)^\frac{1}{2} \]

and the proof is completed. \( \square \)

**Remark 2.3.** If we take \( h(\alpha) = \alpha \) in Theorem 2.3, then the inequality (2.5) becomes the inequality (1.4) of Theorem 1.3.

**Corollary 2.3.** If we take \( h(\alpha) = \alpha^\epsilon \) in Theorem 2.3, we have

\[ \left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| \]
\[ \quad + \left\{ \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \right\} \]
\[ \times \left[ \int_{a,c}^{a,b} f(b,d) + \int_{a,d}^{a,b} f(b,c) + \int_{b,c}^{b,d} f(a,d) + \int_{b,d}^{b,c} f(a,c) \right] - A \]
\[ \leq \frac{(b-a)(d-c)}{[(ap+1)(\beta p+1)]^\frac{1}{2}} \]
\[ \times \left( \frac{1}{(s+1)^2} \left[ \left\| \frac{\partial^2 f}{\partial k \partial t}(a,c) \right\|^q + \left\| \frac{\partial^2 f}{\partial k \partial t}(a,d) \right\|^q + \left\| \frac{\partial^2 f}{\partial k \partial t}(b,c) \right\|^q + \left\| \frac{\partial^2 f}{\partial k \partial t}(b,d) \right\|^q \right] \right)^\frac{1}{2} \]

where

\[ A = \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[ \int_{a,c}^{a,b} f(b,d) + \int_{a,d}^{a,b} f(b,c) + \int_{b,c}^{b,d} f(a,d) + \int_{b,d}^{b,c} f(a,c) \right] \]
\[ + \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[ \int_{a,c}^{a,b} f(b,c) + \int_{a,d}^{a,b} f(b,d) + \int_{b,c}^{b,d} f(a,c) + \int_{b,d}^{b,c} f(a,d) \right] . \]
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