FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 4 (2018), 561–575
https://doi.org/10.22190/FUMI1804561B

CONHARMONIC CURVATURE TENSOR OF A QUARTER-SYMMETRIC METRIC CONNECTION IN A KENMOTSU MANIFOLD

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Abstract. The aim of the present paper is to study Kenmotsu manifolds admitting a quarter-symmetric metric connection whose conharmonic curvature tensor satisfies certain curvature conditions.
Keywords. Kenmotsu manifolds; curvature; conharmonic curvature tensor.

1. Introduction

Manifolds known as Kenmotsu manifolds were studied by K. Kenmotsu in 1972 [17]. They set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension [26]. Consider an almost contact metric manifold M^{2n+1}, with the structure (ϕ, ξ, η, g) given by a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, and a Riemannian metric g such that $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector field X and Y. The fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$ for any vector fields X and Y. The normality of an almost contact metric manifold is expressed by the vanishing of the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [6]. For more details we refer to Blair's books ([6],[7]). A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. It is well known that Kenmotsu manifolds can be characterized through their Levi-Civita connection, by $(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y, Z. Moreover, Kenmotsu proved that such a manifold M^{2n+1} is locally a warped product $]-\varepsilon, \varepsilon[\times_f N^{2n}$, N^{2n} being a Kähler manifold and $f^2 = ce^{2t}$, c is a positive constant.

More recently in ([18],[21]) and [12], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu.

Received December 23, 2017; accepted July 14, 2018
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Kenmotsu manifolds have been studied by Barman and De ([4], [5]) Barman [2], Kim and Pak [18] and many others.

In 1924, Friedmann and Schouten [13] introduced the idea of a semi-symmetric connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ satisfies

$$\tilde{T}(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form and ρ is a vector field defined by

$$u(X) = g(X,\rho), \tag{1.2}$$

for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [14] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M,g). A semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric metric connection if $\tilde{\nabla}g = 0$.

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M,g) was given by Yano [27]: $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)\rho$, where $u(X) = g(X,\rho)$.

In 1975, Golab [15] defined and studied the quarter-symmetric connection in differentiable manifolds with affine connections. A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is called a quarter-symmetric connection [15] if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.3}$$

where η is a 1-form and ϕ is a (1,1) tensor field.

In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [13]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If, moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0, \tag{1.4}$$

for all $X, Y, Z \in \chi(M)$, then the quarter-symmetric connection $\bar{\nabla}$ is said to be a quarter-symmetric metric connection.

After Golab [15] and Rastogi ([23], [24]) the systematic study of quarter-symmetric metric connection have been continued by Mishra and Pandey [19], Yano and Imai [28], Mukhopadhyay, Roy and Barua [20], De and Biswas [10], Taleshian and Parakasha [25], Barman [3] and many others.

Let M be a Riemannian manifold of dimension n equipped with two metric tensors g and $\grave{g}$. If a transformation of M does not change the angle between two tangent vectors at a point with respect to g and $\grave{g}$, then such a transformation is said to be a conformal transformation of the metrics on the Riemannian manifold. Under conformal transformation, the length of the curves are changed but the angles made by the curves remain the same.

Let us consider a Riemannian manifold M with two metric tensors g and $\grave{g}$ such that they are related by

$$\grave{g}(X,Y) = e^{2\sigma} g(X,Y), \tag{1.5}$$

where σ is a real function on M.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general, a harmonic function is not transformed into a harmonic function. The condition under which a harmonic function remains invariant has been studied by Ishii [16] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.5) satisfying the condition

$$\sigma,_i^i + \sigma,_i \sigma,^i = 0, \tag{1.6}$$

where the comma denotes the covariant differentiation with respect to the metric g.

Let C denote the conharmonic curvature tensor of type $(1,3)$ with respect to the Levi-Civita connection which is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[&g(Y,Z)QX - g(X,Z)QY \\ &+S(Y,Z)X - S(X,Z)Y], \end{aligned} \tag{1.7}$$

where $S(Y,Z) = g(QY,Z)$.

Taking the inner product of (1.7) with W, we have

$$\begin{aligned} 'C(X,Y,Z,W) =' R(X,Y,Z,W) - \frac{1}{2n-1}[&g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+S(Y,Z)g(X,W) - S(X,Z)g(Y,W)], \end{aligned} \tag{1.8}$$

where $'C(X,Y,Z,W) = g(C(X,Y)Z,W)$, $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$, R and S are the curvature tensor and the Ricci tensor with respect to the Levi-Civita connection, respectively.

A manifold is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = a'g(X,Y)$, where a' is a constant on the manifold.

A manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on the manifold.

In this paper we study the conharmonic curvature tensor on Kenmotsu manifolds with respect to the quarter-symmetric metric connection. The paper is organized as follows. After the introduction in Section 2, we give a brief account of the Kenmotsu manifolds. In section 3, we express the quarter-symmetric metric connection on Kenmotsu manifolds. Section 4 is devoted to the study of the semi ϕ-conharmonically flat on Kenmotsu manifolds admitting the quarter-symmetric metric connection and we prove that the manifold is an Einstein manifold with respect to the Levi-Civita connection. Section 5 deals with the ξ-conharmonically flat on Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Section 6 contains the ϕ-conharmonically flat on Kenmotsu manifolds admitting the quarter-symmetric metric connection. We get the manifold to be an η- Einstein manifold with respect to the Levi-Civita connection. Finally, we construct an example of a 3-dimensional Kenmotsu manifold admitting the quarter-symmetric metric connection to support the results obtained in Section 5.

2. Kenmotsu Manifolds

Let M be an $(2n+1)$-dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and the Riemannian metric g on M satisfying [6]

$$\eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta(\phi(X)) = 0,\ g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X ,Y on $\chi(M)$. A manifold with an almost contact metric structure (ϕ, ξ, η, g) is an almost Kenmotsu manifold if the following conditions are satisfied

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega,$$

where Ω being the 2-form defined by $\Omega(X,Y) = g(X, \phi Y)$. Any normal almost Kenmotsu manifold is a Kenmotsu manifold. An almost contact metric structure (ϕ, ξ, η, g) is a Kenmotsu manifold [17] if and only if

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.4}$$

Here we denote the Kenmotsu manifold of dimension $(2n+1)$ by M. From the above relations, it follows that

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.9}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.10}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.11}$$

where R and S denote the curvature tensor and the Ricci tensor of M,, respectively, with respect to the Levi-Civita connection.

Let M be a Kenmotsu manifold. M is said to be a η-*Einstein manifold* if there exist real valued functions λ_1, λ_2 such that

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y).$$

For $\lambda_2 = 0$, the manifold M is an Einstein manifold.

Now we state the following:

Lemma 2.1. *[17] Let M be an η-Einstein Kenmotsu manifold of the form $S(X,Y) = \lambda_1 g(X,Y) + \lambda_2\eta(X)\eta(Y)$. If $\lambda_2 = constant$ (or, $\lambda_1 = constant$), then M is an Einstein one.*

3. Quarter-symmetric metric connection on Kenmotsu manifolds

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M,g) has been obtained by Sular, *Özgür* and De [11] which is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.1}$$

Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇ and the curvature tensor $\bar{R}$ of M with respect to the quarter-symmetric metric connection $\bar{\nabla}$ [11] given by

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z + \eta(X)g(\phi Y,Z)\xi - \eta(Y)g(\phi X,Z)\xi - \\ \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X\end{aligned} \tag{3.2}$$

and

$$\bar{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X, \tag{3.3}$$

where $X,Y,Z \in \chi(M)$, the set of all differentiable vector fields on M.

The above equation (3.2) yields

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z.$$

Taking the inner product of (3.2) with W [11], we have

$$\begin{aligned} {}'\bar{R}(X,Y,Z,W) ={}' R(X,Y,Z,W) + \eta(X)\eta(W)g(\phi Y,Z) \\ -\eta(Y)\eta(W)g(\phi X,Z) - \eta(X)\eta(Z)g(\phi Y,W) \\ +\eta(Y)\eta(Z)g(\phi X,W), \end{aligned} \tag{3.4}$$

where ${}'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$, ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

A relation between the Ricci tensor $\bar{S}$ of $\bar{\nabla}$ and the Ricci tensors S of ∇ on (M,g) has been obtained by S [11] which is obtained by

$$\bar{S}(Y,Z) = S(Y,Z) + g(\phi Y,Z) \tag{3.5}$$

and also

$$\bar{S}(Y,\xi) = -2n\eta(Y). \tag{3.6}$$

In view of (3.5) yields

$$\bar{Q}Y = QY + \phi Y, \tag{3.7}$$

where $\bar{S}(Y,Z) = g(\bar{Q}Y,Z)$.

Again the scalar curvature tensor $\bar{r}$ of the quarter-symmetric metric connection $\bar{\nabla}$ and the scalar curvature tensor r of the Levi-Civita connection ∇ on (M,g) is defined by [11], so we get

$$\bar{r} = r, \tag{3.8}$$

From (2.9), it is implied that

$$r = -2n(2n+1).$$

4. Semi ϕ-conharmonically flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Let ${}'\bar{C}$ denote the conharmonic curvature tensor of type $(0,4)$ with respect to the quarter-symmetric metric connection which is defined by

$$\begin{aligned} {}'\bar{C}(X,Y,Z,W) ={}' \bar{R}(X,Y,Z,W) - \frac{1}{2n-1}[g(Y,Z)\bar{S}(X,W) - g(X,Z)\bar{S}(Y,W) \\ +\bar{S}(Y,Z)g(X,W) - \bar{S}(X,Z)g(Y,W)], \end{aligned} \tag{4.1}$$

where$'\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W)$, $'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$ and $X,Y,Z \in \chi(M)$, the set of all differentiable vector fields on M.

Let $\mathbf{C}$ be the Weyl conformal curvature tensor of a $(2n+1)$-dimensional manifold M. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into a direct sum $\chi_p(M) = \phi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $\chi_p(M)$ generated by ξ_p. Then we have a map:

$\mathbf{C} : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M)) \oplus L(\xi_p).$

It may be natural to consider the following particular cases:

(1)$\mathbf{C} : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow L(\xi_p)$, i.e, the projection of the image of $\mathbf{C}$ in $\phi(\chi_p(M))$ is zero.

(2)$\mathbf{C} : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M))$, i.e, the projection of the image of $\mathbf{C}$ in $L(\xi_p)$ is zero.

$$\mathbf{C}(X,Y)\xi = 0.$$

(3)$\mathbf{C} : \phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M)) \longrightarrow L(\xi_p)$, i.e, when $\mathbf{C}$ is restricted to $\phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M))$, the projection of the image of $\mathbf{C}$ in $\phi(\chi_p(M))$ is zero. This condition is equivalent to

$$\phi^2 \mathbf{C}(\phi X, \phi Y)\phi Z = 0.$$

Here the cases 1, 2 and 3 are conformally symmetric, ξ-conformally flat and ϕ-conformally flat, respectively. The cases (1) and (2) were considered in [9] and [29], respectively. The case (3) was considered in [8] for the case M is a K-contact manifold. Furthermore, in [1], the authors studied contact metric manifolds satisfying (3). Analogous to the definition of ξ-conformally flat and ϕ-conformally flat, we give the following definitions:

Definition 4.1. A Kenmotsu manifold is said to be semi-ϕ-conharmonically flat with respect to the quarter-symmetric metric connection if

$$g(\bar{C}(\phi X, Y)Z, \phi W) = 0. \tag{4.2}$$

Definition 4.2. A Kenmotsu manifold is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X,Y) = a' g(X,Y),$$

where a' is a constant on the manifold.

Putting $X = \phi X$ and $W = \phi W$ in (4.1), we get

$$'\bar{C}(\phi X, Y, Z, \phi W) =' \bar{R}(\phi X, Y, Z, \phi W) - \frac{1}{2n-1}[g(Y, Z)\bar{S}(\phi X, \phi W) - g(\phi X, Z)\bar{S}(Y, \phi W) + \bar{S}(Y, Z)g(\phi X, \phi W) - \bar{S}(\phi X, Z)g(Y, \phi W)]. \tag{4.3}$$

In view of (2.1), (2.2), (3.4) and (4.3) yields

$$'\bar{C}(\phi X, Y, Z, \phi W) =' R(\phi X, Y, Z, \phi W) - \eta(Y)\eta(Z)g(X, \phi W) - \frac{1}{2n-1}[g(Y, Z)\bar{S}(\phi X, \phi W) - g(\phi X, Z)\bar{S}(Y, \phi W) + \bar{S}(Y, Z)g(\phi X, \phi W) - \bar{S}(\phi X, Z)g(Y, \phi W)]. \tag{4.4}$$

Applying (2.1), (2.2), (2.3), (2.5) and (3.5) in (4.4), it follows that

$$\begin{aligned}'\bar{C}(\phi X, Y, Z, \phi W) =' R(\phi X, Y, Z, \phi W) - \eta(Y)\eta(Z)g(X, \phi W) \\ - \frac{1}{2n-1}[g(Y, Z)S(X, W) + 2n\eta(X)\eta(W)g(Y, Z) - g(Y, Z)g(X, \phi W) \\ -g(\phi X, Z)S(Y, \phi W) + g(Y, W)g(X, \phi Z) - \eta(Y)\eta(W)g(X, \phi Z) \\ +g(X, W)S(Y, Z) - \eta(X)\eta(W)S(Y, Z) + g(X, W)g(\phi Y, Z) \\ -\eta(X)\eta(W)g(\phi Y, Z) - g(Y, \phi W)S(\phi X, Z) - g(X, Z)g(\phi Y, W) \\ +\eta(X)\eta(Z)g(\phi Y, W)].\end{aligned} \tag{4.5}$$

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.5) and summing over $i = 1$ to $2n$, we obtain

$$\begin{aligned}\sum_{i=1}^{2n} g(\bar{C}(\phi e_i, Y)Z, \phi e_i) = \sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) - \sum_{i=1}^{2n} \eta(Y)\eta(Z)g(e_i, \phi e_i) \\ - \frac{1}{2n-1}\sum_{i=1}^{2n}[g(Y, Z)S(e_i, e_i) + 2n\eta(e_i)\eta(e_i)g(Y, Z) - g(Y, Z)g(e_i, \phi e_i) \\ -g(\phi e_i, Z)S(Y, \phi e_i) + g(Y, e_i)g(e_i, \phi Z) - \eta(Y)\eta(e_i)g(e_i, \phi Z) \\ +g(e_i, e_i)S(Y, Z) - \eta(e_i)\eta(e_i)S(Y, Z) + g(e_i, e_i)g(\phi Y, Z) \\ -\eta(e_i)\eta(e_i)g(\phi Y, Z) - g(Y, \phi e_i)S(\phi e_i, Z) - g(e_i, Z)g(\phi Y, e_i) \\ +\eta(e_i)\eta(Z)g(\phi Y, e_i)].\end{aligned} \tag{4.6}$$

Again using (2.1), (2.2), (2.5) and (4.2) in (4.6), we see that

$$S(Y,Z) = -2n^2 g(Y,Z) - (n - \frac{3}{2})g(Y,\phi Z). \tag{4.7}$$

Interchanging Y with Z in (4.7), implies that

$$S(Y,Z) = -2n^2 g(Y,Z) - (n - \frac{3}{2})g(Z,\phi Y). \tag{4.8}$$

By adding (4.7) and (4.8) and using (2.5), we have

$$S(Y,Z) = -2n^2 g(Y,Z).$$

Therefore, $S(Y,Z) = a' g(Y,Z)$,
where $a' = -2n^2$.

This means that the manifold is an Einstein manifold with respect to the Levi-Civita connection.

Summing up we can state the following:

Theorem 4.1. *If a Kenmotsu manifold is semi-ϕ-conharmonically flat with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold.*

5. ξ -conharmonically flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Let $\bar{C}$ denote the conharmonic curvature tensor of type $(1,3)$ with respect to the quarter-symmetric metric connection which is defined by

$$\begin{aligned}\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y \\ +\bar{S}(Y,Z)X - \bar{S}(X,Z)Y],\end{aligned} \tag{5.1}$$

where $\bar{S}(Y,Z) = g(\bar{Q}Y,Z)$ and $X,Y,Z \in \chi(M)$, the set of all differentiable vector fields on M.

Definition 5.1. A Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be ξ-conharmonically flat if $\bar{C}(X,Y)\xi = 0$.

Putting $Z = \xi$ in (5.1), it follows that

$$\begin{aligned}\bar{C}(X,Y)\xi = \bar{R}(X,Y)\xi - \frac{1}{2n-1}[g(Y,\xi)\bar{Q}X - g(X,\xi)\bar{Q}Y \\ +\bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y].\end{aligned} \tag{5.2}$$

Using (2.1), (2.2), (3.3), (3.6) and (3.7) in (5.2), we get

$$\bar{C}(X,Y)\xi = C(X,Y)\xi + \frac{2(n-1)}{2n-1}[\eta(Y)\phi X - \eta(X)\phi Y]. \tag{5.3}$$

If $n = 1$, then the above equation (5.3) implies that

$$\bar{C}(X,Y)\xi = C(X,Y)\xi.$$

Now, we are in a position to state the following:

Theorem 5.1. *A three-dimensional Kenmotsu manifold is ξ -conharmonically flat with respect to the quarter-symmetric metric connection if the manifold is also ξ -conharmonically flat with respect to the Levi-Civita connection.*

6. ϕ -conharmonically flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Definition 6.1. A Kenmotsu manifold is said to be ϕ-conharmonically flat with respect to the quarter-symmetric metric connection if

$$g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{6.1}$$

where $X, Y, Z, W \in \chi(M)$, the set of all differentiable vector fields on M.

Definition 6.2. A Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold .

Putting $Y = \phi Y$ and $Z = \phi Z$ in (4.3), we get

$$\begin{aligned} {}'\bar{C}(\phi X, \phi Y, \phi Z, \phi W) &= {}'\bar{R}(\phi X, \phi Y, \phi Z, \phi W) \\ -\frac{1}{2n-1}&[g(\phi Y, \phi Z)\bar{S}(\phi X, \phi W) - g(\phi X, \phi Z)\bar{S}(\phi Y, \phi W) \\ &+\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{6.2}$$

Using (2.1), (2.2) and (3.4) in (6.2), we have

$$\begin{aligned} {}'\bar{C}(\phi X, \phi Y, \phi Z, \phi W) &= {}'R(\phi X, \phi Y, \phi Z, \phi W) \\ -\frac{1}{2n-1}&[g(\phi Y, \phi Z)\bar{S}(\phi X, \phi W) - g(\phi X, \phi Z)\bar{S}(\phi Y, \phi W) \\ &+\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{6.3}$$

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (6.3) and summing over $i = 1$ to $2n$, we obtain

$$\sum_{i=1}^{2n} g(\bar{C}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \sum_{i=1}^{2n} g(R(\phi e_i, \phi Y,)\phi Z, \phi e_i)$$
$$-\frac{1}{2n-1}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)\bar{S}(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)\bar{S}(\phi Y, \phi e_i)$$
$$+\bar{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \bar{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \tag{6.4}$$

In view of (3.8), (3.), (6.1) and (6.4), we take the form

$$S(\phi Y, \phi Z) - \frac{1}{2n-1}[-2n(2n+1)g(\phi Y, \phi Z)$$
$$+2(n-1)\bar{S}(\phi Y, \phi Z)] = 0. \tag{6.5}$$

Applying (2.1), (2.2) and (3.5) in (6.5), it is implied that

$$S(\phi Y, \phi Z) = -2n(2n+1)g(\phi Y, \phi Z) - 2(n-1)g(Y, \phi Z). \tag{6.6}$$

Interchanging Y with Z in (6.6), we get

$$S(\phi Y, \phi Z) = -2n(2n+1)g(\phi Y, \phi Z) - 2(n-1)g(Z, \phi Y). \tag{6.7}$$

By adding (6.6) and (6.7) and using (2.5), we have

$$S(\phi Y, \phi Z) = -2n(2n+1)g(\phi Y, \phi Z). \tag{6.8}$$

By virtue of (2.3) and (6.8) we yield

$$S(Y, Z) = -2n(2n+1)g(Y, Z) + 4n^2\eta(Y)\eta(Z).$$

Therefore, $S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z)$,
where $a = -2n(2n+1)$ and $b = 4n^2$.
From which it follows that the manifold is an η-Einstein manifold.

This leads us to state the following:

Theorem 6.1. *If a Kenmotsu manifold is ϕ -conharmonically flat with respect to the quarter-symmetric metric connection, then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

Since a and b are both constant, in view of Lemma 2.1, we conclude the following:

Corollary 6.1. *If a Kenmotsu manifold is ϕ -conharmonically flat with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold one.*

7. Example

In this section we construct an example on a Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ which verify the result in Section 3 and Section 5 of $\bar{\nabla}$.

We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. We choose the vector fields

$$e_1 = z\frac{\partial}{\partial x}, \ \ e_2 = z\frac{\partial}{\partial y}, \ \ e_3 = -z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be a 1-form defined by

$$\eta(Z) = g(Z, e_3)$$

for any $Z \in \chi(M)$.

Let ϕ be a $(1, 1)$-tensor field defined by

$$\phi e_1 = -e_2, \ \phi e_2 = e_1, \ \phi e_3 = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = 1$$

$$\phi^2(Z) = -Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $U, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. The 1-form η is closed. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold and is also normal. So, it is a Kenmotsu manifold.

Then we have

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = e_1, \;\; [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by [22]

$$\begin{aligned} 2g(\nabla_X Y, W) &= Xg(Y, W) + Yg(X, W) - Wg(X, Y) - g(X, [Y, W]) \\ &\quad - g(Y, [X, W]) + g(W, [X, Y]). \end{aligned} \tag{7.1}$$

Using Koszul's formula we get the following

$$\nabla_{e_1} e_1 = -e_3, \; \nabla_{e_1} e_2 = 0, \; \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \; \nabla_{e_2} e_2 = -e_3, \; \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \; \nabla_{e_3} e_2 = 0, \; \nabla_{e_3} e_3 = 0.$$

Using (3.1) in the above equation, we obtain

$$\bar{\nabla}_{e_1} e_1 = -e_3, \; \bar{\nabla}_{e_1} e_2 = 0, \; \bar{\nabla}_{e_1} e_3 = e_1,$$

$$\bar{\nabla}_{e_2} e_1 = 0, \; \bar{\nabla}_{e_2} e_2 = -e_3, \; \bar{\nabla}_{e_2} e_3 = e_2,$$

$$\bar{\nabla}_{e_3} e_1 = e_2, \; \bar{\nabla}_{e_3} e_2 = -e_1, \; \bar{\nabla}_{e_3} e_3 = 0.$$

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1, e_2)e_2 = -e_1, \;\; R(e_1, e_3)e_3 = -e_1, \;\; R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \;\; R(e_3, e_1)e_1 = -e_3, \;\; R(e_3, e_2)e_2 = -e_3,$$

and

$$\bar{R}(e_1, e_2)e_2 = -e_1, \;\; \bar{R}(e_1, e_3)e_3 = -e_2 - e_1, \;\; \bar{R}(e_2, e_1)e_1 = -e_2,$$

$$\bar{R}(e_2, e_3)e_3 = e_1 - e_2, \;\; \bar{R}(e_3, e_1)e_1 = -e_3, \;\; \bar{R}(e_3, e_2)e_2 = -e_3.$$

With the help of the above results we can express the Ricci tensor as follows:

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2$$

and

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = \bar{S}(e_3, e_3) = -2.$$

From the above expressions we can easily verify the equation (3.5). Also, it follows that the scalar curvature with respect to the Levi-Civita connection and quarter-symmetric metric connection is equal to -6.

Let X and Y be any two vector fields given by
$X = a_1 e_1 + a_2 e_2 + a_3 e_3$ and $Y = b_1 e_1 + b_2 e_2 + b_3 e_3$ where a_i, b_i, for all $i = 1, 2, 3$

are all non-zero real numbers.

Using the above curvature tensors and the Ricci tensors of the Levi-Civita connection and quarter-symmetric metric connection, respectively, we obtain

$$\bar{C}(X,Y)\xi = 3(a_1b_3 - a_3b_1)e_1 + 3(a_2b_3 - a_3b_2)e_2 = C(X,Y)\xi.$$

Hence, the manifold under consideration satisfies the Theorem 5.1 of Section 5.

Acknowledgements. The author wishes to express his sincere thanks and gratitude to the referee for his valuable suggestions towards the improvement of the paper.

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