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GENERALIZED FUGLEDE-PUTNAM THEOREM AND m-QUASI-CLASS $A(k)$ OPERATORS

Mohammad H.M. Rashid

Abstract. For a bounded linear operator T acting on an complex infinite dimensional Hilbert space $\mathcal{H}$, we say that T is an m-quasi-class $A(k)$ operator for $k > 0$ and m is a positive integer (abbreviation $T \in \mathbb{Q}(A(k), m)$) if $T^{*m}\left((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T|^2\right)T^m \geq 0$. The famous *Fuglede-Putnam theorem* asserts that: the operator equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal operators. In this paper, we prove that if $T \in \mathbb{Q}(A(k), m)$ and S^* is an operator of class $A(k)$ for $k > 0$. Then $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$ is an injective with a dense range which implies $XT^* = S^*X$.
Keywords. Bounded linear operator; Hilbert space; Fuglede-Putnam theorem; Normal operator.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on $\mathcal{H}$. Throughout this paper, the range and the null space of an operator T will be denoted by $ran(T)$ and $\ker(T)$, respectively. Let $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ be the norm closure and the orthogonal complement of the subspace $\mathcal{M}$ of $\mathcal{H}$. The classical Fuglede-Putnam theorem [12, Problem 152] asserts that if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ are normal operators such that $TX = XS$ for some operator $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS^*$. The references [16, 17, 18, 19, 20, 21] are among the various extensions of this celebrated theorem for non-normal operators.

Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([7], [12], [14] and [31]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *positive*, $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

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An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators ([1, 4, 5, 8, 9] and [13]). An operator T is said to be *p-hyponormal* if$(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$ and an operator T is said to be log-*hyponormal* if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal* if it satisfies the following norm inequality

$$\|T^2\|\|x\| \geq \|Tx\|^2$$

for all $x \in \mathcal{H}$. Ando [3] proved that every log-hyponormal operator is paranormal. It was originally introduced as an intermediate class between hyponormal and normaloid operators.

In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators, Furuta el al. [9] introduced a class A defined by $|T^2| \geq |T|^2$ and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example [9, 10]. Fujii et al. [10] introduced a new class $A(t,s)$ of operators: For $t > 0$ and $s > 0$, the operator T belongs to class $A(s,t)$ if it satisfies the operator inequality

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{t+s}} \geq |T^*|^{2t}.$$

Furuta el al. [9] introduced class $A(k)$ for $k > 0$ as a class of operators including p-hyponormal and log-hyponormal operators, where $A(1)$ coincides with class A operator. We say that an operator T is class $A(k)$, $k > 0$ (Abbreviation, $T \in A(k)$) if $(T^*|T^{2k}|T)^{\frac{1}{k+1}} \geq |T|^2$. The inclusion relations among these classes are known as follows:

$$\begin{aligned}
\{hyponormal\ operators\} &\subset \{p - hyponormal\ operators\ for\ 0 < p \leq 1\} \\
&\subset \{class\ A(s,t)\ operators\ for\ s,t \in [0,1]\} \\
&\subset \{class\ A\ operators\} \\
&\subset \{paranormal\ operators\}.
\end{aligned}$$

and

$$\begin{aligned}
\{hyponormal\ operators\} &\subset \{p - hyponormal\ operators\ for\ 0 < p \leq 1\} \\
&\subset \{class\ A\ operators\} \\
&\subset \{class\ A(k)\ for\ k \geq 1\}.
\end{aligned}$$

2. Spectral properties of k-quasi class $A(m)$ operators

Throughout this article we would like to present some known results as propositions which will be used in the sequel. Firstly, we begin with the following definition.

Definition 2.1. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is of m-quasi class $A(k)$ (abbreviate $\mathbb{Q}(A_k, m)$), if

$$T^{*m}(T^*|T|^{2k}T)^{1/(k+1)}T^m \geq T^{m*}|T|^2T^m,$$

where m is a positive integers and $k > 0$. If $m = 1$, then T is called a quasi-class $A(k)$ and if $k = m = 1$, then $\mathbb{Q}(A_k, m)$ coincides with quasi-class A operator.

Lemma 2.1. *[6, Hansen's Inequality] If $A, B \in \mathcal{B}(\mathcal{H})$ satisfying $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\alpha \geq B^*A^\alpha B \qquad \forall \alpha \in (0, 1].$$

Proposition 2.1. *[23, Lemma 2.2]Let $T \in \mathbb{Q}(A_k, m)$ and T^m not have a dense range. Then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ran(T^m)} \oplus \ker(T^{*m}),$$

where $T_1 = T|_{\overline{ran(T^m)}}$ is the restriction of T to $\overline{ran(T^m)}$, and $T_1 \in A(k)$ and T_3 is nilpotent of nilpotency m. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proposition 2.2. *[23, Theorem 2.3] Let $T \in \mathcal{B}(\mathcal{H})$ be a $\mathbb{Q}(A_k, m)$ operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_\mathcal{M}$ of T to $\mathcal{M}$ is also $\mathbb{Q}(A_k, m)$ operator.*

Proposition 2.3. *[23, Theorem 2.4] Let $T \in \mathbb{Q}(A_k, m)$. Then the following assertions holds:*

(a) If $\mathcal{M}$ is an invariant subspace of T and $T|_\mathcal{M}$ is an injective normal operator, then $\mathcal{M}$ reduces T.

(b) If $(T - \lambda)x = 0$, then $(T - \lambda)^*x = 0$ for all $\lambda \neq 0$.

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathcal{H}$ such that $(T - \lambda)x = 0$. If, in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly, $\sigma_p(T) \subseteq \sigma_{jp}(T)$. In general, $\sigma_p(T) \neq \sigma_{jp}(T)$.

In [33], Xia showed that if T is a semi-hyponormal operator, then $\sigma_p(T) = \sigma_{jp}(T)$; Tanahashi extended this result to log-hyponormal operators in [27]. Aluthge [2] showed that if T is w-hyponormal, then nonzero points of $\sigma_p(T)$ and $\sigma_{jp}(T)$ are identical; Uchiyama extended this result to class A operators in [28]. In the following, we will point out that if T is a quasi-$*$-class (A, k) operator for a positive integer k, then nonzero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are also identical and the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal.

Corollary 2.1. *If $T \in \mathbb{Q}(A_k, m)$, then $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.*

Corollary 2.2. *If $T \in \mathbb{Q}(A_k, m)$ and $\alpha, \beta \in \sigma_p(T) \setminus \{0\}$ with $\alpha \neq \beta$. Then* $\ker(T-\alpha) \perp \ker(T-\beta)$.

Proof. Let $x \in \ker(T-\alpha)$ and $y \in \ker(T-\beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha\langle x, y\rangle = \langle \alpha x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\beta}y\rangle = \beta\langle x, y\rangle.$$

Hence $\alpha\langle x, y\rangle = \beta\langle x, y\rangle$ and so $(\alpha-\beta)\langle x, y\rangle = 0$. But $\alpha \neq \beta$, hence $\langle x, y\rangle = 0$. Consequently, $\ker(T-\alpha) \perp \ker(T-\beta)$. □

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{Q}(A_k, m)$ with a dense range, then T is a class $A(k)$ operator for $k > 0$.*

Proof. Since T has a dense range, $\overline{ran(T^m)} = \mathcal{H}$. Then there exists a sequence $\{x_n\} \subset \mathcal{H}$ such that $\lim\limits_{n\to\infty} T^m x_n = y$. Since $T \in \mathbb{Q}(A_k, m)$, we have

$$\begin{aligned}
\langle T^{*m}(T^*|T|^{2k}T)^{\frac{1}{k+1}}T^m x_n, x_n\rangle &\geq \langle T^{*m}|T|^2 T^m x_n, x_n\rangle \\
\langle T^{*m}(T^*|T|^{2k}T)^{\frac{1}{k+1}}T^m x_n, x_n\rangle &\geq \langle T^{*m}|T|^2 T^m x_n, x_n\rangle \\
\langle (T^*|T|^{2k}T)^{\frac{1}{k+1}}T^m x_n, T^m x_n\rangle &\geq \langle |T|^2 T^m x_n, T^m x_n\rangle \ \forall n \in \mathbb{N}
\end{aligned}$$

By the continuity of the inner product, we have

$$\langle ((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T|^2)y, y\rangle \geq 0,$$

for all $y \in \mathcal{H}$. Therefore T is a class $A(k)$ operator for $k > 0$. □

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{Q}(A_k, m)$ and not class $A(k)$, then T is not invertible.*

3. Generalized Fuglede-Putnam Theorem

For $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, we say that the FP-theorem holds for the pair (T, S) if $TX = XS$ implies $T^*X = XS^*$, $\overline{ran(X)}$ reduces T, and $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{ran(X)}}$ and $S|_{\ker(X)^\perp}$ are unitary equivalent normal operators for all $X \in \mathcal{B}(\mathcal{H})$. The following result is very useful in the sequel.

Proposition 3.1. *[26] Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent.*

1. *If $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS^*$,*
2. *If $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$, then $\overline{ran(X)}$ reduces T, $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{ran(X)}}$ and $S|_{\ker(S)^\perp}$ are normal.*

The numerical range of an operator T, denoted by $W(T)$, is the set defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

In general, the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(T)}$ do not imply that T is normal. If $T = SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal. Therefore the following question arises naturally.
Question: Which operator T satisfying the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ is normal?

In 1966, Sheth [24] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for any operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint. We extend the result of Sheth to the class $A(k), k > 0$ operators as follows.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{K})$. If T or T^* belongs to class $A(k)$ for every $k > 0$ and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

To prove Theorem 3.1 we need the following Lemmas.

Lemma 3.1. *[30] If $T \in \mathcal{B}(\mathcal{H})$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let T belongs to the class $A(s,t)$ for some $s > 0$ and $t > 0$, we have*

(a) If $\widetilde{T}_{s,t}$ is normal, then T is normal [29].

(b) If $m_2(\sigma(T)) = 0$, where m_2 means the planer Lebsegue measure, then T is normal [22].

Proof. [Proof of Theorem 3.1] Suppose that T or T^* is a class $A(k), k > 0$ operator. Since $\sigma(T) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Apply Lemma 3.1 to T^* to get $\sigma(T^*) \subseteq \mathbb{R}$. Then $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subseteq \mathbb{R}$. Thus $m_2(\sigma(T)) = m_2(\sigma(T^*))) = 0$ for the planer Lebesgue measure m_2. It follows from Lemma 3.2 that T or T^* is normal. Since $\sigma(T) = \sigma(T^*) \subseteq \mathbb{R}$. Therefore, T is self-adjoint. □

We can extend the result of Theorem 3.1 to the class of $\mathbb{Q}(A_k, m)$ as follows:

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{Q}(A_k, m)$ and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is a direct sum of self-adjoint and nilpotent operator.*

Proof. Since T is m-quasi-class $A(k)$. then by Proposition 2.1, T has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ran(T^m)} \oplus \ker(T^{*m}),$$

where $T_1 = T|_{\overline{ran(T^m)}}$ is the restriction of T to $\overline{ran(T^m)}$, and T_1 is a class $A(k)$ and T_3 is nilpotent of nilpotency m. Since $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, we have $\sigma(T) \subseteq \mathbb{R}$ by Lemma 3.1. Therefore $\sigma(T_1) \subseteq \mathbb{R}$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$ and hence T_1 is self-adjoint by Theorem 3.1 because T_1 belongs to class $A(k)$. Now let Q be the orthogonal projection of $\mathcal{H}$ onto $\overline{ran(T^m)}$. Since $T \in \mathbb{Q}(A_k, m)$ we have

$$\begin{aligned} \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix} &= Q|T|^2Q \leq Q(T^*|T|^{2k}T)^{1/(k+1)}Q \\ &\leq \big(QT^*|T|^{2k}T)Q\big)^{1/(k+1)} \\ &\leq \big(QT^*(QT^*TQ)^kT)Q\big)^{1/(k+1)} = \begin{pmatrix} (T_1^*|T_1|^{2k}T_1)^{\frac{1}{k+1}} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

by Lemma 2.1. Therefore,

$$Q(T^*|T|^{2k}T)^{1/(k+1)}Q = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = Q|T|^2Q.$$

Since S is normal, we can write $(T^*|T|^{2k}T)^{1/(k+1)} = \begin{pmatrix} |T_1|^2 & C \\ C^* & D \end{pmatrix}$. Since

$$\begin{pmatrix} |T_1|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix} = Q(T^*|T|^{2k}T)Q = Q((T^*|T|^{2k}T)^{k+1})^{1/(k+1)}Q,$$

we can easily show that $C = 0$. Therefore,

$$(T^*|T|^{2k}T)^{1/(k+1)} = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & D \end{pmatrix}$$

and hence

$$T^*|T|^{2k}T = \begin{pmatrix} |T_1|^{2(k+1)} & 0 \\ 0 & D^{k+1} \end{pmatrix} = T^*(T^*T)^kT.$$

This implies that $D = (T_3^*|T_3|^{2k}T_3)^{1/(k+1)}$, and by the matrix representation of T we also have

$$T^*T = \begin{pmatrix} T_1T_1^* & T_1^*T_2 \\ T_2^*T_1 + T_3^*T_3 & T_2^*T_2 \end{pmatrix}.$$

Therefore $T_2^*T_2 = 0$ and hence $T_2 = 0$, which completes the proof. $\square$

The following corollary is an extension of the result of Theorem 3.1 to the class of quasi-class $A(k)$ operators.

Corollary 3.1. *If T is a quasi-class $A(k)$ operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

Proof. If T is a quasi-class $A(k)$ operator, T has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{ran(T)} \oplus \ker(T^*),$$

where T_1 is a class $A(k)$ on $\overline{ran(T)}$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since T_1 is self-adjoint and $T_2 = 0$ by Theorem 3.2, T is also self-adjoint. $\square$

In 1976, Stampfli and Wadhwa [25] showed that if $T^* \in \mathcal{B}(\mathcal{H})$ is hyponormal, $S \in \mathcal{B}(\mathcal{H})$ is dominant, $X \in \mathcal{B}(\mathcal{H})$ is injective and has a dense range, and if $XT = SX$, then T and S are normal. on the other hand, in 1981, Gupta and Ramanujan [11] showed that if $T \in \mathcal{B}(\mathcal{H})$ is k-quasihyponormal operator and $S \in \mathcal{B}(\mathcal{H})$ is normal operator for which $TY = YS$ where $Y \in \mathcal{B}(\mathcal{H})$ is injective with dense range, then T is normal operator unitarily equivalent to S. In the following theorem , we extend the result of Gupta and Ramanujan to the *class* $\mathbb{Q}(A_k, m)$ operators. We need the following Lemmas.

Lemma 3.3. [15] Let T, S be normal operators. If there exist injective operators X and Y such that $XT = SX$ and $YS = TY$, then T and S are unitarily equivalent.

Lemma 3.4. *Let $T = U|T|$ be the polar decomposition of T which belong to class $A(p,p)$ for $p > 0$. Then $\widetilde{T}_{p,p} = |T|^p U |T|^p$ is semi-hyponormal and $\widetilde{\widetilde{T}}_{p,p}$ is hyponormal.*

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be class $A(k)$ and $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. If $X \in \mathcal{B}(\mathcal{H})$ has dense range and satisfies $TX = XN$, then T is also a normal operator.*

Proof. Since $TX = XN$ and X has dense range, we have $\overline{X\overline{ran(N)}} = \overline{ran(T)}$. If we denote the restriction of X to $\overline{ran(N)}$ by X_1, then $X_1 : \overline{ran(N)} \to \overline{ran(T)}$ has dense range and for every $x \in \overline{ran(N)}$

$$X_1 N x = XNx = TXx = TX_1 x$$

so that $X_1 N = TX_1$. Since T is of class $A(k)$ then T belongs to class $A(p,p)$, where $p = \max\{1, k\}$. Hence it follows from Lemma 3.4 that $\widetilde{T}_{p,p}$ is semi-hyponormal and hence there is a quasiaffinity Y such that $\widetilde{T}_{p,p} Y = YT$. Thus we have

$$\widetilde{T}_{p,p} Y X_1 = YTX_1 = YX_1 N$$

since YX_1 has dense range, $\widetilde{T}_{p,p}$ is normal, and so T is normal by Lemma 3.2. $\square$

Theorem 3.4. *Let $T^* \in \mathcal{B}(\mathcal{H})$ be of class $A(k)$ for $k > 0$ and let $S \in \mathcal{B}(\mathcal{H})$ be of class $A(k)$ for $k > 0$. If $XT = SX$, where $X : \mathcal{H} \to \mathcal{H}$ is an injective bounded linear operator with dense range, then T is a normal operator unitarily equivalent to S.*

Proof. Since T^* and S are class $A(k)$, then T^* and S are class $A(p,p)$, where $p = \max\{1, k\}$. Now, decompose S and T^* into their normal and pure parts by $S = W \oplus J$ and $T^* = L^* \oplus Q^*$. Let $X_1 = \widetilde{\widetilde{X}} = |\tilde{J}_{p,p}|^{\frac{1}{2}} |\tilde{J}_{p,p}|^{\frac{1}{2}} X |\tilde{Q^*}_{p,p}|^{\frac{1}{2}} |\widetilde{Q^*}_{p,p}|^{\frac{1}{2}}$.

Since $XQ = JX$, $X_1\tilde{\tilde{Q}}_{p,p} = \tilde{\tilde{J}}_{p,p}X_1$, where $\tilde{\tilde{Q}}_{p,p}$, $\tilde{\tilde{J}}_{p,p}$ are hyponormal operators by Lemma 3.4 and X_1 is quasi-affinity. Now by Fuglede-Putnam Theorem for hyponormal operators, $X_1\tilde{\tilde{Q}}_{p,p} = \tilde{\tilde{J}}^*{}_{p,p}X_1$ and $\overline{ran(X_1)}$ reduces $\tilde{\tilde{J}}_{p,p}$ and $(\ker X_1)^\perp$ reduces $\tilde{\tilde{Q}}_{p,p}$ and $\tilde{\tilde{J}}_{p,p}|_{\overline{ran(X_1)}}$ and $\tilde{\tilde{Q}}_{p,p}|_{(\ker X_1)^\perp}$ are unitarily equivalent normal operators. Since X_1 is quasiaffinity, then $\overline{ran(X_1)} = \mathcal{H}$ and $(\ker X_1)^\perp = \{0\}$ and $\tilde{\tilde{Q}}_{p,p}$ and $\tilde{\tilde{J}}_{p,p}$ are unitarily equivalent normal operators. In particular, $\tilde{\tilde{Q}}_{p,p}$ and $\tilde{\tilde{J}}_{p,p}$ are normal operators and by Lemmas 3.3, 3.3, the result follows. □

Theorem 3.5. *If $T^* \in \mathcal{B}(\mathcal{H})$ is of class $A(k)$ for $k > 0$, $S \in \mathcal{B}(\mathcal{H})$ is of class $A(k)$ for $k > 0$ and $XT = SX$ for $X \in \mathcal{B}(\mathcal{H})$ is quasiaffinity, then $XT^* = S^*X$*

Proof. Since by assumption $XT = SX$, we can see that $(\ker(X))^\perp$and $\overline{ran(X)}$ are invariant subspaces of T^* and S , respectively. Then $T^*|_{(\ker X)^\perp}$ is of class $A(k)$ and $S|_{\overline{ran(X)}}$ is also of class $A(k)$. Now consider the decomposition $\mathcal{H} = (\ker X)^\perp \oplus \ker X$ and $\mathcal{H} = \overline{ran(X)} \oplus (\overline{ran(X)})^\perp$. Then we have the following matrix representation:

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where T_1^* is of class $A(k)$, S_1 is of class $A(k)$ and X_1 is injective with dense range. Therefore, we have $X_1T_1x = XTx = SXx = S_1X_1x$ for $x \in (\ker X)^\perp$. That is, $X_1T_1 = S_1X_1$ and T_1 and S_1 are normal by Theorem 3.4. By Fuglede-Putnam theorem we have $X_1T_1^* = S_1^*X_1$. Therefore, $(\ker X)^\perp$ and $(\overline{ran(X)})$ reduces T^* and S, respectively. Hence, we obtain the $XT^* = S^*X$. □

Theorem 3.6. *Let $T \in \mathbb{Q}(A_k, m)$ and let S^* be an operator of class $A(k)$ for $k > 0$. If $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$ is an injective with dense range. Then $XT^* = S^*X$.*

Proof. Let $T_1 = T|_{\overline{ran(T^m)}}$ and $S_1 = S|_{\overline{ran(S^m)}}$. Then we have the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

where T_1 is class $A(k)$, $T_3^m = 0$ and $S_1^* = 0$. Notice that $T^mX = XS^m$ for all positive integer m. Thus $\overline{X(\overline{ran(S^m)})} = \overline{ran(T^m)}$. If we denote the restriction of X to $\overline{ran(S^m)}$ by N then $N : \overline{ran(S^m)} \to \overline{ran(S^m)}$ is an injective and has a dense range. Since $NS_1x = XSx = TXx = T_1Nx$ for all $x \in \overline{ran(S^m)}$, it follows that $NS_1 = T_1N$. On the other hand, since T_1 and S_1^* are belong to class $A(k)$, it follows from Theorem 3.5 that T_1 is a normal operator unitarily equivalent to S_1. Now let E be the orthogonal projection of $\mathcal{H}$ onto $\overline{ran(T^m)}$. Since $T \in \mathbb{Q}(A_k, m)$ and T_1 is a normal operator, from the argument of the proof of Theorem 3.2 we have

$T_2 = 0$ and hence $\overline{ran(T^m)}$ reduces T. Since $X^*(\ker(T^{m^*})) \subseteq \ker(S^{m^*}) = \ker(S^*)$, we have that for each $x \in \ker(T^{m^*})$,

$$X^* T_3^* x = X^* T^* x = S^* X^* x = 0. \tag{3.2}$$

But since X has a dense range, X^* is an injective and hence $T_3^* x = 0$ for every $x \in \ker(T^{k^*})$. Thus $T_3 = 0$, so that $T = T_1 \oplus 0$. Therefore, the proof is achieved. □

Theorem 3.7. *If $T^* \in \mathcal{B}(\mathcal{H})$ is of class $A(k)$ for $k > 0$, $S \in \mathcal{B}(\mathcal{H})$ is injective m-quasi-class $A(k)$, and if $XT = SX$ for $X \in \mathcal{B}(\mathcal{H})$, then $XT^* = S^*X$.*

Proof. Since by assumption $XT = SX$, we can see that $(\ker X)^\perp$ and $\overline{ranX}$ are invariant subspace of T^* and S, respectively. Therefore, by Lemma 2.2 we have that $T^*|_{(\ker X)^\perp}$ is class $A(k)$ and $S|_{\overline{ran(X)}} \in \mathbb{Q}(A_k, m)$. Now consider the decomposition $\mathcal{H} = (\ker X)^\perp \oplus \ker X$. Then we have the matrix representations:

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}, S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}, X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

where T_1^* is of class $A(k)$ and S_1 is injective m-quasi-class $A(k)$ and X_1 is an injective with dense range. Therefore, we have

$$X_1 T_1 x = XTx = SXx = S_1 X_1 x \quad for \quad x \in (\ker X)^\perp. \tag{3.4}$$

that is, $X_1 T_1 = S_1 X_1$ and hence, T_1 and S_1 are normal by Theorem 3.6 and $X_1 T_1^* = S_1^* X_1$ by the Fuglede-Putnam Theorem. Therefore, it follows from Lemma 2.3 that $(\ker X)^\perp$ and $\overline{ran(X)}$ reduces T^* and S , respectively. Hence, we obtain the $XT^* = S^*X$. □

Let $T \in \mathcal{B}(\mathcal{H})$ be compact, and let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if

$$\|T\|_p = \left(\sum_{j=1}^{\infty} (s_j(T))^p \right)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty, 1 \leq p < \infty,$$

where $tr(.)$ denote the trace functional. Hence $C_1(\mathcal{H})$ is the trace class, $C_2(\mathcal{H})$ is the Hilbert-Schmidt class, and C_∞ is the class of compact operator with $\|T\|_\infty = s_1(T)$ denoting the usual norm.

For each pairs of operators A and B in $\mathcal{B}(\mathcal{H})$, an operator τ in $(B_2(\mathcal{H}))$ is defined by

$$\tau X = AXB.$$

Evidently $\|\tau\| \leq \|A\|\|B\|$. And the adjoint of τ is given by the formula $\tau^* X = A^* X B^*$. In particular, if A and B are both positive, then τ is positive and $\tau^{\frac{1}{2}} = A^{\frac{1}{2}} X B^{\frac{1}{2}}$, as one sees from the calculation

$$\begin{aligned} \langle \tau X, X \rangle &= tr(AXBX^*) = tr(A^{\frac{1}{2}} XBX^* A^{\frac{1}{2}}) \\ &= tr\left((A^{\frac{1}{2}} X B^{\frac{1}{2}})(A^{\frac{1}{2}} X B^{\frac{1}{2}})^* \right) \geq 0. \end{aligned}$$

Since $|\tau|^2X = |A|^2X|B^*|^2$ and $|\tau^*|^2X = |A^*|^2X|B|^2$, we have

$$|\tau|^{\frac{1}{2^n}} = |A|^{\frac{1}{2^n}}X|B^*|^{\frac{1}{2^n}}$$

and

$$|\tau^*|^{\frac{1}{2^n}} = |A^*|^{\frac{1}{2^n}}X|B|^{\frac{1}{2^n}}$$

for each integer $n \geq 1$.

Now, we need the following lemma.

Lemma 3.5. *Let A and B be operators in $\mathcal{B}(\mathcal{H})$. If A and B^* are m-quasi-class $A(k)$ for $k > 0$. Then the operator $\tau : C_2(\mathcal{H}) \to C_2(\mathcal{H})$ defined by $\tau X = AXB$ is m-quasi-class $A(k)$ for $k > 0$.*

Proof. For $X \in C_2(\mathcal{H})$, we have

$$\begin{aligned}\tau^{*m}\left(\left(\tau^*|\tau|^{2k}\tau\right)^{\frac{1}{k+1}}\right. &- \left.|\tau|^2\right)\tau^m X \\ &= A^{*m}\left[(A^*|A|^{2k}A)^{\frac{1}{k+1}} - |A|^2\right]A^m X B^m \left(B|B^*|^{2k}B^*\right)^{\frac{1}{k+1}} B^{*m} \\ &+ A^{*m}|A|^2A^m X B^m \left((B|B^*|^{2k}B^*)^{\frac{1}{k+1}} - |B^*|^2\right)B^{*m}\end{aligned}$$

Since A and B^* are m-quasi-class $A(k)$ operators, we have

$$\tau^{*m}\left(\left(\tau^*|\tau|^{2k}\tau\right)^{\frac{1}{k+1}} - |\tau|^2\right)\tau^m \geq 0.$$

□

Theorem 3.8. *Let A be m-quasi-class $A(k)$ operator for $k > 0$ and B^* be an invertible class $A(k)$ operator for $k > 0$. If $AX = XB$ for $X \in C_2(\mathcal{H})$, then $A^*X = XB^*$.*

Proof. Let τ be defined on $C_2(\mathcal{H})$ by $\tau X = AXB^{-1}$. Since B^* is an invertible class $A(k)$ operator, then it follows that B^* is also a class $A(k)$ operator for $k > 0$. Since A is an m-quasi-class $A(k)$ operator and $(B^{-1})^* = (B^*)^{-1}$ is an m-quasi-class $A(k)$ operator, we have that τ is an m-quasi-class $A(k)$ operator on $B_2(\mathcal{H})$ by Lemma 3.5. Moreover, we have $\tau X = AXB^{-1} = X$ because of $AX = XB$. Hence X is an eigenvector of τ. By Proposition 2.3 part (b), we have $\tau^*X = A^*X(B^{-1})^* = X$, that is, $A^*X = XB^*$. So, the proof is achieved. □

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Mohammad H.M.Rashid
Faculty of Science
Department of Mathematics and Statistics
P. O. Box (7)
malik_okasha@yahoo.com