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MULTIDISKCYCLIC OPERATORS ON BANACH SPACES

Nareen Bamerni

Abstract. In this paper, we define and study multidiskcyclic operators and find some of their properties. Peris (2001) proved that every multihypercyclic operator is hypercyclic. We show the corresponding result for multidiskcyclic operators. In particular, we show that every multidiskcyclic operator is diskcyclic too.
Keywords. Multidiskcyclic operators; Banach space; hypercyclic operator.

1. Introduction

An operator T is called hypercyclic if there is a vector $x \in \mathcal{H}$ such that $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, such a vector x is called hypercyclic for T. The first example of hypercyclic operators in a Banach space was constructed by Rolewicz in 1969 [13]. He proved that if B is a backward shift on the Banach space $\ell^p(\mathbb{N})$ then λB is hypercyclic for any complex number λ; $|\lambda| > 1$. Motivated by Rolewicz's example, supercyclic operators and diskcyclic operators were defined. An operator T is supercyclic if there is a vector $x \in \mathcal{H}$ such that $\mathbb{C}Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where x is called supercyclic vector [9]. An operator T is called diskcyclic if there is a vector $x \in \mathcal{H}$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$ is dense in $\mathcal{H}$, such a vector x is called diskcyclic for T [15]. For more information on these concepts, one may refer to [5, 4, 2].

Recently, these operators were extended to subspaces of Banach spaces, which are called subspace-hypercyclic, subspace-supercyclic and subspace-diskcyclic. For more details on these operators, we refer the reader to [10, 1, 14, 3].

In 1992, Herrero [8] generalized the concepts of hypercyclicity and supercyclicity to multihypercyclicity and multisupercyclicity, respectively as follows:

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Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{X})$ is called multihypercyclic (or multisupercyclic), if there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of $\mathcal{X}$ such that $\bigcup_{k=1}^{n} Orb(T, x_k)$ (or $\mathbb{C} \bigcup_{k=1}^{n} Orb(T, x_k)$, respectively) is dense in $\mathcal{X}$.

Herrero [8] posed the following conjecture:

if T is multihypercyclic (or multisupercyclic), then T is hypercyclic (or supercyclic, respectively)

Costakis [7] and Peris [12] independently proved Herrero's conjecture positively. For more information on these concepts, the reader may be refered to [8, 6, 7, 12, 11].

Now, since both multihypercyclic operators and multisupercyclic operators have been defined and studied, then it is natural to define and study multidiskcyclic operators as well. Therefore, the purpose of this section is to define multidiskcyclic operators and find some of their properties which are similar to those of multihypercyclicity and multisupercyclicity. We show that if T is multidiskcyclic, then every positive integer power of T is multidiskcyclic and T^* has at most one eigenvalue; and that one has to have a modulus greater than one. Finally, we show that every multidiskcyclic operator is diskcyclic.

2. Main results

Definition 2.1. Let $L = \{x_1, \cdots, x_m\} \subset \mathcal{X}$, $T \in \mathcal{B}(\mathcal{X})$ and $\mathbb{D}Orb(T, L) = \bigcup_{i=1}^{m} \mathbb{D}Orb(T, x_i)$. If L is minimal such that $\mathbb{D}Orb(T, L)$ is dense, then T is called a multidiskcyclic operator and L is called a diskcyclic set for T.

It is clear form the above definition, that every diskcyclic operator is multidiskcyclic.

The following two results give the common properties between multidiskcyclic operators and diskcyclic operators.

Theorem 2.1. *If T is multidiskcyclic, then T^n is multidiskcyclic for all $n \geqslant 2$.*

Proof. Let L be a diskcyclic set for T, then it is clear that

$$\bigcup_{i=1}^{m} \bigcup_{j=0}^{n-1} \mathbb{D}Orb(T^n, T^j x_i) = \bigcup_{i=1}^{m} \mathbb{D}Orb(T, x_i).$$

It follows that T^n is multidiskcyclic with a multidiskcyclic set $\{T^j x_i : 1 \leqslant i \leqslant m, 0 \leqslant j \leqslant n-1\}$. $\square$

Proposition 2.1. *If T is a multidiskcyclic operator on a Hilbert space $\mathcal{H}$, then T^* has at most one eigenvalue. If $\sigma_p(T^*) = \{\lambda\}$, then λ has a modulus greater than one.*

Proof. Since each multidiskcyclic operator is multisupercyclic, then the adjoint of a multidiskcyclic operator has at most one eigenvalue [11, Theorem 5]. Now, suppose that $\sigma_p(T^*) = \{\lambda\}$. Towards a contradiction assume that $|\lambda| \leqslant 1$.
Let $L = \{x_1, \cdots, x_m\}$ be diskcyclic set for T. Then there exists a unit vector z in which $T^*z = \lambda z$ and

$$\left\{ \bigcup_{i=1}^{m} |\langle \mu T^n x_i, z\rangle| : n \geqslant 0, \mu \in \mathbb{D},\, x_i \in L \right\} \text{ is dense in } \mathbb{R}^+ \cup \{0\}\,. \tag{2.1}$$

Since $|\langle \mu T^n x_i, z\rangle| \leqslant |\mu|\, |\lambda|^n \, \|x_i\| \|z\|$ for all $1 \leqslant i \leqslant m$, and since $|\lambda| \leqslant 1$, then

$$|\langle \mu T^n x_i, z\rangle| \leqslant \|x_i\| \|z\|,$$

that is, $\{\bigcup_{i=1}^{m} |\langle \mu T^n x_i, z\rangle| : n \geqslant 0, \mu \in \mathbb{D},\, x_i \in L\}$ is bounded above, a contradiction to the equation (2.1). □

Miller [11] proved that if T is multihypercyclic (or multisupercyclic) then there exists a vector x such that $Orb(T, x)$ (or $\mathbb{C}Orb(T, x)$, respectively) is somewhere dense. Later on, Bourdon and Feldman [6] showed that the somewhere density of orbit and dense orbit imply to everywhere density of them. It follows that every multihypercyclic (or multisupercyclic) operator is hypercyclic (or supercyclic, respectively). The next theorem shows the analogue of Miller's result for multidiskcyclicity.

Proposition 2.2. *If T is multidiskcyclic, then there exists a vector $x \in \mathcal{X}$ such that the disk orbit of x under T is somewhere dense.*

Proof. Let L be a diskcyclic set for T. Towards a contradiction, suppose that $\mathbb{D}Orb(T, x)$ is nowhere dense for all $x \in \mathcal{X}$. Then, there is $x_k \in L$ such that $\mathbb{D}Orb(T, x_k)$ is nowhere dense. It follows that, $\bigcup_{\substack{i=1 \\ i \neq k}}^{m} \mathbb{D}Orb(T, x_i)$ is dense in $\mathcal{X}$, which is contradiction to the minimality of L. Thus, there exists a vector $x \in \mathcal{X}$ such that $\mathbb{D}Orb(T, x)$ is somewhere dense. □

Since the somewhere density of disk orbit does not imply to everywhere density of it [4, Example 3.14], then we can not apply Bourdon's and Feldman's result [6] to show that every multidiskcyclic is diskcyclic. However, we follow Peris' approach [12] to show that every multidiskcyclic operator is diskcyclic. First, we need the following lemmas.

Lemma 2.1. *[12] If p is a polynomial and α is an eigenvalue of T^*, then $p(T)$ has a dense range if and only if $p(\alpha) \neq 0$.*

The following lemma can be proved by the same way of proving [12, Lemma 3].

Lemma 2.2. *If the interior closure of two disk orbits intersect each other, then they coincide.*

Theorem 2.2. *Let T be a multidiskcyclic operator, then T is diskcyclic.*

Proof. Let n be a positive integer and $L = \{x_1, \cdots, x_n\}$ be a diskcyclic set for T, then

$$\mathcal{X} = \bigcup_{i=1}^{n} \overline{\mathbb{D}Orb(T, x_i)}.$$

Let $n > 1$ (otherwise T is diskcyclic) and $x \in \mathcal{X}$ with $int(\overline{\mathbb{D}Orb(T, x)}) \neq \phi$, then there exists $x_h \in L$ such that $int(\overline{\mathbb{D}Orb(T, x)}) \cap int(\overline{\mathbb{D}Orb(T, x_h)}) \neq \phi$. It follows by Lemma (2.2) that

$$int(\overline{\mathbb{D}Orb(T, x)}) = int(\overline{\mathbb{D}Orb(T, x_h)}).$$

Claim. $Orb(T, x) \subset int(\overline{\mathbb{D}Orb(T, x)})$.
Proof of Claim:
Towards a contradiction, suppose that there exists $T^m x \in Orb(T, x)$ such that $T^m x \notin int(\overline{\mathbb{D}Orb(T, x)})$. It follows that $T^m x \notin int(\overline{\mathbb{D}Orb(T, x_h)})$, thus there exists $1 \leqslant k \leqslant n; k \neq h$ such that $T^m x \in \overline{\mathbb{D}Orb(T, x_k)}$. Since $\overline{\mathbb{D}Orb(T, x_k)}$ is T-invariant, then

$$int(\overline{\mathbb{D}\{T^{m+q}x : q \geqslant 0\}}) \subset int(\overline{\mathbb{D}Orb(T, x_k)}). \tag{2.2}$$

Now, we get

$$int(\overline{\mathbb{D}Orb(T, x_h)}) = int(\overline{\mathbb{D}Orb(T, x)}) = int(\overline{\mathbb{D}\{T^{m+q}x : q \geqslant 0\}}) \subset int(\overline{\mathbb{D}Orb(T, x_k)}).$$

By Lemma 2.2, it follows that

$$int(\overline{\mathbb{D}Orb(T, x_h)}) = int(\overline{\mathbb{D}Orb(T, x_k)}).$$

which is a contradiction. Thus the claim is proved.

To prove the theorem, by applying Proposition 2.1 and Lemma 2.1, we have

$$\mathcal{X} = \overline{P(T)(\mathcal{X})} = \bigcup_{i=1}^{n} \overline{P(T)(\mathbb{D}Orb(T, x_i))}.$$

for every polynomial P with $P(\alpha) \neq 0$ (if $\sigma_p(T^*) = \alpha$). Now, since

$$\overline{P(T)(\mathbb{D}Orb(T, x_i))} = \overline{\mathbb{D}Orb(T, P(T)x_i)},$$

and since L is minimal, then

$$int\left(\overline{\mathbb{D}Orb(T, P(T)x_i)}\right) \neq \phi, \text{ for all } x_i \in L.$$

Thus, for each $i \in \{1, \cdots, n\}$ there exists $j \in \{1, \cdots, n\}$ such that

$$Orb(T, P(T)x_i) \subset int\left(\overline{\mathbb{D}Orb(T, P(T)x_i)}\right) = int\left(\overline{\mathbb{D}Orb(T, x_j)}\right)$$

Let

$$B = \bigcup_{P(\lambda)\neq 0} Orb(T, P(T)x_1) \subset \bigcup_{i=1}^{n} int\left(\overline{\mathbb{D}Orb(T, x_i)}\right)$$

Moreover, $B = \text{span}\,(Orb(T, x_1)) \setminus \overline{(T-\lambda I)\,(\mathcal{X})}$. It follows that B is connected and hence $B \subset int\left(\overline{\mathbb{D}Orb(T, x_1)}\right)$. By [12, Lemma 2], we have B is dense, thus

$$\mathcal{X} = \overline{\mathbb{D}Orb(T, x_1)},$$

which means that, T is diskcyclic. □

Corollary 2.1. *An operator is diskcyclic if and only if it is multidiskcyclic.*

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Nareen Bamerni
College of Science
Department of Mathematics
Kurdistan Regional, Iraq
nareen_bamerni@yahoo.com