Structural Theorems for \((m,n)\)-Quasi-Ideal Semigroups

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Abstract. The definitions of \((m,n)\)-ideal and \((m,n)\)-quasi-ideal on a semigroup are given in [5]. They have been studied in many papers, recently in papers [7], [8] and [9]. In this paper we introduce the notion of an \((m,n)\)-quasi-ideal semigroup and consider some general properties of this class of semigroups. Also, we introduce the notion of an \((m,n)\)-duo quasi-ideal semigroup and give its structural description. In section 3 we describe a class of \((m,n)\)-quasi-ideal semigroups by matrix representation.

Keywords: \((m,n)\)-quasi-ideal, periodic semigroup, rectangular band, power breaking partial semigroup

1. Introduction and some general properties for \((m,n)\)-quasi ideal semigroups

Let \(S\) be a semigroup, \(a \in S\), then by \(\langle a \rangle\) we denote monogenic subsemigroup on \(S\) generated by \(a\). Either \(\langle a \rangle\) is isomorphic to the natural numbers \(\mathbb{N}\) under addition or there exist positive integers \(r, m\) such that \(\langle a \rangle = \{a, a^2, \ldots, a^r, \ldots, a^{r+m-1}\}\) with \(K_a = \{a^r, a^{r+1}, \ldots, a^{r+m-1}\}\) a cyclic group of order \(m\). The integers \(r\) and \(m\) are called index and period of \(\langle a \rangle\), respectively. A semigroup \(S\) is periodic if all its monogenic subsemigroups are finite. By \(|\langle a \rangle|\) we denote the cardinal of \(\langle a \rangle\).

By \(E(S)\) we denote a set of all idempotents on \(S\).

For nondefined notions we refer to [2] and [3].

A subsemigroup \(A\) of a semigroup \(S\) is an \((m,n)\)-ideal on \(S\) if \(A^m SA^n \subseteq A\), where \(m, n \in \mathbb{N} \cup \{0\} \) \(A^0 SA^n = SA^n\), \(A^m SA^0 = A^m S\), [5].

A semigroup \(S\) is an \((m,n)\)-ideal semigroup if each subsemigroup of \(S\) is an \((m,n)\)-ideal on \(S\), [11, 13].

Definition 1.1. A subsemigroup \(Q\) of a semigroup \(S\) is an \((m,n)\)-quasi-ideal on \(S\) if \(Q^m S \cap SQ^n \subseteq Q\), where \(m, n \in \mathbb{N} \cup \{0\} \) \(Q^0 S \cap SQ^n = SQ^n\), \(Q^m S \cap SQ^0 = Q^m S\).
Remark 1.1. Let \( Q \) be an \((m, n)\)-quasi-ideal on \( S \), then
\[
Q^n S Q^n = Q^n S Q^n \cap Q^n S Q^n \subseteq Q^n S \cap S Q^n \subseteq Q
\]
and so \( Q \) is an \((m, n)\) ideal on \( S \).

Definition 1.2. A semigroup \( S \) is an \((m, n)\)-quasi-ideal semigroup if each subsemigroup of \( S \) is an \((m, n)\)-quasi-ideal on \( S \).

For \( m = 1 \), \( n = 0 \) \((m = 0, n = 1)\) we have right (left) ideal semigroups considered in [4] and [5]. For \( m = n = 1 \) we have quasi-ideal semigroups.

It is clear that if \( S \) is an \((m, n)\)-quasi-ideal semigroup, then each subsemigroup of \( S \) is an \((m, n)\)-quasi-ideal semigroup as well.

Remark 1.2. By Remark 1 we conclude that a class of \((m, n)\)-quasi-ideal semigroups is a subclass of the class of \((m, n)\)-ideal semigroups.

Remark 1.3. Let \( S \) be an \((m, n)\)-quasi-ideal semigroup, \( a \in S \), then
\[
a^m S \cap S a^n \subseteq a^m S \cap a > a^n \subseteq a > .
\]

The following proposition is proved in [11].

Proposition 1.1. Let \( S \) be an \((m, n)\)-ideal semigroup. Then the following statements are true:

(i) \( S \) is periodic, \( K_a = \{e\} \), \( e \) is zero in \( a \) for every \( a \in S \) and \( | a | \leq 2m + 2n + 1 \);

(ii) The set \( E(S) \) is a rectangular band and an ideal of \( S \);

(iii) \( S \) is a disjoint union of the maximal unipotent subsemigroups \( S_e = \{ x \in S \mid \exists p \in N \ x^p = e, e \in E(S) \} \) and \( e \) is zero in \( S_e \).

Remark 1.4. By Remark 2 the above proposition is valid for \((m, n)\)-quasi-ideal semigroups.

Theorem 1.1. Let \( S \) be an \((m, n)\)-quasi-ideal semigroup, then for every \( a \in S \) is \( | a | \leq 2 \max\{m, n\} + 1 \).

Proof. By Proposition 1 \( a > a = [a, a^2, a^3, \ldots, a^p] \) and \( | a | = p \). Suppose that \( p > 2m + 2k = 2n + 2s + 1 \) where \( k, s \in N \cup \{0\} \) are minimal such that \( m + k = n + s \). Let \( B = [a^2, a^3, \ldots, a^p] \), then \( B \) is a subsemigroup and \((m, n)\)-quasi-ideal of \( a \). So
\[
B^m a \cap a B^n \subseteq B^m \ a > \cap \ a > B^n \subseteq B
\]
and
\[
\{a^{2m+1}, \ldots, a^{2n+2k+1}, \ldots, a^p\} \cap \{a^{2m+1}, \ldots, a^{2n+2s+1}, \ldots, a^p\} \subseteq B.
\]

Now \( a^{2m+2k+1} = a^{2m+2s+1} \in B \) where \( k, s \in N \cup \{0\} \) are minimal such that \( m + k = n + s \) and \( a^{2m+2k+1} \neq a^p \) which is a contradiction.

Hence, \( p \leq 2m + 2k + 1 = 2m + 2s + 1 \) or \( p \leq \max\{m, n\} + 1 \). \( \square \)
2. A structural theorems for \((m,n)\)-quasi-ideal semigroup

Let \(P\) be a partial semigroup, then \(Q \subseteq P\) is a partial subsemigroup of \(P\) if for \(x, y \in Q\) and \(xy \in P\) it follows that \(xy \in Q\). Then if \(Q^mP \cap PQ^n \subseteq Q\) we say that \(Q\) is a partial \((m,n)\)-quasi-ideal of \(P\).

A partial semigroup \(P\) is a partial \((m,n)\)-quasi-ideal semigroup if each partial subsemigroup of \(P\) is an \((m,n)\)-quasi-ideal.

A partial semigroup \(P\) is called a power breaking partial semigroup if for every \(a \in P\) there exists some \(k \in \mathbb{N}\) such that \(a^k \notin P\).

Example 2.1. Let semigroup \(S = \{i, j, k, m, n, p, q, e, f, g, h\}\) be a given by the following table:

\[
\begin{array}{cccccccc}
  i & j & k & m & n & p & q & e & f & g & h \\
  i & j & k & m & n & p & q & e & e & g & e \\
  j & k & m & n & p & q & e & e & g & e \\
  k & m & n & p & q & e & e & e & e & g & e \\
  m & n & p & q & e & e & e & e & e & g & e \\
  n & p & q & e & e & e & e & e & e & g & e \\
  p & q & e & e & e & e & e & e & e & g & e \\
  q & e & e & e & e & e & e & e & e & g & e \\
  e & e & e & e & e & e & e & e & e & g & e \\
  f & h & h & h & h & h & h & h & h & f & h \\
  g & e & e & e & e & e & e & e & e & g & e \\
  h & h & h & h & h & h & h & h & h & f & h .
\end{array}
\]

Then \(S\) is a \((4,1)\)-quasi-ideal semigroup, \(E(S) = \{e, f, g, h\}\) is a rectangular band and an ideal of \(S\) and set \(P = \{i, j, k, m, n, p, q\}\) is a power breaking partial \((4,1)\)-quasi-ideal semigroup.

Theorem 2.1. Let \(S\) be an \((m,n)\)-quasi-ideal semigroup, then \(S = P \cup E(S)\) where \(P\) is a partial power breaking \((m,n)\)-quasi-ideal semigroup and \(E(S)\) is a rectangular band and an ideal of \(S\), i.e. \(S\) is a retractive extension of the rectangular band \(E(S)\) by the partial power breaking \((m,n)\)-quasi-ideal semigroup \(P\).

Proof. Let \(S\) be an \((m,n)\)-quasi-ideal semigroup, then by Remark 1.2 and Proposition 1.1 we have that \(S\) is a periodic semigroup, \(E(S)\) is a rectangular band and an ideal of \(S\) and \(P = S \setminus E(S)\) is a power breaking partial semigroup.

Let \(Q\) be a subsemigroup of \(P\). We denote by \(R = \langle Q \rangle\) a subsemigroup of \(S\) generated by \(Q\). Then \(R\) is an \((m,n)\)-quasi-ideal of \(S\), and so

\[
Q^mP \cap PQ^n \subseteq R^mS \cap SR^n \subseteq R,
\]

whence

\[
Q^mP \cap PQ^n \subseteq R \setminus E(S) = Q.
\]
Hence, $Q$ is a partial $(m, n)$-quasi-ideal of $P$ and so $P$ is an $(m, n)$-quasi-ideal semigroup.

By Proposition 1 the mapping $\varphi : S \to E(S)$ defined by $\varphi(a) = e_a$, where $e_a$ is a zero in $< a >$, is a retraction. 

Next, we give the following construction.

**Construction 1.** Let $P$ be a power breaking partial $(m, n)$-quasi-ideal semigroup, $E$ be a rectangular band, $P \cap E \neq \emptyset$ and $\varphi$ be a partial homomorphism from $P$ into $E$. Let us put $\varphi(e) = e$ for each $e \in E$. Then the mapping $\varphi : S = P \cup E \to E$ is a surjective.

Let us define an operation $\cdot$ on $S$ by

$$x \cdot y = \begin{cases} xy, & \text{if } x, y \in P \text{ and } xy \text{ is defined in } P, \\ \varphi(x)\varphi(y) & \text{otherwise.} \end{cases}$$

Then $(S, \cdot)$ is a semigroup which will be denoted by $[P, E, \varphi]$.

**Theorem 2.2.** The semigroup $S = [P, E, \varphi]$ given by the above construction is an $(m, n)$-quasi-ideal semigroup.

**Proof.** Let $Q$ be a subsemigroup of $S$ and $Q^* = Q \setminus E$. Let $s \in Q^mS \cap SQ^n$. Then it is $s =pd = tq$ where $p = x_1x_2 \ldots x_m \in Q^m$, $q = y_1y_2 \ldots y_n \in Q^n$, $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in Q$ and $d, t \in S$. It is enough to consider the following cases:

1. $p, q \in Q^*$, $d, t \in P$

1.1. $s = pd = tq \in P$, then $s \in (Q^*)^mP \cap P(Q^*)^n \subseteq Q^*$ since $P$ is an $(m, n)$-quasi-ideal partial semigroup.

1.2. $s = pd = tq \in E$, then

$$s = s^2 = pdtq = \varphi(p)\varphi(d)\varphi(t)\varphi(q) = \varphi(p)\varphi(q) = \varphi(p^{\max[m,n]+1})\varphi(q) = \varphi(p^{\max[m,n]+1}) = p^{\max[m,n]+1} q \in Q.$$

Other cases can be considered analogously. Hence, $Q$ is an $(m, n)$-quasi-ideal of $S$ and so $S$ is an $(m, n)$-quasi-ideal semigroup. 

**Definition 2.1.** A subsemigroup $Q$ of a semigroup $S$ is an $(m, n)$-duo quasi-ideal of $S$ if

$$Q^mS \cap SQ^n \subseteq Q \quad \text{and} \quad Q^nS \cap SQ^m \subseteq Q$$

where $m, n \in \mathbb{N} \cup \{0\}$. A semigroup $S$ is an $(m, n)$-duo quasi-ideal semigroup if each subsemigroup of $S$ is an $(m, n)$-duo quasi-ideal of $S$. 
It is clear that a class of \((m, n)\)-duo quasi-ideal semigroups is a subclass of a class of \((m, n)\)-quasi-ideal semigroups. Putting \(n = 1\), we have \((m, 1)\)-duo quasi-ideal semigroups.

If \(P\) is a partial semigroup then the notions of \((m, n)\)-duo quasi-ideal and \((m, n)\)-duo quasi-ideal partial semigroup are defined analogously by the above.

The semigroup \(S\) given in Example 1 is an \((4, 1)\)-duo quasi-ideal semigroup and \(P = S \setminus E(S)\) is an \((4, 1)\)-duo quasi-ideal partial semigroup.

Let in Construction 1 \(P\) be an \((m, 1)\)-duo quasi-ideal power breaking partial semigroup. Then we prove the following

**Theorem 2.3.** A semigroup \(S\) is an \((m, 1)\)-duo quasi-ideal semigroup if and only if \(S \cong [P, E, \varphi]\).

**Proof.** Let \(S\) be an \((m, 1)\)-duo quasi-ideal semigroup. The proof that \(S = P \cup E(S)\) where \(P\) is a partial power breaking \((m, 1)\)-duo quasi-ideal semigroup and \(E(S)\) is a rectangular band and an ideal of \(S\) is analogous with the proof in Theorem 1.

Let \(\varphi : S = P \cup E(S) \rightarrow E(S)\) be a mapping defined by \(\varphi(x) = e_x\) such that \(e_x\) is the idempotent in \(\langle x \rangle\). Since a class of \((m, 1)\)-duo quasi-ideal semigroups is a subclass of a class of \((m, 1)\)-duo ideal semigroup, we have that mapping \(\varphi\) is epimorphism [9].

The operation on \(S\) is defined in the following way:

\[
x y = \begin{cases} 
xy, & \text{as in } P \text{ if } x, y \in P \text{ and } xy \text{ is defined in } P, \\
\text{otherwise.} & 
\end{cases}
\]

It follows that the operation on \(S\) is defined in the same way as Construction 1, and so \(S \cong [S, P, \varphi]\).

Conversely, suppose that \(S \cong [P, E, \varphi] = T\) where \(P\) is an \((m, 1)\)-duo quasi-ideal power breaking partial semigroup. The proof that \(T\) is an \((m, 1)\)-duo quasi-ideal semigroup is analogous with the proof of Theorem 2. Consequently, \(S\) is an \((m, 1)\)-duo quasi-ideal semigroup. \(\square\)

### 3. The matrix representation for \((m, n)\)-quasi ideal semigroups

In this section we describe the class of \((m, n)\)-quasi-ideal semigroups by matrix representation.

**Construction 2.** Let \(E = I \times Y\) be a rectangular band and let \(Q\) be a partial \((m, n)\)-quasi-ideal power breaking semigroup such that \(E \cap Q = \emptyset\). Let \(\xi : p \rightarrow \xi_p\) be a mapping from \(Q\) into the semigroup \(T(I)\) of all mappings from \(I\) into itself and \(\eta : p \rightarrow \eta_p\) be a mapping from \(Q\) into \(T(J)\).

For all \(p, q \in Q\) let:

(i) \(pq \in Q \implies \xi_{pq} = \xi_p \xi_q, \eta_{pq} = \eta_p \eta_q\).
(ii) \( pq \notin Q \implies \xi_p \xi_q = \text{cons.}, \eta_p \eta_q = \text{cons.} \)

Let us define a multiplication on \( S = E \cup Q \) with:

(a) \((i, j)(k, l) = (i, l),\)

(b) \(p(i, j) = (i \xi_p, j),\)

(c) \((i, j)p = (ij \eta_p),\)

(d) \(pq = r \in Q \implies pq = r \in S,\)

(e) \(pq \notin Q \implies pq = (i \xi_q \xi_p, j \eta_p \eta_q).\)

Then \( S \) with this multiplication is a semigroup, \([1], [6] \) and \([12, 14] \).

A subsemigroup \( B \) of \( S \) is of the form \( B = EB \cup QB \), where \( EB = I_B \times J_B, (I_B \subseteq I, J_B \subseteq J) \) is a rectangular band and \( Q \) is a partial subsemigroup of \( Q \).

If for \( p, q \in QB, p \in Q^m_B, q \in Q^n_B \) the following condition holds:

(iii) \( \xi_p : I \rightarrow I_B, \eta_q : J \rightarrow J_B \)

then a semigroup which is constructed in this way will be denoted by \( M(I, J, Q, \xi, \eta) \).

**Theorem 3.1.** Semigroup \( S \) is an \((m, n)\)-quasi-ideal semigroup if and only if \( S \) is isomorphic to some \( M(I, J, Q, \xi, \eta) \).

**Proof.** Let \( S \) be an \((m, n)\)-quasi-ideal semigroup, then \( S \) is an \((m, n)\)-ideal semigroup. By Theorem, \([1] \), we have that \( S \) is isomorphic to a semigroup \( M(I, J, Q, \xi, \eta) \).

Conversely, let \( S = M(I, J, Q, \xi, \eta) \) and let \( B \) be a subsemigroup of \( S \). Then \( B = EB \cup QB \), where \( EB = I_B \times J_B, (I_B \subseteq I, J_B \subseteq J) \) is a rectangular band and \( Q \) is a partial subsemigroup of \( Q \).

We will prove that \( B \) is an \((m, n)\)-quasi-ideal of \( S \).

Since \( EB \) is an ideal in \( B \), we have

\[ B^2 = (EB \cup QB)(EB \cup QB) = E^2_B \cup E_B Q_B \cup Q_B E_B \cup Q^2_B = E_B \cup Q^2_B \]

and so, by induction, we conclude that \( B^p = (EB \cup QB)^p = E_B \cup Q^p_B \) for every \( p \in \mathbb{N} \).

Now

\[ B^m S \cap SB^n = (E_B \cup Q_B)^m S \cap S(E_B \cup Q_B)^n \]

\[ = (E_B \cup Q^m_B) S \cap S(E_B \cup Q^n_B) \]

\[ = (E_B S \cap SE_B) \cup (E_B S \cap SQ^n_B) \cup (Q^m_B \cap SE_B) \cup (Q^p_B \cap SQ^n_B). \]

Now we will consider the set

\[ E_B S \cap SE_B = \{ ea = be, e \in E_B, a, b \in S \}. \]
Since $E = I \times J$ is a rectangular band and ideal on $S$ we have
\[ ea = e^2a = eea = ebf = e(fb) = ef \in E_B. \]
Hence
\[ (3.1) \quad E_BS \cap SE_B \subseteq E_B. \]

We will consider the set
\[ E_BS \cap SQ^n_B = \{ ea = bq \mid e \in E_B, a, b \in S, q \in Q^n_B \}. \]
Then $ea = bq \in E$. Let this be $e = (i, j) \in E_B = I_B \times J_B$. We have the following cases:

(2.1) If $a, b \in Q, q \in Q^n_B \subseteq Q_B$ then $(i, j)a = bq$ and
\[ (i, j)a = (i, j)^2a = (i, j)bq = (i, j\eta)bq = (i, j\eta\eta bq). \]
so by (iii) it follows that
\[ (i, j\eta) = (i, j\eta\eta bq) \in E_B \subseteq B. \]
(2.2) If $a, b \in Q, q \in E_B$, then it is case (1).
(2.3) Let $a \in E, b \in Q, q \in Q_B$, then $a = (k, l)$ and $(i, j)(k, l) = bq$. Now
\[ (i, j)^2 = (i, j)bq = (i, l\eta bq) \in E_B. \]
(2.4) Let $a \in Q, b \in E, q \in Q_B$, then $b = (k, l)$ and $(i, j)a = (k, l)q$. Now
\[ (i, j\eta) = (i, j\eta\eta bq) = (i, j\eta\eta b) = (i, l\eta) \in E_B. \]
(2.5) Let $a, b \in E, q \in Q_B$, then $a = (k, l), b = (k', l'), (i, j)(k, l) = (k', l')q$ and
\[ (i, l) = (k', l'\eta b) \implies (i, l'\eta b) \in E_B. \]
(2.6) If $a \in E, b \in Q, q \in E_B$, then it is case (1).
(2.7) If $a \in Q, b \in E, q \in E_B$, then it is case (1).
(2.8) If $a, b \in E, q \in E_B$, then it is case (1).

By the above it follows that
\[ (3.2) \quad E_BS \cap SQ^n_B \subseteq E_Q \subseteq B. \]
Similarly we have that

\[ \quad Q^m B \cap SE_B \subseteq B. \]

Finally, we will consider the set

\[ \quad Q^m B \cap SQ^m B = \{ p a = b q \mid p \in Q^m B, \quad q \in Q^m B, \quad a, b \in S \} \]

Then we have the following cases:

(4.1) If \( p, q \in Q_B, \quad a, b \in Q \) then

\[ \quad p a = b q \in Q^m B \cap QQ^m B \subseteq Q_B \subseteq B. \]

(4.2) If \( p, q \in Q_B, \quad a, b \in Q \) and \( p a = b q \in E \), then by (e) it follows

\[ \quad p a = (i \xi a \xi p, j \eta_b \eta_a) = b q = (i \xi q \xi b, j \eta_b \eta_q) \]

and so by (iii) we have

\[ \quad p a = b q = p a b q = (i \xi a \xi p, j \eta_b \eta_q) \in E_B \subseteq B. \]

(4.3) If \( p \in E_B, \quad a, b \in Q, \quad b \in Q_B \) then it is case (2).

(4.4) From \( p, q \in Q_B, \quad a \in E, \quad b \in Q \) follows \( b q \in E \) and, for \( a = (i, j) \) and (e), it follows that \( p a = (i \xi p, j), \quad b q = (i \xi q \xi b, j \eta_b \eta_q) \). So

\[ \quad p a = b q = p a b q = (i \xi p, j \eta_b \eta_q) \in E_B \subseteq B. \]

(4.5) If \( p, q \in Q_B, \quad a \in Q, \quad b = (i, j) \in E \) then \( p a = (i \xi a \xi p, j \eta_b \eta_a) = b q = (i, j \eta_q) \) and so \( (i \xi a \xi p, j \eta_q) \in E_B \).

(4.6) If \( p = (i, j) \in E_B, \quad a = (k, l) \in E, \quad b \in Q, \quad q \in Q_B \) then it is case (2).

(4.7) If \( p = (i, j) \in E_B, \quad a \in Q, \quad b \in E, \quad q \in Q_B \) then it is case (2).

(4.8) If \( p = (i, j) \in E_B, \quad a, b \in Q, \quad q = (k, l) \in E_B \) then it is case (1).

(4.9) If \( p \in Q_B, \quad a = (k, l) \in E, \quad b \in Q, \quad q \in E_B \) then it is case (3).

(4.10) If \( p \in Q_B, \quad a = \in Q, \quad b \in E, \quad q \in E_B \) then it is case (3).

(4.11) If \( p \in Q_B, \quad a = (i, j) \in E, \quad b = (k, l) \in E, \quad q \in E_B \) then \( p a = p(i, j) = (i \xi p, j) = b q = (k, l) q = (k, l \eta_q) \) and so

\[ \quad p a = b q = (i \xi p, l \eta_q) \in E_B \subseteq B. \]

(4.12) If \( p \in E_B, \quad a, b \in E, \quad q \in Q_B \) then it is case (2).
(4.13) If $p, q \in E_B$, $a \in E$, $b \in Q$ then it is case (1).

(4.14) If $p, q \in E_B$, $a \in Q$, $b \in E$ then it is case (1).

(4.15) If $p \in Q_B$, $a, b \in E$, $q \in E_B$ then it is case (3).

(4.16) If $p, q \in E_B$, $a, b \in E$ then it is case (1).

There are no other cases. Hence

\[(3.4) \quad Q^m_S \cap SQ^n_B \subseteq B.\]

By (1), (2), (3) and (4) we have that $B^mS \cap SB^n \subseteq B$, $B$ is an $(m, n)$-ideal of $S$ and $S$ is an $(m, n)$-quasi-ideal semigroup.

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