PROJECTIVE CHANGE BETWEEN RANDERS METRIC AND EXPONENTIAL \((\alpha, \beta)\)-METRIC

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Abstract. In this paper, we find conditions to characterize the projective change between two \((\alpha, \beta)\)-metrics, such as exponential \((\alpha, \beta)\)-metric, \(L = \alpha e^{\beta}\) and Randers metric \(L = \alpha + \beta\) on a manifold with \(\dim n > 2\), where \(\alpha\) and \(\alpha\) are two Riemannian metrics, \(\beta\) and \(\beta\) are two non-zero 1-forms. We also discuss special curvature properties of two classes of \((\alpha, \beta)\)-metrics.

Keywords: Finsler space, \((\alpha, \beta)\)-metric, projective change, Randers metric, Berwald, Riemannian metric.

1. Introduction

M. Matsumoto [10] introduced the concept of \((\alpha, \beta)\)-metric on a differentiable manifold with local coordinates \(x^i\), where \(\alpha^2 = a_{ij}(x)y^iy^j\) is a Riemannian metric and \(\beta = b_i(x)y^i\) is a 1-form on \(M^n\). M. Hashiguchi and Y. Ichijyo [6] studied some special \((\alpha, \beta)\)-metrics and obtained interesting results. In the projective Finsler geometry, there is a remarkable theorem called Rapcsak [14] theorem, which plays an important role in the projective geometry of Finsler spaces. In fact, this theorem gives the necessary and sufficient condition for a Finsler space to be projective to another Finsler space.

The projective change between two Finsler spaces has been studied by many authors ([2], [5], [8], [11], [12], [16]). In 1994, S. Bacso and M. Matsumoto [2] studied the projective change between Finsler spaces with \((\alpha, \beta)\)-metric. In 2008, H. S. Park and Y. Lee [11] studied the projective changes between a Finsler space with \((\alpha, \beta)\)-metric and the associated Riemannian metric. The authors Z. Shen and Civi Yildirim [16] studied a class of projectively flat metrics with a constant flag curvature. In 2009, Ningwei Cui and Yi-Bing Shen [5] studied projective change between two classes of \((\alpha, \beta)\)-metrics. Also the author N. Cui [4] studied the \(S\)-curvature of some \((\alpha, \beta)\)-metrics. In this paper, we find conditions to characterize
the projective change between two \((\alpha,\beta)\)-metrics, such as the exponential \((\alpha,\beta)\)-metric, \(L = \alpha e^{\frac{\beta}{\alpha}}\) and Randers metric \(L = \alpha + \beta\) on a manifold with \(\dim n > 2\), where \(\alpha\) and \(\overline{\alpha}\) are two Riemannian metrics, \(\beta\) and \(\overline{\beta}\) are two non-zero 1-forms. In addition, we discuss special curvature properties of two classes of \((\alpha,\beta)\)-metrics.

2. Preliminaries

The terminology and notation are referred to ([15], [9], [1]). Let \(M^n\) be a real smooth manifold of dimension \(n\) and let \(F^n = (M^n, L)\) be a Finsler space on the differentiable manifold \(M^n\) endowed with the fundamental function \(L(x,y)\). We use the following notation:

\[
\begin{aligned}
g_{ij} &= \frac{1}{2} \partial_i \partial_j L^2, \\
C_{ijk} &= \frac{1}{2} \partial_k g_{ij}, \\
h_{ij} &= g_{ij} - l_i l_j, \\
\gamma^i_{jk} &= \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}), \\
G^i &= \frac{1}{2} \gamma^i_{jk} y^j y^k, \\
G^i_{jk} &= \partial_k G^i_j, \\
G^i_{jkl} &= \partial_l G^i_{jk},
\end{aligned}
\]

where \(\partial_i \equiv \frac{\partial}{\partial y^i}\).

**Definition 2.1.** A change \(L \rightarrow \overline{L}\) of a Finsler metric on the same underlying manifold \(M\) is called projective change if any geodesic in \((M,L)\) remains to be geodesic in \((M,\overline{L})\) and vice versa.

A Finsler metric is projectively related to another metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics \(\alpha\) and \(\overline{\alpha}\) are projectively related if and only if their spray coefficients have the relation [5]

\[
G^i_\alpha = G^i_{\overline{\alpha}} + \lambda x^k y^k y^i,
\]

where \(\lambda = \lambda(x)\) is a scalar function on the based manifold.

Two Finsler metric \(F\) and \(\overline{F}\) are projectively related if and only if their spray coefficients have the relation [5]

\[
G^i = \overline{G}^i + P(y) y^i,
\]

where \(P(y)\) is a scalar function and homogeneous of degree one in \(y^i\).

**Definition 2.2.** A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

For a given Finsler metric \(L = L(x, y)\), the geodesic of \(L\) is given by

\[
\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,
\]
where \( G^i = G^i(x, y) \) are called geodesic coefficients, which are given by

\[
G^i = \frac{g^{ij}}{4} \left( [L^2]_{x=y} y^m - [L^2]_{x^i} \right).
\]

Let \( \phi = \phi(s), \ |s| < b_0 \), be a positive \( C^\infty \) satisfying the following

\[
\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ (|s| \leq b < b_0).
\]

Let \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) be a Riemannian metric, \( \beta = b_i y^i \) is a 1-form satisfying

\[
\|\beta\|_\alpha < b_0 \text{ for all } x \in M,
\]

then \( L = \alpha \phi(s) \), \( s = \beta \alpha \), is called an (regular) \((\alpha, \beta)\)-metric. In this case, the fundamental form of the metric tensor induced by \( L \) is positive definite. Let \( \nabla \beta = b_{ij} dx^i \otimes dx^j \) be the covariant derivative of \( \beta \) with respect to \( \alpha \).

Denote

\[
r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \ s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}).
\]

\( \beta \) is closed if and only if \( s_{ij} = 0 \) \[17\]. Let \( s_j = b^i s_{ij}, \ s^i_j = a_{ij} s_{ij}, \ s_0 = s_i y^i, \ s_0^i = s^i_j y^j \) and \( r_{00} = r_{ij} y^i y^j \).

The relation between the geodesic coefficient \( G^i \) of \( L \) and the geodesic coefficient \( G^i_\alpha \) of \( \alpha \) is given by

\[
G^i = G^i_\alpha + \alpha Q s_0^i + \{r_{00} - 2Q \alpha s_0\}\{\psi b^i + \Theta \alpha^{-1} y^i\},
\]

where

\[
\Theta = \frac{\phi \phi' - s(\phi \phi'' + \phi' \phi'')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},
\]

\[
Q = \frac{\phi'}{\phi - s\phi'},
\]

\[
\psi = \frac{1}{2}(\phi - s\phi') + (b^2 - s^2)\phi''.
\]

**Definition 2.3.** \[5\] Let

\[
D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( G^i - \frac{1}{n + 1} \frac{\partial G^m}{\partial y^m} y^i \right),
\]

where \( G^i \) is the spray coefficient of \( L \). The tensor \( D^i_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l \) is called the Douglas tensor. A Finsler metric is called a Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant \[13\]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes \(2.8\). This shows that the Douglas tensor is a non-Riemannian quantity. In what follows, we use quantities with a bar to denote the
corresponding quantities of metric $\mathcal{L}$. We compute the Douglas tensor of a general $(\alpha, \beta)$-metric. Let

\begin{equation}
\hat{G}^i = G^i + \alpha Q s^i_0 + \psi \{ r_{00} - 2Q \alpha s_0 \} b^i. 
\end{equation}

Using (2.9) in (2.7), we have

\begin{equation}
G^i = \hat{G}^i + \Theta \{ r_{00} - 2Q \alpha s_0 \} \alpha^{-1} y^i.
\end{equation}

Clearly, $G^i$ and $\hat{G}^i$ are projective equivalents according to (2.3). They have the same Douglas tensor. Let

\begin{equation}
T^i = \alpha Q s^i_0 + \psi \{ r_{00} - 2Q \alpha s_0 \} b^i.
\end{equation}

Then $\hat{G}^i = G^i + T^i$, thus

\begin{equation}
\frac{\partial}{\partial y^i \partial y^j \partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G^n_m}{\partial y^m} y^i + T^i - \frac{1}{n+1} \frac{\partial T^n_m}{\partial y^m} y^i \right) = \frac{\partial}{\partial y^i \partial y^j \partial y^l} \left( T^i - \frac{1}{n+1} \frac{\partial T^n_m}{\partial y^m} y^i \right).
\end{equation}

To simplify (2.12), we use the following identities

\begin{align*}
\alpha y^i &= \alpha^{-1} y^i, \quad s y^i = \alpha^{-2} (b \alpha - s y^i),
\end{align*}

where $y_i = a_i y^i$, $\alpha_y = \frac{\partial y}{\partial y^i}$. Then

\begin{align*}
[\alpha Q s^m_0]_y^m &= \alpha^{-1} y_m Q s^m_0 + \alpha^{-2} Q s^i_m [b_m \alpha^2 - \beta y_m] s^m_0 \\
&= \alpha Q s_0
\end{align*}

and

\begin{align*}
\psi \{ r_{00} - 2Q \alpha s_0 \} b^m_0 y^m &= \psi \alpha^{-1} (b^2 - s^2) [r_{00} - 2Q \alpha s_0] + 2\psi \{ r_0 - Q (b^2 - s^2) s_0 - Q s_0 \},
\end{align*}

where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.11), we get

\begin{equation}
T^m_0 = \alpha Q s_0 + \psi \alpha^{-1} (b^2 - s^2) [r_{00} - 2Q \alpha s_0]
\begin{align*}
&+ 2\psi \{ r_0 - Q (b^2 - s^2) s_0 - Q s_0 \}.
\end{align*}

We assume that the $(\alpha, \beta)$-metrics $L$ and $\mathcal{L}$ have the same Douglas tensor, i.e.,

\begin{equation}
D_{jkl} = \hat{D}_{jkl}.
\end{equation}

Thus from (2.8) and (2.12), we get

\begin{equation}
\frac{\partial}{\partial y^i \partial y^j \partial y^l} \left( T^i - \hat{T}^i - \frac{1}{n+1} (T^m_0 - \hat{T}^m_0) y^i \right) = 0.
\end{equation}
Then there exists a class of scalar functions $H_{i}^{j} = H_{i}^{j}(x)$, such that

$$H_{00}^{i} = T_{i}^{i} - \frac{1}{n+1}(T_{ym}^{m} - T_{ym}^{m})y^{i},$$

where $H_{00}^{i} = H_{j}^{i} y^{j} y^{k}$.

**Theorem 2.1.** [4] For the special form of $(\alpha, \beta)$-metric, $L = \alpha + \epsilon \beta + k \left( \frac{\beta^{2}}{\alpha} \right)$, where $\epsilon, k$ are non-zero constant, the following are equivalent:

- $L$ has an isotropic $S$-curvature, i.e., $S = (n+1)c(x)L$ for some scalar function $c(x)$ on $M$.
- $L$ has an isotropic mean Berwald curvature.
- $\beta$ is a killing one form of constant length with respect to $\alpha$. This is equivalent to $r_{00} = s_{0} = 0$.
- $L$ has a vanished $S$-curvature, i.e., $S = 0$.
- $L$ is a weak Berwald metric, i.e., $E = 0$.

### 3. Projective Change between Randers Metric and Exponential $(\alpha, \beta)$-Metric

In this section, we find the projective relation between two $(\alpha, \beta)$-metrics on the same underlying manifold $M$ of dimension $n > 2$. For $(\alpha, \beta)$-metric $L = \alpha e^{\beta}$, one can prove by (2.6) that $L$ is a regular Finsler metric if and only if 1-form $\beta$ satisfies the condition $\|\beta_{x}\|_{\alpha} < 1$ for any $x \in M$. The geodesic coefficients are given by (2.7) with

$$\begin{align*}
\Theta &= \frac{1}{2(1+s)^{2}} - 2s, \\
Q &= \frac{1}{1-s}, \\
\psi &= \frac{1}{2(1+s)^{2}}.
\end{align*}$$

Using (3.1) in (2.7), we get

$$G^{i} = G_{\alpha}^{i} + \frac{2}{2(\alpha^{2} - \beta^{2} + \alpha^{2}b^{2} - \alpha \beta)} \left[ r_{00} - \frac{2\alpha^{2}}{\alpha - \beta} s_{0} \right] \left[ a^{2}b^{i} + (\alpha - 2\beta)y^{i} \right].$$

For the Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$, one can also prove by (2.6) that $\bar{L}$ is a regular Finsler metric if and only if $\|\beta_{x}\|_{\alpha} < 1$ for any $x \in M$. The geodesic coefficients are given by (2.7) with

$$\begin{align*}
\overline{\Theta} &= \frac{1}{2(1+s)} , \\
\overline{Q} &= 1, \\
\overline{\psi} &= 0.
\end{align*}$$

First, we prove the following lemma:
Lemma 3.1. Let \( L = \alpha \pi \) and \( \overline{L} = \pi + \overline{\beta} \) be two \((\alpha, \beta)\)-metrics on a manifold \( M \) with dimension \( n > 2 \). Then they have the same Douglas tensor if and only if both metrics are Douglas metrics.

Proof. First, we prove the sufficient condition. Let \( L \) and \( \overline{L} \) be Douglas metrics and the corresponding Douglas tensor \( D^j_\text{jkl} \) and \( \overline{D}^j_\text{jkl} \). Then by the definition of Douglas metric, we have \( D^i_\text{jkl} = 0 \) and \( \overline{D}^i_\text{jkl} = 0 \), that is, both metrics have the same Douglas tensor. Next, we prove the necessary condition. If \( L \) and \( \overline{L} \) have the same Douglas tensor, then (2.14) holds. Using (2.13), (3.1) and (3.3) in (2.14), we have

\[
H^i_{00} = \frac{A^i \alpha^7 + B^i \alpha^6 + C^i \alpha^5 + D^i \alpha^4 + E^i \alpha^3 + F^i \alpha^2}{K \alpha^6 + U \alpha^5 + M \alpha^4 + N \alpha^3 + V \alpha^2 + R} - \overline{\pi} \overline{\pi}_{00},
\]

where

\[
A^i = (1 + b^2)[2s_0(1 + b^2) - 2s_0],
B^i = (1 + b^2)[r_00b^i - 2\beta(3 + b^2)s_0^i + 2\beta s_0 b^i] - 2\lambda ss_0(1 + s)y^i - 2\lambda y_0 y^i - 2\lambda s_0 y^i,
C^i = \beta(3 + 2b^2)(2\lambda r_0 y^i - r_00b^i) - 2\lambda ss_0(2 + b^2)y^i + 4\lambda_0 s_0 (1 + b^2)y^i + 2\beta(1 - 2b^2)s_0^i - \lambda b^2 r_00 y^i,
D^i = 2\beta^3(3 + 2b^2)s_0 + r_00b^2(2 + b^2)b^i + 2\lambda_0^2(\beta s_0 + 2\beta s s_0 - \beta b r_0 - 2\beta r_0 - s^2 r_00)y^i,
E^i = \beta^2 r_00[\beta b^i + \lambda(3b^2 - 4s^2 - 2\beta b^2)y^i] + 2\lambda_0^2(s_0 - r_0 - s_0)y^i - 2\beta^4 s_0^i,
F^i = 2\beta_0^2(\beta r_0 - \beta s_0 + s^2 r_00)y^i + 2\beta_0^2 s_0 - \beta^4 r_00 b^i,
\]

\[
\lambda = \frac{1}{n + 1}
\]

and

\[
K = 2(1 + b^2)^2, \quad U = 4\beta(b^4 - 3b^2 - 2), \quad M = 2\beta^2(b^2 + 2)^2,
N = 4\beta^3(1 + b^2), \quad V = -4\beta^4(2 + b^2), \quad R = 2\beta^6.
\]

Then (3.4) is equivalent to

\[
A^i \alpha^7 + B^i \alpha^6 + C^i \alpha^5 + D^i \alpha^4 + E^i \alpha^3 + F^i \alpha^2 = (K \alpha^6 + U \alpha^5 + M \alpha^4 + N \alpha^3 + V \alpha^2 + R)(H^i_{00} - \overline{\pi} \overline{\pi}_{00}).
\]

Replacing \( y^i \) in (3.5) by \(-y^i\), we have

\[
- A^i \alpha^7 + B^i \alpha^6 - C^i \alpha^5 + D^i \alpha^4 - E^i \alpha^3 + F^i \alpha^2 = (K \alpha^6 - U \alpha^5 + M \alpha^4 - N \alpha^3 + V \alpha^2 + R)(H^i_{00} - \overline{\pi} \overline{\pi}_{00}).
\]
Subtracting (3.6) from (3.5), we get
\[
A'\alpha^i + C'\alpha^3 + E'\alpha^3 = (U\alpha^5 + N\alpha^3)H_{00}^i + (K\alpha^6 + M\alpha^4 + V\alpha^2 + R)\bar{\pi} \bar{\pi}^0_i.
\]
(3.7)

From (3.7), we have
\[
\alpha^2[A'\alpha^5 + C'\alpha^3 + E'\alpha - (U\alpha^3 + N\alpha)H_{00}^i - \bar{\pi} \bar{\pi}^0_i(K\alpha^4 + M\alpha^2 + V)] = R\bar{\pi} \bar{\pi}^0_i.
\]
(3.8)

From (3.8), $R\bar{\pi} \bar{\pi}^0_i$ has the factor $\alpha^2$, i.e., the term $R\bar{\pi} \bar{\pi}^0_i = 2\beta^6 \bar{\pi} \bar{\pi}^0_i$ has the factor $\alpha^2$. We can study two cases for Riemannian metric.

Case (i): If $\bar{\pi} \neq \mu(x)\alpha$, then $R\bar{\pi} \bar{\pi}^0_i = 2\beta^6 \bar{\pi} \bar{\pi}^0_i$ has the factor $\alpha^2$. Note that $\beta^2$ has no factor $\alpha^2$. Then for each $i$ there exists a scalar function $\eta^i = \eta^i(x)$ such that $\beta\bar{\pi}^0_i = \eta^i\alpha^2$ which is equivalent to $b_j\bar{\pi}^0_k + b_k\bar{\pi}^0_j = 2\eta^i\alpha_{jk}$. When $n > 2$ and we assume that $\eta^i \neq 0$, then
\[
2 \geq \text{rank}(b_j\bar{\pi}^0_k) + \text{rank}(b_k\bar{\pi}^0_j)
\]
\[
> \text{rank}(b_j\bar{\pi}^0_k + b_k\bar{\pi}^0_j)
\]
\[
= \text{rank}(2\eta^i\alpha_{jk}) > 2,
\]
which is impossible unless $\eta^i = 0$. Then $\beta\bar{\pi}^0_i = 0$. Since $\beta \neq 0$, we have $\bar{\pi}^0_i = 0$, which says that $\bar{\pi}$ is closed.

Case (ii): If $\bar{\pi} = \mu(x)\alpha$, then (3.7), becomes
\[
R\mu(x)\bar{\pi}^0_i = \alpha^2[A'\alpha^5 + C'\alpha^3 + E'\alpha - (U\alpha^3 + N\alpha)H_{00}^i - \mu(x)\bar{\pi}^0_i(K\alpha^4 + M\alpha^2 + V)]
\]
(3.9)

From (3.9), we can see that $\mu(x)R\bar{\pi}^0_i$ has the factor $\alpha^2$. i.e., $\mu(x)R\bar{\pi}^0_i = 2\mu(x)\bar{\pi}^0_i\beta^6$ has the factor $\alpha^2$. Note that $\mu(x) \neq 0$ for all $x \in M$ and $\beta^2$ has no factor $\alpha^2$. The only possibility is that $\beta\bar{\pi}^0_i$ has the factor $\alpha^2$. As the similar reason in case (i), we have $\bar{\pi}^0_i = 0$, when $n > 2$, which says that $\bar{\pi}$ is closed.

M. Hashiguchi [7] proved that the Randers metric $\bar{L} = \bar{\pi} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed. Thus $\bar{L}$ is a Douglas metric. Since $L$ is projectively related to $\bar{L}$, then both $L$ and $\bar{L}$ are Douglas metrics. \[\square\]

**Theorem 3.1.** The Finsler metric $L = a\alpha^3$ is projectively related to $\bar{L} = \bar{\pi} + \bar{\beta}$ if and only if the following conditions are satisfied
\[
\left\{
\begin{align*}
G^i_\alpha &= G^i_\tau + \theta y^i = -\tau \xi \alpha^2 b^i, \\
b_{ij} &= \tau[(1 + 2y^2)a_{ij} - 3h_i b_j], \\
d\beta &= 0,
\end{align*}
\right.
\]
(3.10)

where $b^i = a^i j b_j$, $b = ||\beta||_\alpha$, $b_{ij}$ denotes the coefficient of the covariant derivatives of $\beta$ with respect to $\alpha$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on a manifold $M$ with dimension $n > 2$. 
Proof. First, we prove the necessary condition. Since the Douglas tensor is invariant under projective changes between two Finsler metrics, if $L$ is projectively related to $\overline{L}$, then they have the same Douglas tensor. According to Lemma 3.1, we obtain that both $L$ and $\overline{L}$ are Douglas metrics.

We know that the Randers metric $\overline{L} = \alpha + \beta$ is a Douglas metric if and only if $\beta$ is closed, i.e., $d\beta = 0$.

The Finsler metric $L = \alpha e^\beta$ is a Douglas metric if and only if
\begin{equation}
\tag{3.11}
b_{ij} = \tau[(1 + 2b^2) - 3b_ibj],
\end{equation}
for some scalar function $\tau = \tau(x)$ [3], where $b_{ij}$ denotes the coefficient of the covariant derivatives of $\beta = b_iy^i$ with respect to $\alpha$. In this case, $\beta$ is closed. Since $\beta$ is closed, $s_{ij} = 0 \Rightarrow b_{ij} = b_{ji}$. Thus $s_0 = 0$ and $s_0 = 0$.

By using (3.11), we have $r_{00} = \tau[(1 + 2b^2)\alpha^2 - 3\beta^2]$. Substituting all these in (3.2), we get
\begin{equation}
\tag{3.12}
G^i = G^i_\alpha + \tau\left[\frac{(1 + 2b^2)(\alpha^3 - 2\alpha^2\beta) - 3\alpha\beta^2 + 6\beta^3}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right]y^i + \tau\xi\alpha^2b^i,
\end{equation}
where $\xi = \frac{\tau[(1 + 2b^2)\alpha^2 - 3\beta^2]b^i}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}$.

Since $L$ is projectively related to $\overline{L}$, this is a Randers change between $L$ and $\overline{L}$. Noticing that $\overline{\beta}$ is closed, then $L$ is projectively related to $\overline{\alpha}$. Thus, there is a scalar function $P = P(y)$ on $TM - \{0\}$ such that
\begin{equation}
\tag{3.13}
G^i = G^i_\overline{\alpha} + Py^i.
\end{equation}

From (3.12) and (3.13), we have
\begin{equation}
\tag{3.14}
\left[P + \frac{3\alpha\beta^2 - 6\beta^3 - (1 + 2b^2)(\alpha^3 - 3\alpha^2\beta)}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right]y^i = G^i_\alpha - G^i_\overline{\alpha} + \tau\xi\alpha^2b^i.
\end{equation}

Note that the RHS of the above equation is a quadratic form. Then there must be one form $\theta = \theta_iy^i$ on $M$, such that
\begin{equation}
P + \frac{3\alpha\beta^2 - 6\beta^3 - (1 + 2b^2)(\alpha^3 - 3\alpha^2\beta)}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)} = \theta.
\end{equation}

Thus (3.14) becomes
\begin{equation}
\tag{3.15}
G^i_\alpha = G^i_\overline{\alpha} + \theta y^i - \tau\xi\alpha^2b^i.
\end{equation}

Equations (3.11) and (3.12) together with (3.15) complete the proof of the necessity. Since $\overline{\beta}$ is closed, it suffices to prove that $L$ is projectively related to $\overline{\alpha}$. From (3.12) and (3.15), we have
\begin{equation}
G^i = G^i_\overline{\alpha} + \left[\theta + \frac{\tau[(1 + 2b^2)(\alpha^3 - 3\alpha^2\beta) - 3\alpha\beta^2 + 6\beta^3]}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right]y^i,
\end{equation}
that is, $L$ is projectively related to $\overline{\alpha}$. \qed
From the above theorem, we get the following corollaries.

**Corollary 3.1.** The Finsler metric $L = \alpha e^\beta$ is projectively related to $\overline{L} = \overline{\alpha} + \overline{\beta}$ if and only if they are Douglas metrics and the spray coefficients of $\alpha$ and $\overline{\alpha}$ have the following relation

$$G^i_\alpha = G^i_{\overline{\alpha}} + \theta y^i - \tau \xi \alpha^2 b^i,$$

where $b^i = a^{ij} b_j$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is one form on a manifold $M$ with dimension $n \geq 2$.

Further, we assume that the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ is locally Minkowskian, where $\overline{\alpha}$ is a Euclidean metric and $\overline{\beta} = \overline{b}_i y^i$ is one form with $\overline{b}_i = \text{constant}$. Then (3.10) can be written as

$$G^i_\alpha = \theta y^i - \tau \xi \alpha^2 b^i,$$

$$b_{ij} = \tau [(1 + 2 \beta^2) a_{ij} - 3 b_i b_j].$$

Thus, we state

**Corollary 3.2.** The Finsler metric $\alpha e^\beta$ is projectively related to $\overline{L} = \overline{\alpha} + \overline{\beta}$ if and only if $L$ is projectively flat, that is, $L$ is projectively flat if and only if (3.16) holds.

### 4. Special Curvature Properties of two $(\alpha, \beta)$-metrics

We know that the Berwald curvature tensor of a Finsler metric $L$ is defined by [9]

$$G = G^i_{jkl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l,$$

where $G^i_{jkl} = [G^i]_{y^j y^k y^l}$ and $G^i$ are the spray coefficients of $L$. The mean Berwald curvature tensor is defined by

$$E = E_{ij} dx^i \otimes dx^j,$$

where $E_{ij} = \frac{1}{2} G_{mij}$. A Finsler space is said to be of the isotropic mean Berwald curvature if

$$E_{ij} = \frac{n+1}{2} c(x) L y^i y^j,$$

where $c(x)$ is scalar function on $M$.

In this section, we assume that $(\alpha, \beta)$-metric $L = \alpha e^\beta$ has some special curvature properties.

**Theorem 4.1.** The Finsler metric $L = \alpha e^\beta$ having an isotropic $S$-curvature or isotropic mean Berwald curvature is projectively related to $\overline{L} = \overline{\alpha} + \overline{\beta}$ if and only if the following conditions hold:
α is projectively related to τ,
β is parallel with respect to α, i.e., $b_{ij} = 0$,
β̄ is closed, i.e., $dβ = 0$,

where $b_{ij}$ denotes the coefficient of the covariant derivative of β with respect to α.

Proof. The sufficiency is obvious from Theorem (3.2). For the necessary condition, from Theorem 3.1, if $L$ is projectively related to $\mathcal{T}$, then

$$b_{ij} = \tau[(1 + 2\beta^2)\alpha_{ij} - 3\beta b_j],$$

where $\tau = \tau(x)$ is scalar function. Transvecting the above equation with $y^i$ and $y^j$, we have

$$(4.4) \quad r_{00} = \tau[(1 + 2\beta^2)\alpha^2 - 3\beta^2].$$

From Theorem 2.4, if $L$ has an isotropic $S$-curvature or an equivalently isotropic mean Berwald curvature, then $r_{00} = 0$. If $\tau \neq 0$, then (4.4) gives

$$(4.5) \quad (1 + 2\beta^2)\alpha^2 - 3\beta^2 = 0,$$

which is equivalent to

$$(4.6) \quad (1 + 2\beta^2)\alpha_{ij} - 3\beta b_j = 0.$$

Transvecting (4.6) with $a^l$, we get

$$(4.7) \quad (1 + 2\beta^2)\delta^l_j - 3\delta^l b_j = 0.$$

Contracting $l$ and $j$ in (4.7), we have $n + (2n - 3)b^2 = 0$, which is impossible. Thus $\tau = 0$. Substituting in Theorem 3.2, we complete the proof. ☐

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