SOME RESULTS ON GENERALIZED \((k, \mu)\)-PARACONTACT METRIC MANIFOLDS

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Abstract. The aim of this paper is to study the Codazzi type of the Ricci tensor in generalized \((k, \mu)\)-paracontact metric manifolds. We also study the cyclic parallel Ricci tensor in generalized \((k, \mu)\)-paracontact metric manifolds. Further, we characterize generalized \((k, \mu)\)-paracontact metric manifolds whose structure tensor \(\phi\) is \(\eta\)-parallel. Finally, we investigate locally \(\phi\)-Ricci symmetric generalized \((k, \mu)\)-paracontact metric manifolds.

Keywords: Generalized \((k, \mu)\)-paracontact metric manifold, Codazzi type of tensor, cyclic parallel Ricci tensor, \(\eta\)-parallel \(\phi\)-tensor, locally \(\phi\)-Ricci symmetric.

1. Introduction

In 1985, Kaneyuki and Williams [8] introduced the idea of paracontact geometry. A systematic investigation on paracontact metric manifolds was done by Zamkovoy [12]. Recently, Cappelletti-Montano et al [5] introduced a new type of paracontact geometry, the so-called paracontact metric \((k, \mu)\) space, where \(k\) and \(\mu\) are constants. This is known [2] about the contact case \(k \leq 1\), but in the paracontact case there is no restriction of \(k\). Recently, three-dimensional generalized \((k, \mu)\)-paracontact metric manifolds were studied by Kupeli Erken et al [9, 10].

Zamkovoy [12] studied paracontact metric manifolds and some remarkable subclasses named para-Sasakian manifolds. In particular, in recent years, many authors have pointed to the importance of paracontact geometry and, in particular, para-Sasakian geometry. Several papers have established relationships with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. A normal paracontact metric manifold is a para-Sasakian manifold. An almost paracontact metric manifold is a para-sasakian manifold if and only if

\[
(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X.
\]

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A. Gray [7] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. The Ricci tensor \( S \) of type (0,2) is said to be cyclic parallel if it is non-zero and satisfies the condition

\[
(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0.
\]

Again, a Riemannian or a pseudo-Riemannian manifold is said to be of Codazzi type if its Ricci tensors of type (0,2) is non-zero and satisfy the following condition

\[
(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z),
\]

for all vector fields \( X, Y, Z \). On a contact metric manifold there is an associated CR-structure which is integrable if and only if the structure tensor \( \phi \) is \( \eta \)-parallel, that is,

\[
g((\nabla_X \phi)Y,Z) = 0,
\]

for all vector fields \( X, Y, Z \) in the contact distribution \( D(\eta = 0) \). In 2005, Boeckx and Cho [3] considered a milder condition that \( h \) is \( \eta \)-parallel, that is,

\[
g((\nabla_X h)Y,Z) = 0,
\]

for all vector fields \( X, Y, Z \) in the contact distribution \( D \).

The paper is organized in the following way:

In Section 2, we discuss some basic results of paracontact metric manifolds. Further, we characterize the Codazzi type of the Ricci tensor in generalized \((k,\mu)\)-paracontact metric manifolds. In Section 4, we investigate the cyclic parallel Ricci tensor in generalized \((k,\mu)\)-paracontact metric manifolds. In the next section we study \( \eta \)-parallel \( \phi \)-tensor in a generalized \((k,\mu)\)-paracontact metric manifold. Finally, we investigate locally \( \phi \)-Ricci symmetric generalized \((k,\mu)\)-paracontact metric manifolds.

### 2. Preliminaries

An odd dimensional smooth manifold \( M^n (n > 1) \) is said to be an almost paracontact manifold [8] if it carries a \((1,1)\)-tensor \( \phi \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying :

(i) \( \phi^2 X = X - \eta(X)\xi \), for all \( X \in \chi(M) \),

(ii) \( \eta(\xi) = 1, \phi(\xi) = 0, \eta \circ \phi = 0 \),

(iii) the tensor field \( \phi \) induces an almost paracomplex structure on each fiber of \( D = \text{ker}(\eta) \), that is, the eigen distributions \( D^+ \) and \( D^- \) of \( \phi \) corresponding to the eigenvalues 1 and \( -1 \), respectively, have an equal dimension \( n \).

An almost paracontact structure is said to be normal [8] if and only if the \((1,2)\) type torsion tensor \( N_\phi = [\phi, \phi] - 2d\eta \otimes \xi \) vanishes identically, where \( [\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \). A para-Sasakian manifold is a normal paracontact metric manifold. If an almost paracontact manifold admits a pseudo-Riemannian metric \( g \) such that

\[
g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y),
\]
for $X, Y \in \chi(M)$, then we say that $(M, \phi, \xi, \eta, g)$ is an almost paracontact metric manifold. Any such pseudo-Riemannian metric is of signature $(n+1, n)$. An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ [12]. In a paracontact metric manifold we define $(1,1)$-type tensor fields $h$ by $h = \frac{1}{2} \mathcal{L}_\xi \phi$, where $\mathcal{L}_\xi \phi$ is the Lie derivative of $\phi$ along the vector field $\xi$. Then we observe that $h$ is symmetric and anti-commutes with $\phi$. Also $h$ satisfies the following conditions [12]:

\begin{equation}
(2.2) \quad h\xi = 0, \quad tr(h) = tr(\phi h) = 0,
\end{equation}

\begin{equation}
(2.3) \quad \nabla_X \xi = -\phi X + \phi h X,
\end{equation}

for all $X \in \chi(M)$, where $\nabla$ denotes the Levi-Civita connection of the pseudo-Riemannian manifold.

Moreover, $h$ vanishes identically if and only if $\xi$ is a Killing vector field. In this case, $(M, \phi, \xi, \eta, g)$ is said to be a $K$-paracontact manifold [11].

Generalized $(k, \mu)$-paracontact metric manifolds were studied by Erken et al. [10] and Erken [9]. A generalized $(k, \mu)$-paracontact metric manifold means a three-dimensional paracontact metric manifold which satisfies the curvature condition

\begin{equation}
(2.4) \quad R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h X - \eta(X)h Y),
\end{equation}

where $k$ and $\mu$ are smooth functions.

In a generalized $(k \neq -1, \mu)$-paracontact manifold the following results hold [4, 5, 9, 10]

\begin{equation}
(2.5) \quad h^2 = (1 + k)\phi^2,
\end{equation}

\begin{equation}
(2.6) \quad \xi(k) = 0,
\end{equation}

\begin{equation}
(2.7) \quad Q\xi = 2k\xi,
\end{equation}

\begin{equation}
(2.8) \quad (\nabla_\xi h)(Y) = \mu h(\phi Y),
\end{equation}

\begin{equation}
(2.9) \quad (\nabla_X h)Y - (\nabla_Y h)X = -(1 + k)[2g(X, \phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X]
+ (1 - \mu)(\eta(X)\phi h Y - \eta(Y)\phi h X),
\end{equation}

\begin{equation}
(2.10) \quad (\nabla_X \phi)Y = -g(X - h X, Y)\xi + \eta(Y)(X - h X), \text{ for } k \neq -1
\end{equation}

\begin{equation}
(2.11) \quad h \text{ grad } \mu = \text{ grad } k,
\end{equation}
\[ (\nabla \eta)(Y) = -g(\phi X, Y) + g(\phi h X, Y), \]

\[ Q X = \left( \frac{r}{2} - k \right) X + \left( -\frac{r}{2} + 3k \right) \eta(X) \xi + \mu h X, \quad k \neq -1, \]

where \( X \) is any vector fields on \( M \), \( Q \) is the Ricci operator of \( M \), \( r \) denotes the scalar curvature of \( M \).

From (2.13), we have

\[ S(X, Y) = \left( \frac{r}{2} - k \right) g(X, Y) + \left( -\frac{r}{2} + 3k \right) \eta(Y) \eta(X) + \mu g(h X, Y), \quad k \neq -1. \]

3. The Codazzi type of the Ricci tensor in generalized \((k, \mu)\)-paracontact metric manifolds

In this section we characterize generalized \((k, \mu)\)-paracontact metric manifolds whose Ricci tensor is of Codazzi type.

Then we have

\[ (\nabla X S)(Y, Z) = (\nabla Y S)(X, Z), \]

which implies \( r = \text{constant} \).

Now from (2.14) we have

\[ (\nabla X S)(Y, Z) = \left\{ \frac{(X r)}{2} - (X k) \right\} g(Y, Z) + \left\{ -\frac{(X r)}{2} + 3(X k) \right\} \eta(Y) \eta(Z) \]
\[ + \left\{ -\frac{r}{2} + 3k \right\} ((\nabla X \eta)(Y) \eta(Z) + \eta(Y)(\nabla X \eta)(Z)) + (X \mu) g(h Y, Z) \]
\[ + \mu g((\nabla X h)(Y), Z) \]

and

\[ (\nabla Y S)(X, Z) = \left\{ \frac{(Y r)}{2} - (Y k) \right\} g(X, Z) + \left\{ -\frac{(Y r)}{2} + 3(Y k) \right\} \eta(X) \eta(Z) \]
\[ + \left\{ -\frac{r}{2} + 3k \right\} ((\nabla Y \eta)(X) \eta(Z) + \eta(X)(\nabla Y \eta)(Z)) + (Y \mu) g(h X, Z) \]
\[ + \mu g((\nabla Y h)(X), Z). \]

Using (3.2) and (3.3) in (3.1) yields

\[ \left\{ \frac{(X r)}{2} - (X k) \right\} g(Y, Z) + \left\{ -\frac{(X r)}{2} + 3(X k) \right\} \eta(Y) \eta(Z) \]
\[ + \left\{ -\frac{r}{2} + 3k \right\} ((\nabla X \eta)(Y) \eta(Z) + \eta(Y)(\nabla X \eta)(Z)) + (X \mu) g(h Y, Z) \]
\[ + \mu g((\nabla X h)(Y), Z) = \left\{ \frac{Y r}{2} - Y k \right\} g(X, Z) \]
\[ + \left\{ -\frac{(Y r)}{2} + 3(Y k) \right\} \eta(X) \eta(Z) + \left\{ -\frac{r}{2} + 3k \right\} ((\nabla Y \eta)(X) \eta(Z) \]
\[ + \eta(X)(\nabla Y \eta)(Z)) + (Y \mu) g(h X, Z) + \mu g((\nabla Y h)(X), Z). \]
Substituting $Z = \xi$ in (3.4) gives
\[
\left( \frac{Xr}{2} - (Xk) \right) \eta(Y) + \left\{ - \frac{(Xr)}{2} + 3(Xk) \right\} \eta(Y) + \left\{ - \frac{r}{2} + 3k \right\} \{(\nabla_X \eta)(Y) + \eta(Y)(\nabla_X \eta)\} = \left\{ \frac{Yr}{2} - Yk \right\} \eta(X)
\]
\[
+ \left\{ - \frac{(Yr)}{2} + 3(Yk) \right\} \eta(X) + \left\{ - \frac{r}{2} + 3k \right\} \{(\nabla_Y \eta)(X) + \eta(X)(\nabla_Y \eta)\}(\xi)
\]
(3.5)
\[
+ \mu \eta((\nabla_Y h)(X)).
\]

Putting $X = \xi$ in (3.5) and using $r = \text{constant}$, we obtain
(3.6)
\[
\mu \eta((\nabla\xi h)(Y)) = 2(Yk) + \mu \eta((\nabla_Y h)(\xi)) = 0.
\]

Applying (2.8) in (3.6), we have $(Yk) = 0$, which implies $k = \text{constant}$. Hence from (2.11), we get either $h = 0$ or $\mu = \text{constant}$. Thus, we can state the following

**Theorem 3.1.** If in a generalized $(k, \mu)$-paracontact metric manifold with $k \neq -1$ the Ricci tensor is of Codazzi type, then the manifold is either a $(k, \mu)$-paracontact metric manifold or a $K$-paracontact manifold.

4. The cyclic parallel Ricci tensor in generalized $(k, \mu)$-paracontact metric manifolds

This section is devoted to the study of the cyclic parallel Ricci tensor in generalized $(k, \mu)$-paracontact metric manifolds

If the Ricci tensors is cyclic parallel, then we have
(4.1)
\[
(\nabla_Z S)(X, Y) + (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) = 0,
\]
which implies $r = \text{constant}$.

Now from the equation (2.14), we obtain
\[
\left( \frac{Zr}{2} - (Zk) \right) g(X, Y) + \left\{ - \frac{(Zr)}{2} + 3(Zk) \right\} \eta(X) \eta(Y)
\]
\[
+ \left\{ - \frac{r}{2} + 3k \right\} \{(\nabla_Z \eta)(X) \eta(Y) + \eta(X)(\nabla_Z \eta)(Y)\} + (Z\mu) g(hX, Y)
\]
\[
+ \mu g((\nabla_X h)(Y)) + \left\{ \frac{Xr}{2} - Xk \right\} g(Y, Z) + \left\{ - \frac{(Xr)}{2} + 3(Xk) \right\} \eta(Y) \eta(Z)
\]
\[
+ \left\{ - \frac{r}{2} + 3k \right\} \{(\nabla_Y \eta)(Y) \eta(Z) + \eta(Y)(\nabla_Y \eta)(Z)\} + (X\mu) g(hY, Z)
\]
\[
+ \mu g((\nabla_X h)(Y)) + \left\{ (Yr) - (Yk) \right\} g(Z, X) + \left\{ - \frac{(Yr)}{2} + 3(Yk) \right\} \eta(Z) \eta(X)
\]
\[
+ \left\{ - \frac{r}{2} + 3k \right\} \{(\nabla_Y \eta)(Z) \eta(X) + \eta(Z)(\nabla_Y \eta)(X)\} + (Y\mu) g(hZ, X)
\]
(4.2) $+ \mu g((\nabla_Y h)(Z), X) = 0.$
Substituting $X = Y = \xi$ and applying (2.8) in (4.2) yields

$$2(Zk) + (\frac{r}{2} + 3k)(\nabla_\xi \eta)(Z) + (\frac{r}{2} + 3k)(\nabla_\xi \eta)(Z) = 0.$$  

Now using (2.12) in (4.3), we have

$$Zk = 0.$$

Therefore, $k =$constant. Hence from (2.11), we have either $h = 0$ or $\mu =$constant.

This leads to the following:

**Theorem 4.1.** If in a generalized $(k, \mu)$-paracontact metric manifold with $k \neq -1$ the Ricci tensor is cyclic parallel, then the manifold is either a $(k, \mu)$-paracontact metric manifold or a $K$-paracontact manifold.

5. The $\eta$-parallel $\phi$-tensor in generalized $(k, \mu)$-paracontact metric manifolds

In this section we study the $\eta$-parallel $\phi$-tensor in generalized $(k, \mu)$-paracontact metric manifolds

If the $(1, 1)$ tensor $\phi$ is $\eta$-parallel, then we have [1]

$$g(\nabla_X \phi Y, Z) = 0.$$  

From (2.10) and (5.1), we get

$$-g(X, Y)\eta(Z) + g(hX, Y)\eta(Z) + g(X, Z)\eta(Y) - g(hX, Z)\eta(Y) = 0.$$

Putting $Z = \xi$ in (5.2) yields

$$-g(X, Y) + g(hX, Y) + \eta(X)\eta(Y) = 0.$$  

Substituting $X = hX$ in (5.3), we have

$$-g(hX, Y) - (k + 1)g(X, Y) + (k + 1)\eta(X)\eta(Y) = 0.$$  

Adding (5.3) and (5.4), we obtain

$$(k + 2)\{g(X, Y) - \eta(X)\eta(Y)\} = 0.$$  

Thus we have $k = -2$, that is, $k =$constant. Using (2.11) we have $h \text{ grad} \mu = 0$. Therefore, either $h = 0$ or $\mu =$constant.

Thus we can state the following:

**Theorem 5.1.** If in a generalized $(k, \mu)$-paracontact metric manifold with $k \neq -1$, the tensor $\phi$ is $\eta$-parallel, then the manifold is either a $(k, \mu)$-paracontact metric manifold or a $K$-paracontact manifold.
6. Locally $\phi$-Ricci symmetric generalized $(k, \mu)$-paracontact manifolds

A paracontact metric manifold is said to be locally $\phi$-Ricci symmetric [6] if it satisfies

\begin{equation}
\phi^2(\nabla_X Q)(Y) = 0,
\end{equation}

for all vector fields $X$, $Y$ orthogonal to $\xi$, where $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

Taking the covariant derivative of (2.13) with respect to $Y$ and applying $\phi^2$ we get

\begin{equation}
-\frac{(Y_k)}{2}X - \frac{(X_k)}{2}Y - (Y_\mu)hX + \mu\phi^2((\nabla_Y h)X) = 0.
\end{equation}

Interchanging $X$ and $Y$ in (6.2), we have

\begin{equation}
-\frac{(X_k)}{2}X - \frac{(Y_k)}{2}Y - (X_\mu)hY + \mu\phi^2((\nabla_Y h)X) = 0.
\end{equation}

Subtracting (6.3) from (6.2), we obtain

\begin{equation}
\frac{(Y_r)}{2}X - \frac{(X_r)}{2}Y - \frac{(Y_k)}{2}X - \frac{(X_k)}{2}Y + (Y_\mu)hX - (X_\mu)hY - \frac{1}{2}(\nabla_Y h)X) = 0.
\end{equation}

Applying (2.9) in (6.4), we get

\begin{equation}
\frac{(Y_k)}{2}X - \frac{(X_k)}{2}Y - \frac{(Y_\mu)}{2}X + \frac{(X_\mu)}{2}Y = 0.
\end{equation}

Substituting $X = \xi$ in (6.5) yields

\begin{equation}
-\frac{1}{2}(\xi_r)Y - (\xi_\mu)hY + \frac{(Y_r)}{2} - \frac{(Y_k)}{2} = 0.
\end{equation}

Taking the inner product with $Z$ from (6.6), we have

\begin{equation}
-\frac{1}{2}(\xi_r)g(Y, Z) - (\xi_\mu)g(hY, Z) = 0.
\end{equation}

Let \{\epsilon_i\}, $i = 1, 2, 3$ be a local orthonormal basis in the tangent space $T_pM$ at each point $p \in M$. Substituting $Y = Z = \epsilon_i$ in (6.7) and summing over $i = 1$ to 3, we infer that $\xi_r = 0$, since $k \neq -1$.

This leads to the following:

**Theorem 6.1.** If a generalized $(k, \mu)$-paracontact metric manifold with $k \neq -1$, is locally $\phi$-Ricci symmetric, then the characteristic vector field $\xi$ leaves the scalar curvature invariant.
REFERENCES


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