ON TZITZEICA CURVES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

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Abstract. In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. We characterize such curves according to their curvatures. We show that there is no Tz-curve with constant curvatures (W-curves). We consider Salkowski (TC-curve) and anti-Salkowski curves.

Keywords: Tz-curves, W-curves, TC-curves

1. Introduction

Gheorgha Tzitzeica, a Romanian mathematician (1872-1939), introduced a class of curves, nowadays called Tzitzeica curves, and a class of surfaces of the Euclidean 3-space called Tzitzeica surfaces. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x = x(s)$ for which the ratio of its torsion $\kappa_2$ and the square of the distance $d_{\text{osc}}$ from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e.,

$$\frac{\kappa_2}{d_{\text{osc}}^2} = a$$

where $d_{\text{osc}} = \langle N_2, x \rangle$ and $a \neq 0$ is a real constant, $N_2$ is the binormal vector of $x$.

In [3] the authors gave the connections between the Tzitzeica curve and the Tzitzeica surface in a Minkowski 3-space and the original ones from the Euclidean 3-space. In [7] the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in a Euclidean space. In [12], the elliptic cylindrical curves verifying Tzitzeica condition were adapted to the Minkowski 3-space. In [2], the authors gave the necessary and sufficient condition for a space curve to become a Tzitzeica curve. The new classes of symmetry reductions for the Tzitzeica curve equation were determined. In [1], the authors were interested in the curves of Tzitzeica type and they investigated the conditions for non-null general helices, pseudo-spherical curves and pseudo-spherical general helices to become of Tzitzeica type in a Minkowski space $\mathbb{E}_1^3$. 

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A Tzitzeica surface in \( \mathbb{E}^3 \) is a spatial surface \( M \) given with the parametrization \( X(u, v) \) for which the ratio of its Gaussian curvature \( K \) and the distance \( d_{\tan} \) from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e.,

\[
\frac{K}{d_{\tan}^4} = a_1
\]

for a constant \( a_1 \). The orthogonal distance from the origin to the tangent plane is defined by

\[
d_{\tan} = \langle X, \vec{U} \rangle
\]

where \( X \) is the position vector of the surface and \( \vec{U} \) is a unit normal vector of the surface.

The asymptotic lines of a Tzitzeica surface with a negative Gaussian curvature are Tzitzeica curves \([7]\). In \([18]\), the authors gave the necessary and sufficient condition for the Cobb-Douglas production hypersurface to be a Tzitzeica hypersurface. In addition, a new Tzitzeica hypersurface was obtained in parametric, implicit and explicit forms in \([8]\).

In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space \( \mathbb{E}^3 \). Furthermore, we investigate a Tzitzeica curve in a Euclidean 3-space \( \mathbb{E}^3 \) whose position vector \( x(s) \) satisfies the parametric equation

\[
x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s),
\]

for some differentiable functions, \( m_i(s), 0 \leq i \leq 2 \), where \( \{T, N_1, N_2\} \) is the Frenet frame of \( x \). We characterize such curves according to their curvatures. We show that there is no Tzitzeica curve in \( \mathbb{E}^3 \) with constant curvatures (W-curves). We give the relations between the curvatures of the Tz-Salkowski curve (TC-curve) and the Tz-anti-Salkowski curve.

2. Basic Notations

Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in a Euclidean 3-space \( \mathbb{E}^3 \). Let us denote \( T(s) = x'(s) \) and call \( T(s) \) a unit tangent vector of \( x \) at \( s \). We denote the curvature of \( x \) by \( \kappa_1(s) = ||x''(s)|| \). If \( \kappa_1(s) \neq 0 \), then the unit principal normal vector \( N_1(s) \) of the curve \( x \) at \( s \) is given by \( x''(s) = \kappa_1(s)N_1(s) \). The unit vector \( N_2(s) = T(s) \times N_1(s) \) is called the unit binormal vector of \( x \) at \( s \). Then we have the Serret-Frenet formulae:

\[
\begin{align*}
T'(s) &= \kappa_1(s)N_1(s), \\
N_1'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)N_2(s), \\
N_2'(s) &= -\kappa_2(s)N_1(s),
\end{align*}
\]

where \( \kappa_2(s) \) is the torsion of the curve \( x \) at \( s \) (see, \([10]\)).
If the Frenet curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$ of $x$ are constant functions then $x$ is called a screw line or a helix [9]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations then F. Klein and S. Lie called them $W$-curves [14]. It is known that a curve $x$ in $\mathbb{E}^3$ is called a general helix if the ratio $\kappa_2(s)/\kappa_1(s)$ is a nonzero constant [16]. Salkowski (resp. anti-Salkowski) curves in a Euclidean space $\mathbb{E}^3$ are generally known as the family of curves with a constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization [15, 17] (for T.C.-curve see also [13]).

For a space curve $x: I \subset \mathbb{R} \to \mathbb{E}^3$, the planes at each point of $x(s)$ spanned by $\{T, N_1\}$, $\{T, N_2\}$ and $\{N_1, N_2\}$ are known as the osculating plane, the rectifying plane and normal plane, respectively. If the position vector $x$ lies on its rectifying plane, then $x(s)$ is called rectifying curve [5]. Similarly, the curve for which the position vector $x$ always lies in its osculating plane is called osculating curve. Finally, $x$ is called normal curve if its position vector $x$ lies in its normal plane.

Rectifying curves characterized by the simple equation
\begin{equation}
(2.2) \quad x(s) = \lambda(s)T(s) + \mu(s)N_2(s),
\end{equation}
where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $N_2(s)$ are tangent and binormal vector fields of $x$, respectively [5, 6].

For a regular curve $x(s)$, the position vector $x$ can be decomposed into its tangential and normal components at each point:
\begin{equation}
(2.3) \quad x = x^T + x^N.
\end{equation}

A curve in $\mathbb{E}^3$ is called $N$-constant if the normal component $x^N$ of its position vector $x$ is of constant length [4, 11]. It is known that a curve in $\mathbb{E}^3$ is congruent to an $N$-constant curve if and only if the ratio $\frac{\kappa_2}{\kappa_1}$ is a non-constant linear function of an arc-length function $s$, i.e., $\frac{\kappa_2}{\kappa_1}(s) = c_1 s + c_2$ for some constants $c_1$ and $c_2$ with $c_1 \neq 0$ [4]. Further, an $N$-constant curve $x$ is called first kind if $\|x^N\| = 0$, otherwise second kind [11].

3. Tzitzeica Curves in $\mathbb{E}^3$

In the present section we characterize Tzitzeica curves in $\mathbb{E}^3$ in terms of their curvatures.

**Definition 3.1.** Let $x: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s) \neq 0$. If the torsion of $x$ satisfies the condition
\begin{equation}
(3.1) \quad \kappa_2(s) = a d_{osc}^2,
\end{equation}
for some real constant $a$ then $x$ is called Tzitzeica curve (Tz-curve), where
\begin{equation}
(3.2) \quad d_{osc} = \langle N_2, x \rangle
\end{equation}
is the orthogonal distance from the origin to the osculating plane of $x$. 
We have the following result.

**Proposition 3.1.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in \( \mathbb{E}^3 \). If \( x \) is a Tz-curve, then the equation

\[
(3.3) \quad \kappa'_2 \langle x, N_2 \rangle + 2\kappa^2_2 \langle x, N_1 \rangle = 0
\]

holds.

**Proof.** Let \( x \) be a unit speed curve in \( \mathbb{E}^3 \), then by the use of the equations (3.1) and (3.2) we get

\[
(3.4) \quad \frac{\kappa_2(s)}{\langle N_2, x \rangle^2} = a \neq 0.
\]

Further, differentiating the equation (3.4), we obtain the result. \( \square \)

**Definition 3.2.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve with curvatures \( \kappa_1(s) > 0 \) and \( \kappa_2(s) \neq 0 \). Then \( x \) is a spherical curve if and only if

\[
(3.5) \quad \frac{\kappa_2(s)}{\kappa_1(s)} = \left( \frac{\kappa'_1(s)}{\kappa_2(s)\kappa_1^2(s)} \right)
\]

holds [9].

**Theorem 3.1.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed spherical curve in \( \mathbb{E}^3 \). If \( x \) is a Tz-curve then the equation

\[
(3.6) \quad \frac{\kappa'_2(s)}{2\kappa^2_2(s)} = \frac{\kappa_1(s)}{\kappa'_1(s)}
\]

holds between the curvatures of \( x \).

**Proof.** Let \( x \) be a unit speed spherical curve in \( \mathbb{E}^3 \). Then we have

\[
(3.7) \quad ||x|| = r
\]

where \( r \) is the radius of the sphere. Differentiating the equation (3.7) with respect to \( s \), we get

\[
(3.8) \quad \langle x, T \rangle = 0.
\]

Further, differentiating the equation (3.8), we have

\[
(3.9) \quad \langle x, N_1 \rangle = -\frac{1}{\kappa_1}.
\]

By differentiating the equation (3.9), we obtain

\[
(3.10) \quad \langle x, N_2 \rangle = \frac{\kappa'_1}{\kappa^2_1\kappa_2}.
\]

Finally, substituting (3.9) and (3.10) into (3.3), we get the result. \( \square \)
Corollary 3.1. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed spherical Tz-curve in \( \mathbb{E}^3 \). Then the torsion of \( x \) satisfies the equation

\[
\kappa_2 = \sqrt{\frac{\kappa_1''\kappa_1 - 2(\kappa_1')^2}{3\kappa_1'^2}}.
\]

Proof. Substituting (3.6) into (3.5), we get the result. \( \square \)

Corollary 3.2. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski spherical Tz-curve in \( \mathbb{E}^3 \). Then the curvature of \( x \) is given by

\[
\kappa_1 = \frac{\sqrt{3}\kappa_2}{c_1 \sin (\sqrt{3}\kappa_2 s) - c_2 \cos (\sqrt{3}\kappa_2 s)}
\]

where \( c_1, c_2 \) are integral constants and \( \kappa_2 \) is the constant torsion of \( x \).

Proof. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski spherical Tz-curve in \( \mathbb{E}^3 \). Then from (3.11), we obtain the differential equation

\[
\kappa_1''\kappa_1 - 2(\kappa_1')^2 - 3\kappa_2^2 = 0
\]

which has the solution (3.12). \( \square \)

Lemma 3.1. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in \( \mathbb{E}^3 \) whose position vector satisfies the parametric equation

\[
x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s)
\]

for some differentiable functions, \( m_i(s) \), \( 0 \leq i \leq 2 \). If \( x \) is a Tz-curve then we get

\[
\begin{align*}
m_0' - \kappa_1 m_1 &= 1, \\
m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\
m_2' + \kappa_2 m_1 &= 0, \\
\kappa_2' m_2 + 2\kappa_2^2 m_1 &= 0.
\end{align*}
\]

Proof. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in \( \mathbb{E}^3 \). Then, by taking the derivative of (3.14) with respect to the parameter \( s \) and using the Frenet formulae, we obtain

\[
x'(s) = (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\
+ (m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\
+ (m_2'(s) + \kappa_2(s)m_1(s))N_2(s).
\]

Further, using the equations (3.3) and (3.16), we get (3.15). \( \square \)
Theorem 3.2. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski Tz-curve in \( \mathbb{E}^3 \) (with the curvatures \( \kappa_1 > 0 \) and \( \kappa_2 \neq 0 \)) given with the parametrization (3.14). Then \( x \) is congruent to a rectifying curve with the parametrization
\[
(3.17) \quad x(s) = (s + c_1) T(s) + c_2 N_2(s)
\]
where \( c_1 \) and \( c_2 \) are integral constants.

Proof. Let \( x \) be a unit speed anti-Salkowski Tz-curve in \( \mathbb{E}^3 \). Then, the torsion \( \kappa_2 \) of \( x \) is constant. From the equation (3.15), we get
\[
m_0 = s + c_1 \\
m_1 = 0 \\
m_2 = c_2
\]
where \( c_1 \) and \( c_2 \) are integral constants. Finally, substituting (3.18) into (3.14), we get the result.

Corollary 3.3. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski Tz-curve in \( \mathbb{E}^3 \) (with curvatures \( \kappa_1 > 0 \) and \( \kappa_2 \neq 0 \)) given with the parametrization (3.14). Then \( x \) is congruent to \( N \)-constant curve of second kind.

Corollary 3.4. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed Salkowski Tz-curve in \( \mathbb{E}^3 \) (with the curvatures \( \kappa_1 > 0 \) and \( \kappa_2 \neq 0 \)) given with the parametrization (3.14). Then we have
\[
(3.19) \quad m''_1 + (\kappa_1^2 + 3\kappa_2^2) m_1 + \kappa_1 = 0
\]
where the curvature \( \kappa_1 \) of \( x \) is a real constant.

Proof. Let \( x \) be a unit speed Salkowski Tz-curve in \( \mathbb{E}^3 \). Hence, the curvature \( \kappa_1 \) of \( x \) is constant, from the equation (3.15), we get the result.

Corollary 3.5. There is no Tz-curve with a constant curvature and a constant torsion. (i.e. Tz-W-curve)

Proof. Let \( x \) be a unit speed Tz-curve in \( \mathbb{E}^3 \) with a constant curvature and a constant torsion. (i.e. Tz-W-curve). Then, using (3.15), we obtain
\[
(3.20) \quad \kappa_1(s) = \frac{c_2}{s + c_1} \kappa_2(s)
\]
which is a contradiction.
REFERENCES

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