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ON DERIVATIONS SATISFYING CERTAIN IDENTITIES ON RINGS AND ALGEBRAS

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Abstract. The present paper deals with the commutativity of an associative ring R and a unital Banach Algebra A via derivations. Precisely, the study of multiplicative (generalized)-derivations F and G of semiprime (prime) ring R satisfying the identities $G(xy) \pm [F(x), y] \pm [x, y] \in Z(R)$ and $G(xy) \pm [x, F(y)] \pm [x, y] \in Z(R)$ has been carried out. Moreover, we prove that a unital prime Banach algebra A admitting continuous linear generalized derivations F and G is commutative if for any integer $n > 1$ either $G((xy)^n) + [F(x^n), y^n] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [F(x^n), y^n] - [x^n, y^n] \in Z(A)$.
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1. Multiplicative (generalized)-derivations on rings

Throughout this paper $Z(R)$ stands for the center of an associative ring R. Recall that if $aRb = (0)$ (resp. $aRa = (0)$) implies either $a = 0$ or $b = 0$ (resp. $a = 0$) then R is called a prime (resp. semi-prime) ring for all $a, b \in R$. For a positive integer n, a ring R is called n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. The symbol $[x, y]_n = [[x, y]_{n-1}, y]$ represents the nth commutator where $[x, y]_1 = [x, y] = xy - yx$. A mapping $\delta : R \to R$ satisfying $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$ is called a derivation of R. The notion of derivations has been generalized in many ways for instance local derivations, skew derivations, (θ, ϕ)-derivations, Lie derivations, Jordan derivations, multiplicative derivations etc. A set $A_R(S) = \{a \in R : as = sa = 0 \text{ for all } s \in S\}$ is called the annihilator of a non-empty subset S of R. By a left centralizer, we mean an additive mapping $H : R \to R$ such that $H(xy) = H(x)y$ for all $x, y \in R$. A mapping $f : R \to R$ is called centralizing (resp. commuting) on R if $[f(a), a] \in Z(R)$ (resp. $[f(a), a] = 0$) for all $a \in R$. There has been a significant interest in the study of centralizing and commuting mappings in associative rings (for example, see [5], [6] , [18] and references therein).

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Let us turn to the earlier investigation of multiplicative derivation and its generalizations. A map $\delta : R \to R$ is called a multiplicative derivation of R if it satisfies the *Leibniz rule* on R i.e.; $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. Of course these mappings are not necessarily additive. The idea of such mappings was introduced by Daif [8] inspired by the work of Martindale [17]. Further Goldmann and Šemrl [12] provided a complete study of these maps. The following example shows the existence of multiplicative derivation; let $R = C[0, 1]$ be the ring of all continuous real (or complex) valued functions and a map $\delta : R \to R$ defined as:

$$\delta(h)(u) = \left\{ \begin{array}{ll} h(u)log|h(u)| & \text{if } \ h(u) \neq 0 \\ 0 & \text{if } \ h(u) = 0 \end{array} \right\}$$

It is easy to verify that the map δ is not additive but it satisfies the *Leibnitz's rule.* Further, Daif and Tammam-El-Sayiad [10] amplified this notion of multiplicative derivation to multiplicative generalized derivation as; A mapping $D : R \to R$ is said to be a multiplicative generalized derivation if it is uniquely determined by a derivation $\delta : R \to R$ such that $D(ab) = D(a)b + a\delta(b)$ for all $a, b \in R$. Recently, Dhara and Ali [11] made a slight generalization in the definition of multiplicative generalized derivation and hence introduced the notion of multiplicative (generalized)-derivation. Accordingly, a mapping $F : R \to R$ (not necessarily additive) is called multiplicative (generalized)-derivation associated with a map $f : R \to R$ (not necessarily additive nor a derivation) if $F(ab) = F(a)b + af(b)$ for all $a, b \in R$. Very recently, Camci and Aydin [7] proved that if F is a multiplicative (generalized)-derivation of a semiprime ring associated with a map f, then f is a multiplicative derivation. For our convenience, we denote a multiplicative (generalized)-derivation as (F, f) throughout this paper. The multiplicative (generalized)-derivation looks more appropriate than multiplicative generalized derivation as it covers both the concept of multiplicative derivation and multiplicative left multiplier.

During the last two decades, the commutativity of associative rings with derivations have become one of the focus point of several authors and a significant work has been done in this direction (for the references one can see [3], [5], [9], [13], [16], [18], [19], [4] and references therein). In [13], Hongan proved that if d is a derivation of a prime ring R such that $d([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in I$, where I is a nonzero ideal of R, then R is commutative. Further, Qadri et al. [19] extended this result by proving it for generalized derivations of prime rings. In [4], Ashraf et al. explored the commutativity of prime rings that admit generalized derivations satisfying several differential identities on appropriate subsets. Precisely, they proved the following: *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d satisfying any one of the identities: (i) $F(xy)xy \in Z(R)$; (ii) $F(xy) + xy \in Z(R)$; (iii) $F(xy)yx \in Z(R)$; (iv) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, then R is commutative.* Very recently, Tiwari et al. [22] discussed the commutativity of prime rings by studying the following conditions: (i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$; (ii) $G(xy) \pm F(y)F(x) \pm xy \in Z(R)$; (iii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$; (iv) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$; (v) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$ for all $x, y \in I$, where I is a nonzero ideal of R and F, G are the generalized derivation of R.

Clearly, a generalized derivation is a multiplicative (generalized)- derivation but the converse is not true. Thus, it would be a fact of interest to think about the results of generalized derivations for multiplicative (generalized)-derivations. In this direction, the initial results are due to Dhara and Ali [11], where they extended the theorems of Ashraf et al. [4] to the class of multiplicative (generalized)-derivations of semiprime rings. Moreover, Khan [14] studied the following differential identities: (i) $d(x) \circ F(y) \pm (x \circ y) = 0$; (ii) $d(x) \circ F(y) \pm [x, y] = 0$; (iii) $d(x) \circ F(y) = 0$; (iv) $[d(x), F(y)] \pm [x, y] = 0$; (v) $[d(x), F(y)] \pm (x \circ y) = 0$; (vi) $[d(x), F(y)] = 0$ for all x, y in an appropriate subset of a semiprime ring R and (F, d) the multiplicative (generalized)-derivation of R. For a good cross section of this subject, we refer the reader to [1], [15], [7], [20] and references therein. In this paper, our aim is to explore the nature of multiplicative derivations acting on a semiprime rings. More specifically, we investigate the following differential identities:

(i) $G(xy) \pm [F(x), y] \pm [x, y] \in Z(R)$;

(ii) $G(xy) \pm [x, F(y)] \pm [x, y] \in Z(R)$,

where (F, d) and (G, g) are the multiplicative (generalized)-derivations of a semiprime ring R.

1.1. Preliminaries

To achieve our objectives, we make utilization of the following commutator identities: $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$. We also use the following well known results:

Lemma 1.1. [[16] Theorem 2. (ii)] *Let R be a prime ring and I be a nonzero ideal of R. If there exist a derivation d of R such that $x[[d(x), x], x] = 0$ for all $x \in I$, then either $d = 0$ or R is commutative.*

Lemma 1.2. [[6] Theorem 4.] *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in I$, then R is commutative.*

1.2. Main Results

Theorem 1.1. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R such that $G(xy)+[F(x), y]\pm[x, y] \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = 0$ and $z[f(z), z]_2 = 0$ for all $z \in I$.*

Proof. By our hypothesis

$$G(xy) + [F(x), y] \pm [x, y] \in Z(R) \quad \text{for all } x, y \in I. \tag{1.1}$$

On replacing y by yz in (1.1), we get $(G(xy) + [F(x), y] \pm [x, y])z + xyg(z) + y[F(x), z] \pm y[x, z] \in Z(R)$ for any $x, y, z \in I$. On commuting with z and using given hypothesis we obtain

$$[xyg(z), z] + [y[F(x), z], z] \pm [y[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.2}$$

Put zy in the place of y in (1.2) and we find

$$[xzyg(z), z] + z[y[F(x), z], z] \pm z[y[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.3}$$

Left multiply (1.2) by z and subtract from (1.3) to obtain

$$[[x, z]yg(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.4}$$

Replacing x by xt in (1.4) and we get

$$[x[t, z]yg(z), z] + [[x, z]tyg(z), z] = 0 \quad \text{for all } x, y, z, t \in I. \tag{1.5}$$

Put $y = ty$ in (1.4) and subtract from (1.5), we get $0 = [x[t, z]yg(z), z] = x[[t, z]y g(z), z] + [x, z][t, z]yg(z)$ for any $x, y, z, t \in I$. Using (1.4), we obtain

$$[x, z][t, z]yg(z) = 0 \quad \text{for all } x, y, z, t \in I. \tag{1.6}$$

Substituting tk for t in (1.6) in order to get

$$[x, z]t[k, z]yg(z) + [x, z][t, z]kyg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.7}$$

Replace y by ky in (1.6) and subtract from (1.7), we obtain

$$[x, z]t[k, z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.8}$$

Put $x = xg(z)$ in (1.8) and we have

$$x[g(z), z]t[k, z]yg(z) + [x, z]g(z)t[k, z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.9}$$

Replace t by $g(z)t$ in (1.8) and subtract from (1.9) to get

$$x[g(z), z]t[k, z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.10}$$

Putting $kg(z)$ for k in (1.10) and we find

$$x[g(z), z]tk[g(z), z]yg(z) + x[g(z), z]t[k, z]g(z)yg(z) = 0 \text{ for all } x, y, z, t, k \in I. \tag{1.11}$$

Replace y by $g(z)y$ in (1.10) and subtract from (1.11), we have

$$x[g(z), z]tk[g(z), z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.12}$$

Substitute $k = g(z)zk$ in (1.12) and we obtain

$$x[g(z), z]tg(z)zk[g(z), z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.13}$$

Replacing t by $tzg(z)$ in (1.12) to get

$$x[g(z), z]tzg(z)k[g(z), z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.14}$$

Subtract (1.13) and (1.14), we get $x[g(z), z]t[g(z), z]k[g(z), z]yg(z) = 0$ for all $x, y, z, t, k \in I$. It implies that $x[g(z), z]t[g(z), z]k[g(z), z]y[g(z), z] = 0$ for all $x, y, z, t, k \in I$. In particular, $(I[g(z), z])^4 = (0)$ for all $z \in I$. Since R is semiprime ring, so we must have $I[g(z), z] = (0)$ for all $w \in I$. Therefore, semiprimeness of I yields that $[g(z), z] = 0$ for all $z \in I$.

Now, substitute $y = yz$ in (1.2), we get

$$[xyzg(z), z] + [yz[F(x), z], z] \pm [yz[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.15}$$

Right multiply (1.2) by z and subtract from (1.15) and using the fact that $[g(z), z] = 0$, we get

$$[y[[F(x), z], z], z] \pm [y[[x, z], z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.16}$$

Replace x by xz in (1.16) in order to obtain

$$[y[[F(x), z], z], z]z + [y[[xf(z), z], z], z] \pm [y[[x, z], z], z]z = 0, \tag{1.17}$$

for all $x, y, z \in I$. Right multiply (1.16) by z and subtract from (1.17), we get

$$[y[[xf(z), z], z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.18}$$

Replace y by $[xf(z), z]y$ in (1.18) and we find $[xf(z), z][y[[xf(z), z], z], z] + [[xf(z), z], z]y[[xf(z), z], z] = 0$ for any $x, y, z \in I$. Using (1.18), we get $[[xf(z), z], z]y[[x f(z), z], z] = 0$ for all $x, y, z \in I$. That is, $(I[[xf(z), z], z])^2 = 0$ but R is a semiprime ring so we must have $I[[xf(z), z], z] =$ for each $x, z \in I$. Semiprimeness of I implies that $[[xf(z), z], z] = 0$ for all $x, z \in I$. In particular, we obtain $z[f(z), z]_2 = 0$ for all $z \in I$, as desired. □

In Theorem 1.1, substitute $G = -G$ and $g = -g$ we get the following theorem:

Theorem 1.2. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R such that $G(xy) - [F(x), y] \pm [x, y] \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = 0$ and $z[f(z), z]_2 = 0$ for all $z \in I$.*

Corollary 1.1. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm [x, y] \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

Proof. Observe that in Theorem 1.1 and 1.2, if R is prime and f, g are derivations of R, by Lemma 1.1 and Lemma 1.2 the equations $z[[f(z), z], z] = 0$ and $[g(z), z] = 0$ for all $z \in I$ respectively implies that either $f = 0 = g$ or R is commutative. □

In Corollary 1.1, substitute $G \mp I_d$ for G we get the following result:

Corollary 1.2. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm yx \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

In Corollary 1.2, substitute $F \pm I_d$ for F we get the following result:

Corollary 1.3. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm xy \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

Theorem 1.3. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R such that $G(xy)+[x, F(y)]\pm[x, y] \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = -[f(z), z]$ for all $z \in I$.*

Proof. Let us assume that

$$G(xy) + [x, F(y)] \pm [x, y] \in Z(R) \quad \text{for all } x, y \in I. \tag{1.19}$$

Put $y = yz$ in (1.19) and we find $(G(xy)+[x, F(y)]\pm[x, y])z+xyg(z)+F(y)[x, z]+[x, yf(z)] \pm y[x, z] \in Z(R)$ for all $x, y, z \in I$. On commuting with z and using our hypothesis, we obtain

$$[xyg(z), z] + [F(y)[x, z], z] + [[x, yf(z)], z] \pm [y[x, z], z] = 0, \tag{1.20}$$

for all $x, y, z \in I$. Replacing x by xz in (1.20), we get

$$[xzyg(z), z]+[F(y)[x, z], z]z+[[x, yf(z)], z]z+[x[z, yf(z)], z]\pm[y[x, z], z]z=0, \tag{1.21}$$

for all $x, y, z \in I$. Right multiply (1.20) by z and subtract from (1.21), we find $[x[z, yg(z)], z] + [x[z, yf(z)], z] = 0$ where $x, y, z \in I$. That is

$$[x[z, y(g(z) + f(z))], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.22}$$

On substituting ry in the place of y, where $r \in R$ in (1.22), we get

$$[xr[z, y(g(z) + f(z))], z] + [x[z, r]y(g(z) + f(z)), z] = 0, \tag{1.23}$$

for all $x, y, z \in I$, $r \in R$. Replacing x by xr in (1.22) and subtract from (1.23), we get

$$[x[z, r]y(g(z) + f(z)), z] = 0 \quad \text{for all } x, y, z \in I, r \in R. \tag{1.24}$$

Put sx in the place of x, where $s \in R$ in (1.24) in order to find $s[x[z, r]y(g(z) + f(z)), z] + [s, z]x[z, r]y(g(z) + f(z)) = 0$ for all $x, y, z \in I$ and $r, s \in R$. Eq. (1.24) reduces it to

$$[s, z]x[r, z]y(g(z) + f(z)) = 0 \quad \text{for all } x, y, z \in I, r, s \in R. \tag{1.25}$$

Replace y by yz in (1.25), we get

$$[s, z]x[r, z]yz(g(z) + f(z)) = 0 \quad \text{for all } x, y, z \in I, r, s \in R. \tag{1.26}$$

Right multiply (1.25) by z and subtract from (1.26), we get $[s,z]x[r,z]y[(g(z)+f(z)),z]=0$ for each $x,y,z\in I$ and $r,s\in R$. In particular, we have $(I[(g(z)+f(z)),z])^3=(0)$ for all $z\in I$. Since R is semiprime ring, so we must have $I[(g(z)+f(z)),z]=(0)$ for all $z\in I$. Therefore, $[(g(z)+f(z)),z]\in I\cap A_R(I)=(0)$ for any $z\in I$. Hence $[g(z),z]=-[f(z),z]$ for all $z\in I$, as desired. □

In Theorem 1.3, substitute $G=-G$ and $g=-g$ we get the following theorem:

Theorem 1.4. *Let I be a nonzero ideal of a semiprime ring R. If (F,f) and (G,g) are multiplicative (generalized)-derivations of R such that $G(xy)-[x,F(y)]\pm[x,y]\in Z(R)$ holds for all $x,y\in I$, then $[g(z),z]=[f(z),z]$ for all $z\in I$.*

Corollary 1.4. *Let R be a prime ring. If (F,f) and (G,g) are multiplicative generalized derivations of R such that $G(xy)-[x,F(y)]\pm[x,y]\in Z(R)$ holds for all $x,y\in R$ then either $g=f$ or R is commutative.*

Proof. From Theorem 1.4 we have, $[(-g+f)(z),z]=0$ for all $z\in R$. We know that sum of two derivations is a derivation so Posner's second theorem [18] yields that either $g=f$ or R is commutative. □

Corollary 1.5. *Let R be a prime ring with a nonzero ideal I. Suppose that (F,f) and (G,g) are multiplicative generalized derivations of R. If $G(xy)-[x,F(y)]\pm yx\in Z(R)$ holds for all $x,y\in I$ then either $f=g$ or R is commutative.*

Proof. It is easy to check that if G is a multiplicative (generalized)-derivation on R associated with a map g, then $(G\mp I_d)$ is also a multiplicative (generalized)-derivation on R associated with map g. On replacing G by $(G\mp I_d)$ in Theorem 1.4, we obtain that $[(-g+f)(z),z]=0$ for the situation $G(xy)-[F(x),y]\mp yx\in Z(R)$ for all $x,y\in I$. If we assume that F and G are multiplicative generalized derivations associated with non-zero derivations f and g respectively same conclusion i.e.; $(-g+f)$ is commuting on I holds. Hence, Lemma 1.2 implies that either $f=g$ or R is commutative. □

In Corollary 1.5, substitute $F\pm I_d$ for F we get the following results:

Corollary 1.6. *Let I be a nonzero ideal of a prime ring R. If (F,f) and (G,g) are multiplicative generalized derivations of R such that $G(xy)-[x,F(y)]\pm xy\in Z(R)$ holds for all $x,y\in I$, then either $f=g$ or R is commutative.*

Theorem 1.5. *Let I be a nonzero ideal of a semiprime ring R. If (F,f) and (G,g) are multiplicative (generalized)-derivations of R and H is a left centralizer of R such that $G(xy)+[F(x),y]\pm H(xy)\in Z(R)$ holds for all $x,y\in I$, then $[g(z),z]=0$ and $z[f(z),z]_2=0$ for all $z\in I$.*

Proof. By the hypothesis, we have

$$G(xy) + [F(x), y] \pm H(xy) \in Z(R) \quad \text{for all } x, y \in I. \tag{1.27}$$

Taking yz instead of y with $z \in I$ in (1.27), we get $(G(xy) + [F(x), y] \pm H(xy))z + xyg(z) + y[F(x), z] \in Z(R)$ for all $x, y, z \in I$. On commuting with z and using the hypothesis, we get

$$[xyg(z), z] + [y[F(x), z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.28}$$

Replacing y by zy in (1.28), so we have

$$[xzyg(z), z] + z[y[F(x), z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.29}$$

Left multiply (1.28) by z and subtract from (1.29), we obtain

$$[[x, z]yg(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.30}$$

So, same equation with the (1.4) was obtained. Similar proof shows that $[g(z), z] = 0$, for all $z \in I$. If we replace y by yz in (1.28), we get

$$[xyzg(z), z] + [yz[F(x), z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.31}$$

Right multiply (1.28) by z and subtract from (1.31) and using the $[g(z), z] = 0$, we get

$$[y[[F(x), z], z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.32}$$

Replace x by xz and using (1.32), we have

$$[[y[[xf(z), z], z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.33}$$

So, same equation with the (1.18) has obtained. Similar operations applied after this shows that $z[[f(z), z], z] = 0$ for all $z \in I$. □

In Theorem 1.5, substitute $G = -G$ and $g = -g$ we get the following theorem.

Theorem 1.6. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R and H is a left centralizer of R such that $G(xy) - [F(x), y] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = 0$ and $z[f(z), z]_2 = 0$ for all $z \in I$.*

Corollary 1.7. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

By using the similar technique, we obtain the following results. For the sake of brevity, we omit the proofs here.

Theorem 1.7. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations and H is a left centralizer of R such that $G(xy) + [x, F(y)] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = -[f(z), z]$ for all $z \in I$.*

In Theorem 1.7, substitute $G = -G$ and $g = -g$ we get the following theorem.

Theorem 1.8. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations and H is a left centralizer of R such that $G(xy) - [x, F(y)] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = [f(z), z]$ for all $z \in I$.*

Corollary 1.8. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy)-[x, F(y)]\pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then either $f = g$ or R is commutative.*

2. Generalized derivations on Banach algebras

In order to extend the scope of this work, we discuss the commutativity of unital prime Banach algebras with derivations which is directly motivated by the work of Yood [23] and Ali [2]. Since we have already proved that (as in Corollary 1.1 and 1.4) if constraints $G(xy)+[F(x), y]+[x, y] \in Z(R)$ and $G(xy)-[x, F(y)]-[x, y] \in Z(R)$ hold on a prime ring R where F and G are generalized derivations associated with non-zero non-equal derivations f and g respectively, then R is commutative. For an integer $n > 1$, it is natural to consider the constraints: 1. either $G((xy)^n) + [F(x^n), y^n] + [x^n, y^n] \in Z(R)$ or $G((xy)^n) + [y^n, F(x^n)] + [y^n, x^n] \in Z(R)$ and 2. $G((xy)^n)+[x^n, F(y^n)]+[x^n, y^n] \in Z(R)$ or $G((xy)^n)+[x^n, F(y^n)]+[x^n, y^n] \in Z(R)$ on Banach Algebra.

2.1. Preliminaries

Lemma 2.1. [[23]]*Let A is a Banach algebra and M be a closed linear subspace of A. If $p(t) = a_1t + a_2t^2 + ... + a_nt^n$ be a polynomial in real variable t over A such that $p(t) \in M$, then each $a_i \in M$.*

Lemma 2.2. [Open problem 1, [21]] *Let A be a unital prime Banach algebra with non-trivial center $Z(A)$. If $d : A \to A$ be a derivation of A, then $d(e) \in Z(A)$.*

Proof. Let $0 \neq c \in Z(A)$. It is easy to check that $d(c) \in Z(A)$. That means for all $a \in A$, $0 = [d(c), a] = [d(ce), a] = [d(c)e, a] + [cd(e), a] = c[d(e), a]$. Therefore, $cA[d(e), b] = (0)$ for all $b \in A$. Since $c \neq 0$, we get $d(e) \in Z(A)$. □

Lemma 2.3. [Theorem 2, [18]] *A prime ring R admitting a non-zero centralizing derivation is commutative.*

Lemma 2.4. *Let A be a unital prime algebra and $F : A \to A$ be a generalized derivation associated with a derivation f such that $[F(x), x] \in Z(A)$ for all $x \in A$, $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Then A is commutative.*

Proof. By hypothesis, for each $x \in A$, $[F(x), x] \in Z(A)$. Linearizing this relation in order to obtain $[F(x), y] + [F(y), x] \in Z(A)$. Replace x by $xF(e)$ we obtain $([F(x), y] + [F(y), x])F(e) + [x, y]f(F(e)) \in Z(A)$. As $Z(A)$ is a linear subspace of A, we left with $[x, y]f(F(e)) \in Z(A)$. Since $f(F(e)) \neq 0$, we have $[x, y] \in Z(A)$. That means, $0 = [[y, x], z] = [I_y(x), z]$ for all $x, y, z \in A$, where I_y is an inner derivation of A. Hence, Lemma 2.3 completes the proof. □

2.2. Main Results

Theorem 2.1. *Let $F, G : A \to A$ are continuous linear generalized derivations of a unital prime Banach Algebra A associated with non-zero continuous linear derivations $f, g : A \to A$ respectively such that $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Suppose that $G((xy)^n) + [F(x^n), y^n] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [F(x^n), y^n] - [x^n, y^n] \in Z(A)$ for all $x \in P_1$ and $y \in P_2$, where P_1, P_2 are open sets in A and $n = n(x, y) > 1$ is an integer. Then A is commutative.*

Proof. Firstly, we set $\phi_1(x, y, n)=G((xy)^n) + [F(x^n), y^n] + [x^n, y^n]$ and $\phi_2(x, y, n)=G((xy)^n) + [y^n, F(x^n)] + [y^n, x^n]$. By our hypothesis, $\phi_1(x, y, n) \in Z(A)$ and $\phi_2(x, y, n) \in Z(A)$ for all $x \in P_1$ and $y \in P_2$. For an arbitrary fixed element $x \in P_1$, we construct a set $E_n = \{y \in A : \phi_1(x, y, n) \notin Z(A), \phi_2(x, y, n) \notin Z(A)\}$. We claim that E_n is open. For this, we choose a sequence $< s_k >$ in E_n^c that converges to s and prove that $s \in E_n^c$. By our assumption, $s_k \in E_n^c$ i.e. $\phi_1(x, s_k, n) \in Z(A)$ or $\phi_2(x, s_k, n) \in Z(A)$. On making k arbitrarily large, the continuity of G implies that $\phi_1(x, s, n) \in Z(A)$ or $\phi_2(x, s, n) \in Z(A)$. That means, $s \in E_n^c$. Hence, E_n is open. By the Baire Category theorem; if every E_n is dense, then so is their intersection, which contradicts the existence of P_2. Therefore, there must exist a positive integer $m = m(x) > 1$ such that E_m is not dense. Let P_3 be a nonzero open set in E_m^c such that $\phi_1(x, y, m) \in Z(A)$ or $\phi_2(x, y, m) \in Z(A)$ for all $y \in P_3$. Take $q_0 \in P_3$ and $w \in A$ for sufficiently small real t, $q_0 + tw \in P_3$. Therefore, we have

$$\phi_1(x, q_0 + tw, m) \in Z(A) \tag{2.1}$$

or

$$\phi_2(x, q_0 + tw, m) \in Z(A) \tag{2.2}$$

One of these relations must hold for infinitely many real t. If (2.1) holds, the corresponding binomial expansion is a polynomial in t. In the light Lemma 2.1, each coefficient of the polynomial must be in $Z(A)$. On taking the coefficients of t^m, we get $\phi_1(x, w, m) \in Z(A)$. Similarly, if (2.2) holds, $\phi_2(x, w, m) \in Z(A)$. That means, for given $x \in P_1$ there exist an integer $m = m(x) > 1$ such that for each $w \in A$ either $\phi_1(x, w, m) \in Z(A)$ or $\phi_2(x, w, m) \in Z(A)$.

Next, let $y \in A$ be an arbitrary element. Now we want to show that there exists an integer $r > 1$ depending on y such that for each $u \in A$, either $\phi_1(u, y, r) \in Z(A)$ or $\phi_2(u, y, r) \in Z(A)$. Fix $y \in A$ and for each integer $p(y) > 1$, we consider a set $V_p = \{v \in A : \phi_1(v, y, p) \notin Z(A), \phi_2(v, y, p) \notin Z(A)\}$. It is easy to see that V_p is open. The application of the Baire category theorem forces that there exists an integer $r = r(y) > 1$ such that V_r is not dense in A. Let P_4 be a non-empty open subset of V_r^c such that either $\phi_1(x, y, r) \in Z(A)$ or $\phi_2(x, y, r) \in Z(A)$ for all $x \in P_4$. Take $x_0 \in P_4$ and $u \in A$ then $x_0 + tu \in P_4$ for all sufficiently small real t and either $\phi_1(x_0 + tu, y, r) \in Z(A)$ or $\phi_2(x_0 + tu, y, r) \in Z(A)$ for all $u \in A$ and $x_0 \in P_4$. Applying the same argument, we obtain that either $\phi_1(u, y, r) \in Z(A)$ or $\phi_2(u, y, r) \in Z(A)$ for all $u \in A$.

Now, we construct a set $T_j = \{y \in A : \phi_1(w, y, j) \in Z(A)$ or $\phi_2(w, y, j) \in Z(A)$ for all $w \in A\}$. By our above arguments it is clear that $\cup T_j = A$ and each T_j is closed i.e.; each T_j^c is open. Again by the Baire category theorem, if each T_j^c is dense, then their intersection is also dense, which is again a contradiction to the existence of P_2. Thus there must exist an integer $l > 1$ such that T_l contains a non-empty open set P_5 and either $\phi_1(w, y_0, l) \in Z(A)$ or $\phi_1(w, y_0, l) \in Z(A)$ for all $y_0 \in P_5$. If $y_0 \in P_5$ and $z \in A$ then $y_0 + tz \in P_5$ for all sufficiently small real t. Therefore, either $\phi_1(w, y_0 + tz, l) \in Z(A)$ or $\phi_2(w, y_0 + tz, l) \in Z(A)$ for all $w, z \in A$ and $y_0 \in P_5$. By repeating the same argument as earlier, we get either $\phi_1(w, z, l) \in Z(A)$ or $\phi_2(w, z, l) \in Z(A)$ for all $w, z \in A$ and an integer $l > 1$.

As we assumed A a prime Banach algebra with unity and from what that just has been shown, we obtain either $\phi_1(e + tx, y, n) \in Z(A)$ or $\phi_2(e + tx, y, n) \in Z(A)$ for all $x, y \in A$. Explicitly, we have either $G(((e+tx)y)^n) + [F((e+tx)^n), y^n] + [(e+tx)^n, y^n] \in Z(A)$ or $G(((e+tx)y)^n) + [y^n, F((e+tx)^n)] + [y^n, (e+tx)^n] \in Z(A)$ for all $x, y \in A$. The expansions of these expressions are the polynomials in t. Using Lemma 2.1 and taking the coefficients of t, we get either $G(nxy^n) + [F(nx), y^n] + [nx, y^n] \in Z(A)$ or $G(nxy^n) + [y^n, F(nx)] + [y^n, nx] \in Z(A)$ for all $x, y \in A$. Note that $nxy^n = xy^n + \sum_{i=1}^{n-1} y^i xy^{n-i} = xy^n + Q$ where $Q = \sum_{i=1}^{n-1} y^i xy^{n-i}$. Therefore, we have either

$$G(xy^n + Q) + n[F(x), y^n] + n[x, y^n] \in Z(A) \tag{2.3}$$

or

$$G(xy^n + Q) + n[y^n, F(x)] + n[y^n, x] \in Z(A) \tag{2.4}$$

for all $x, y \in A$. Taking $y(e + tx)$ in the place of $(e + tx)y$ and note that $ny^n x = y^n x + Q$, we find either

$$G(y^n x + Q) + n[F(y^n), x] + n[y^n, x] \in Z(A) \tag{2.5}$$

or

$$G(y^n x + Q) + n[x, F(y^n)] + n[x, y^n] \in Z(A) \tag{2.6}$$

for all $x, y \in A$. Thus one of the pair of equations (2.3)-(2.5),(2.3)-(2.6),(2.4)-(2.5) and (2.4)-(2.6) must hold on A. On subtracting these pairs we get either

$$G[x, y^n] + n[(F - i_d)(x), (F + i_d)(y^n)] + 2n[x, y^n] \in Z(A) \tag{2.7}$$

or

$$G[x, y^n] - n[(F - i_d)(x), (F + i_d)(y^n)] - 2n[x, y^n] \in Z(A) \tag{2.8}$$

or

$$G[x, y^n] \pm n[(F - i_d)(x), (F - i_d)(y^n)] \in Z(A) \tag{2.9}$$

holds for all $x, y \in A$ where i_d is the identity map. Firstly, we consider $G[x, y^n] + n[(F - i_d)(x), (F + i_d)(y^n)] + 2n[x, y^n] \in Z(A)$ for all $x, y \in A$. Replacing y by $e + ty$ in this relation. Using Lemma 2.1 and collecting the coefficients of t, we find that $G[x, y] + n[(F - i_d)(x), (F + i_d)(y)] + 2n[x, y] \in Z(A)$ where x, y varies over A. It is easy to check that $F - i_d$ and $F + i_d$ are continuous linear generalized derivations associated with nonzero continuous linear derivations f. Set $F - i_d = H$ and $F + i_d = K$. For each $x, y \in A$, we have $G[x, y] + n[H(x), K(y)] + 2n[x, y] \in Z(A)$. Substitute $yF(e)$ for y in the last expression, we get $(G[x, y] + n[H(x), K(y)] + 2n[x, y])F(e) + [x, y]g(F(e)) + n[H(x), y]f(F(e)) \in Z(A)$ where $x, y \in A$. Since $Z(A)$ is a linear subspace of A, last relation reduces to $[x, y]g(F(e)) + n[H(x), y]f(F(e)) \in Z(A)$ for all $x, y \in A$. In particular, put $x = y$, we have with $n[H(x), x]f(F(e)) \in Z(A)$ where $x, y \in A$. Since $0 \neq f(F(e)) \in Z(A)$, we have $n[H(x), x] \in Z(A)$. That is, for each $x \in A$, $[H(x), x] \in Z(A)$. By Lemma 2.4, A is commutative.

In the same way, we can prove the same conclusion for the equation (2.8) and (2.9). □

Theorem 2.2. *Let $F, G : A \to A$ are continuous linear generalized derivations of a unital prime Banach Algebra A associated with nonzero continuous linear derivations $f, g : A \to A$ respectively such that $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Suppose that $G((xy)^n) + [x^n, F(y^n)] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [x^n, F(y^n)] - [x^n, y^n] \in Z(A)$ for all $x \in P_1$ and $y \in P_2$, where P_1, P_2 are open sets in A and $n = n(x, y) > 1$ is an integer. Then A is commutative.*

Proof. By following the same argument with some necessary variations as in Theorem 2.1, we find either

$$G[x, y^n] + n[(F + i_d)(x), (F + i_d)(y^n)] + 2n[x, y^n] \in Z(A) \tag{2.10}$$

or

$$G[x, y^n] - n[(F + i_d)(x), (F + i_d)(y^n)] - 2n[x, y^n] \in Z(A) \tag{2.11}$$

or

$$G[x, y^n] + n[(F - i_d)(x), (F - i_d)(y^n)] \in Z(A) \tag{2.12}$$

for all $x, y \in A$ and an integer $n > 1$. Again from Theorem 2.1 we can get the desired outcomes. □

Theorem 2.3. *Let $F, G : A \to A$ are continuous linear generalized derivations of a unital prime Banach Algebra A associated with nonzero continuous linear derivations $f, g : A \to A$ respectively such that $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Suppose that $G((xy)^n) + [F(x^n), F(y^n)] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [F(x^n), F(y^n)] - [x^n, y^n] \in Z(A)$ for all $x \in P_1$ and $y \in P_2$, where P_1, P_2 are open sets in A and $n = n(x, y) > 1$ is an integer. Then A is commutative.*

Proof. By following the same argument with some necessary variations as in Theorem 2.1, we find either

$$G[x,y^n] + 2n[F(x),F(y^n)] + 2n[x,y^n] \in Z(A) \tag{2.13}$$

or

$$G[x,y^n] - 2n[F(x),F(y^n)] - 2n[x,y^n] \in Z(A) \tag{2.14}$$

or

$$G[x,y^n] \in Z(A) \tag{2.15}$$

for all $x, y \in A$ and an integer $n > 1$. Let us consider for each $x, y \in A$, $G[x,y^n] \in Z(A)$. This situation is the same as in [Eq. (15), [21]], hence the conclusion follows. For the remaining identities, by applying the same procedure as in Theorem 2.1, we can get the required results. □

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