

ON THE QUASI-CONFORMAL CURVATURE TENSOR OF AN
ALMOST KENMOTSU MANIFOLD WITH NULLITY
DISTRIBUTIONS

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Abstract. The objective of the present paper is to characterize quasi-conformally flat and ξ -quasi-conformally flat almost Kenmotsu manifolds with (k, μ) -nullity and $(k, \mu)'$ -nullity distributions, respectively. Also we characterize almost Kenmotsu manifolds with vanishing extended quasi-conformal curvature tensor and extended ξ -quasi-conformally flat almost Kenmotsu manifolds such that the characteristic vector field ξ belongs to the (k, μ) -nullity distribution.

Keywords: Almost Kenmotsu manifold, Einstein manifold, Weyl conformal curvature tensor, Quasi-conformal curvature tensor, Extended quasi-conformal curvature tensor.

1. Introduction

Let M be a $(2n + 1)$ -dimensional Riemannian manifold with metric g and let $T(M)$ be the Lie algebra of differentiable vector fields in M . The Ricci operator Q of (M, g) is defined by

$$(1.1) \quad g(QX, Y) = S(X, Y),$$

where S denotes the Ricci tensor of type $(0, 2)$ on M and $X, Y \in T(M)$. The Weyl conformal curvature tensor C is defined by

$$(1.2) \quad \begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

for $X, Y, Z \in T(M)$, where R and r denote the Riemannian curvature tensor and scalar curvature of M , respectively.

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For a $(2n + 1)$ -dimensional Riemannian manifold, the quasi-conformal curvature tensor \tilde{C} is given by

$$(1.3) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{r}{2n+1} \left[\frac{a}{2n} + 2b \right] [g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where a and b are two scalars. The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [21]. If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor.

A $(2n + 1)$ -dimensional Riemannian manifold will be called a manifold of the quasi-constant curvature if the Riemannian curvature tensor \tilde{R} of type $(0, 4)$ satisfies the condition

$$(1.4) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &\quad + q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\ &\quad + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)], \end{aligned}$$

where $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, p , q are scalars and there exists a unit vector field ρ satisfying $g(X, \rho) = T(X)$. The notion of the quasi-constant curvature for Riemannian manifolds was introduced by Chen and Yano [4].

At present, the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of k -nullity distribution was introduced by Gray [10] and Tanno [15] in the study of Riemannian manifolds (M, g) , which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$(1.5) \quad N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1, 3)$. Blair, Koufogiorgos and Papantonio [1] introduced the generalized notion of k -nullity distribution, named (k, μ) -nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$(1.6) \quad \begin{aligned} N_p(k, \mu) = \{Z \in T_pM : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \end{aligned}$$

where $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and \mathcal{L} denotes the Lie differentiation.

In [7] Dileo and Pastore introduce the notion of $(k, \mu)'$ -nullity distribution, another generalized notion of k -nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$(1.7) \quad \begin{aligned} N_p(k, \mu)' = \{Z \in T_pM : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \mu[g(Y, Z)h'X - g(X, Z)h'Y]\}, \end{aligned}$$

where $h' = h \circ \phi$.

A differentiable $(2n + 1)$ -dimensional manifold M is said to have a (ϕ, ξ, η) -structure or an almost contact structure, if it admits a $(1, 1)$ tensor field ϕ , a characteristic vector field ξ and a 1-form η satisfying ([2],[3]),

$$(1.8) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where I denotes the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$ hold; both can be derived from (1.8) easily.

If a manifold M with a (ϕ, ξ, η) -structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of $T_p M^{2n+1}$, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \Phi Y)$ for any X, Y of $T_p M^{2n+1}$. The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the $(1, 2)$ -type torsion tensor N_ϕ , defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [2]. Recently in ([7],[8],[9],[13],[14]), an almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also, Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y . It is well known [11] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f , defined by $f = ce^t$ for some positive constant c . Let us denote the distribution orthogonal to ξ by \mathcal{D} and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, \mathcal{D} is an integrable distribution.

At each point $p \in M$, we have

$$T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\}$$

where $\{\xi_p\}$ is 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p . Then the Weyl conformal curvature tensor C is a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M)) \oplus \{\xi\}.$$

Three particular cases can be considered as follows:

- (1) $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \{\xi\}$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero .
- (2) $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M))$, that is, the projection of the image of C in $\{\xi\}$ is zero.
- (3) $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \{\xi\}$, that is, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero, which is equivalent to $\phi^2 C(\phi X, \phi Y)\phi Z = 0$.

Definition 1.1. [22] A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be ξ -conformally flat if the linear operator $C(X, Y)$ is an endomorphism of $\phi(T(M))$, that is, if

$$C(X, Y)\phi(T(M)) \subset \phi(T(M)).$$

Then it follows immediately that

Proposition 1.1. [22] *On a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following conditions are equivalent.*

- (a) M^{2n+1} is ξ -conformally flat,
- (b) $\eta(C(X, Y)Z) = 0$,
- (c) $\phi^2 C(X, Y)Z = -C(X, Y)Z$,
- (d) $C(X, Y)\xi = 0$,

where $X, Y, Z \in T(M)$.

Almost Kenmotsu manifolds have been studied by several authors such as Dileo and Pastore ([7]-[9]), Wang and Liu ([16]-[20]), De and Mandal([5], [6], [12]) and many others. In the present paper we like to study quasi-conformal curvature tensor of almost Kenmotsu manifolds with (k, μ) and $(k, \mu)'$ -nullity distributions, respectively. Also, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended ξ -quasi-conformally flat almost Kenmotsu manifolds with (k, μ) -nullity distribution.

The paper is organized as follows:

In Section 2, we give a brief account on almost Kenmotsu manifolds with ξ belonging to the (k, μ) -nullity distribution and ξ belonging to the $(k, \mu)'$ -nullity distribution. Section 3 deals with quasi-conformally flat and ξ -quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the (k, μ) -nullity distribution. As a consequence of the main result, we obtain several corollaries. Section 4 is devoted to the study of quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the $(k, \mu)'$ -nullity distribution. In the final section, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended ξ -quasi-conformally flat almost Kenmotsu manifolds with (k, μ) -nullity distribution.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1} . The tensor fields l and h are symmetric operators and satisfy the following relations [13]:

$$(2.1) \quad h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0,$$

$$(2.2) \quad \nabla_X \xi = X - \eta(X)\xi - \phi hX (\Rightarrow \nabla_\xi \xi = 0),$$

$$(2.3) \quad \phi l \phi - l = 2(h^2 - \phi^2),$$

$$(2.4) \quad R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,$$

for any vector fields X, Y . The $(1, 1)$ -type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([7], [18])

$$(2.5) \quad h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2).$$

3. Quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k, μ) -nullity distribution

In this section we study quasi-conformally flat and ξ -quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k, μ) -nullity distribution. From (1.6) we obtain

$$(3.1) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

where $k, \mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 3.1. [7] *Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n + 1)$. Suppose that the characteristic vector field ξ belonging to the (k, μ) -nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a wrapped product of an open interval and an almost Kähler manifold.*

In view of Lemma 3.1 it follows from the equation (3.1),

$$(3.2) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(3.3) \quad R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X,$$

$$(3.4) \quad S(X, \xi) = -2n\eta(X),$$

$$(3.5) \quad Q\xi = -2n\xi,$$

for any vector fields X, Y on M^{2n+1} .

Theorem 3.1. *An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (k, μ) -nullity distribution is quasi-conformally flat if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof: Let us first consider the manifold M^{2n+1} which is quasi-conformally flat, that is,

$$(3.6) \quad \tilde{C}(X, Y)Z = 0,$$

for any vector fields X, Y, Z on M^{2n+1} .

From (1.3) we have

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= \frac{b}{a}[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\ &\quad + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\ (3.7) \quad &\quad + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

Putting $Z = \xi$ in the above equation and using (3.2) and (3.4) we get

$$\begin{aligned} \eta(X)g(Y, W) - \eta(Y)g(X, W) &= \frac{b}{a}[-2n\eta(X)g(Y, W) + 2n\eta(Y)g(X, W) \\ &\quad + S(Y, W)\eta(X) - S(X, W)\eta(Y)] \\ (3.8) \quad &\quad + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b][g(X, W)\eta(Y) \\ &\quad - g(Y, W)\eta(X)]. \end{aligned}$$

Putting $Y = \xi$ in the above equation we obtain after simplification

$$(3.9) \quad S(X, W) = \alpha g(X, W) + \beta \eta(X)\eta(W),$$

where $\alpha = \frac{a}{b}[\frac{2bn}{a} + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b] + 1]$ and $\beta = \frac{a}{b}[-\frac{4bn}{a} - \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b] - 1]$.
Therefore, we have $\alpha + \beta = -2n$.

Now using the above relation, (3.9) implies

$$(3.10) \quad r = 2n(\alpha - 1).$$

In [7], Dileo and Pastore proved that in an almost Kenmotsu manifold with ξ belonging to the (k, μ) -nullity distribution the sectional curvature $K(X, \xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k, μ) -nullity distribution the scalar curvature $r = -2n(2n + 1)$. Using this value of r we obtain from (3.10), $\alpha = -2n$. This implies $\beta = 0$.

Hence (3.9) reduces to

$$(3.11) \quad S(X, W) = -2ng(X, W).$$

From (3.7) we obtain

$$\begin{aligned} aR(X, Y)Z &= -b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ (3.12) \quad &\quad + \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Using the value of r and (3.11) in (3.12) yields

$$(3.13) \quad R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y],$$

which implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Conversely, suppose that the manifold is locally isometric to the hyperbolic space

$\mathbb{H}^{2n+1}(-1)$. That is, (3.13) holds.
Contracting X in (3.13) yields

$$(3.14) \quad S(Y, Z) = -2ng(Y, Z).$$

Hence (3.13) and (3.14) together implies $\tilde{C}(X, Y)Z = 0$. That is, the manifold is quasi-conformally flat.
Hence the theorem is proved.

Now, if $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor. Hence we can state the following:

Corollary 3.1. *An almost Kenmotsu manifold with ξ belonging to the (k, μ) -nullity distribution is conformally flat if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

The above corollary has been proved by De and Mandal [5].

Theorem 3.2. *An almost Kenmotsu manifold with ξ belonging to the (k, μ) -nullity distribution is ξ -quasi-conformally flat if and only if the manifold is an Einstein manifold.*

Proof: Let us consider a manifold that is ξ -quasi-conformally flat. That is,

$$\tilde{C}(X, Y)\xi = 0,$$

which implies

$$(3.15) \quad \begin{aligned} aR(X, Y)\xi &= -b[S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY] \\ &+ \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y, \xi)X - g(X, \xi)Y]. \end{aligned}$$

Using (3.2) and (3.4) and $r = -2n(2n + 1)$ we get from the above equation

$$(3.16) \quad \eta(Y)QX - \eta(X)QY = -2n[\eta(Y)X - \eta(X)Y],$$

Putting $Y = \xi$ in the above equation we obtain

$$(3.17) \quad QX = -2nX,$$

which implies $S(X, Y) = -2ng(X, Y)$. That is, the manifold is Einstein.
Conversely, assume that the manifold is Einstein. Then there exists a scalar λ such that

$$(3.18) \quad S(X, Y) = \lambda g(X, Y).$$

In an almost Kenmotsu manifold with (k, μ) -nullity distribution, the scalar curvature $r = -2n(2n + 1)$. This implies $\lambda = -2n$. Now

$$(3.19) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &- \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Using (3.18) we get

$$(3.20) \quad \tilde{C}(X, Y)Z = a[R(X, Y)Z + (g(Y, Z)X - g(X, Z)Y)].$$

Putting $Z = \xi$ in the above equation and using (3.2) we obtain

$$\tilde{C}(X, Y)\xi = 0,$$

which implies that the manifold is ξ -quasi-conformally flat.

If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor.

Thus we are in a position to state the following:

Corollary 3.2. *An almost Kenmotsu manifold with (k, μ) -nullity distribution is ξ -conformally flat if and only if it is Einstein.*

4. Quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k, \mu)'$ -nullity distribution

In this section we study ξ -quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k, \mu)'$ -nullity distribution. Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ . Then from (2.5) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value λ and $-\lambda$ of h' , respectively. Before presenting our main theorem we recall some results:

Lemma 4.1. *(Prop. 4.1 and Prop. 4.3 of [7]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$ -nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and $\text{Spec}(h') = \{0, \lambda, -\lambda\}$, with 0 as a simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvatures are given by the following:*

- (a) $K(X, \xi) = k - 2\lambda$ if $X \in [\lambda]'$ and
 $K(X, \xi) = k + 2\lambda$ if $X \in [-\lambda]'$,
- (b) $K(X, Y) = k - 2\lambda$ if $X, Y \in [\lambda]'$;
 $K(X, Y) = k + 2\lambda$ if $X, Y \in [-\lambda]'$ and
 $K(X, Y) = -(k+2)$ if $X \in [\lambda]'$, $Y \in [-\lambda]'$,
- (c) M^{2n+1} has a constant negative scalar curvature $r = 2n(k-2n)$.

Lemma 4.2. *(Lemma 3 of [16]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$ -nullity distribution. If $h' \neq 0$, then the Ricci operator Q of M^{2n+1} is given by*

$$(4.1) \quad Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'.$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

From (1.7) we have,

$$(4.2) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y],$$

where $k, \mu \in \mathbb{R}$. Also we get from (4.2)

$$(4.3) \quad R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X].$$

Contracting X in (4.2), we have

$$(4.4) \quad S(Y, \xi) = 2nk\eta(Y).$$

Moreover, in an almost Kenmotsu manifold with $(k, \mu)'$ -nullity distribution

$$(4.5) \quad \nabla_X \xi = X - \eta(X)\xi + h'X$$

and

$$(4.6) \quad (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) + g(h'X, Y)$$

holds.

Theorem 4.1. *A $(2n + 1)$ -dimensional $(n > 1)$ quasi-conformally flat almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$ -nullity distribution is either conformally flat or of a quasi-constant curvature.*

Proof: Let us assume that the manifold M^{2n+1} is quasi-conformally flat, that is,

$$(4.7) \quad \tilde{C}(X, Y)Z = 0,$$

for any vector fields X, Y, Z on M^{2n+1} .

From (1.3) we have

$$(4.8) \quad \begin{aligned} a\tilde{R}(X, Y, Z, W) &= b[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\ &\quad + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\ &\quad + \frac{r}{(2n + 1)}\left[\frac{a}{2n} + 2b\right][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

Putting $Z = \xi$ in the above equation and using (4.2) and (4.4) we have

$$(4.9) \quad \begin{aligned} &ak[\eta(Y)g(X, W) - \eta(X)g(Y, W)] + a\mu[\eta(Y)g(h'X, W) - \eta(X)g(h'Y, W)] \\ &= b[2nk\eta(X)g(Y, W) - 2nk\eta(Y)g(X, W) - \eta(Y)S(X, W) + \eta(X)S(Y, W)] \\ &\quad + \frac{r}{(2n + 1)}\left[\frac{a}{2n} + 2b\right][\eta(Y)g(X, W) - \eta(X)g(Y, W)]. \end{aligned}$$

Putting $Y = \xi$ in the above equation and using (4.4) we get after simplifying

$$(4.10) \quad \begin{aligned} S(X, W) &= [-2nk + \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] - \frac{ak}{b}]g(X, W) \\ &+ [4nk - \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] + \frac{ak}{b}]\eta(X)\eta(W) - \frac{a\mu}{b}g(h'X, W). \end{aligned}$$

Let us denote

$$(4.11) \quad A = -2nk + \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] - \frac{ak}{b}$$

and

$$(4.12) \quad B = 4nk - \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] + \frac{ak}{b}.$$

Then, we see that

$$(4.13) \quad A + B = 2nk.$$

Putting $X = W = e_i$ in (4.10), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, i = 1, 2, 3, \dots, (2n+1)$, we get

$$(4.14) \quad r = A(2n+1) + B.$$

From (4.13) and (4.14) we get

$$(4.15) \quad A = \frac{r}{2n} - k.$$

From (4.11) and (4.15), it follows that

$$-2nk + \frac{r}{b(2n+1)}[\frac{a}{2n} + 2b] - \frac{ak}{b} = \frac{r}{2n} - k.$$

The above relation gives

$$(4.16) \quad (a + 2nb - b)(r - 2nk(2n+1)) = 0.$$

Hence, either $a + 2nb - b = 0$ or $r = 2nk(2n+1)$.

Let us suppose that $a + 2nb - b = 0$. Then we see that $b = -\frac{a}{2n-1}$. Hence, from (1.3), it follows that $\tilde{C}(X, Y)Z = aC(X, Y)Z$, where $C(X, Y)Z$ is the Weyl conformal curvature tensor. So, in this case, the quasi-conformally flat manifold is conformally flat.

Now, if $r = 2nk(2n + 1)$, then from (4.10) we obtain

$$(4.17) \quad S(X, W) = 2nkg(X, W) - \frac{a\mu}{b}g(h'X, W).$$

Using (4.17) in (4.8) yields

$$(4.18) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= k[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &\quad - \mu[g(h'X, Z)g(Y, W) - g(h'Y, Z)g(X, W)] \\ &\quad + g(h'Y, W)g(X, Z) - g(h'X, W)g(Y, Z). \end{aligned}$$

From (4.1) and (4.17), it follows that

$$(4.19) \quad g(h'X, W) = l[g(X, W) - \eta(X)\eta(W)],$$

where $l = \frac{2nb(k+1)}{a\mu - 2nb} = -\frac{nb(k+1)}{a+nb}$, by Lemma 4.1.

Using (4.19) in (4.18) we get

$$(4.20) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &\quad + q[g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W)] \\ &\quad + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z), \end{aligned}$$

where $p = k - 4l$ and $q = 2l$.

This completes the proof.

5. Extended quasi-conformal curvature tensor of an almost Kenmotsu manifold with (k, μ) -nullity distribution

In this section we study vanishing extended quasi-conformal curvature tensor and extended ξ -quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to (k, μ) -nullity distribution.

The extended form of quasi-conformal curvature tensor can be written as

$$(5.1) \quad \begin{aligned} \tilde{C}_e(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{r}{2n+1} \left[\frac{a}{2n} + 2b \right] [g(Y, Z)X - g(X, Z)Y] \\ &\quad - \eta(X)\tilde{C}(\xi, Y)Z - \eta(Y)\tilde{C}(X, \xi)Z - \eta(Z)\tilde{C}(X, Y)\xi. \end{aligned}$$

Theorem 5.1. *In an almost Kenmotsu manifold with ξ belonging to (k, μ) -nullity distribution, the extended quasi-conformal curvature tensor vanishes if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof: Putting $Y = Z = \xi$ and supposing that the extended quasi-conformal tensor vanishes, we get from (5.1)

$$(5.2) \quad \begin{aligned} aR(X, \xi)\xi + b[S(\xi, \xi)X - S(X, \xi)\xi + QX - \eta(X)Q\xi] + (a + 4nb)(X - \eta(X)\xi) \\ - \eta(X)\tilde{C}(\xi, \xi)\xi - \tilde{C}(X, \xi)\xi - \tilde{C}(X, \xi)\xi = 0. \end{aligned}$$

Now, using (3.4) and (3.5) the above equation reduces to

$$(5.3) \quad bQX = -2nbX + 2\tilde{C}(X, \xi)\xi.$$

Now, Using (3.2), (3.4) and (3.5) we obtain

$$(5.4) \quad \tilde{C}(X, \xi)\xi = 2nbX + bQX.$$

Putting the value of $\tilde{C}(X, \xi)\xi$ in (5.3) we get

$$(5.5) \quad QX = -2nX,$$

which implies

$$(5.6) \quad S(X, Y) = -2ng(X, Y).$$

This shows that the manifold is Einstein. Since, the extended quasi-conformal curvature tensor vanishes, we have from (5.1)

$$(5.7) \quad \begin{aligned} aR(X, Y)Z &= -b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - (a + 4nb)[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \eta(X)\tilde{C}(\xi, Y)Z + \eta(Y)\tilde{C}(X, \xi)Z + \eta(Z)\tilde{C}(X, Y)\xi. \end{aligned}$$

Now, making use of (3.3), (3.4), (3.5) and (5.5) we obtain

$$\tilde{C}(\xi, Y)Z = 0, \quad \tilde{C}(X, \xi)Z = 0.$$

Again since the manifold is Einstein, we have from Theorem 3.2

$$\tilde{C}(X, Y)\xi = 0.$$

Putting these values in (5.7) and using (5.6) we get

$$(5.8) \quad R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y].$$

This implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

Conversely, suppose that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. That is, (5.8) holds.

Contracting X in (5.8) yields

$$(5.9) \quad S(Y, Z) = -2ng(Y, Z).$$

Now, as shown earlier in this theorem

$$\tilde{C}(\xi, Y)Z = \tilde{C}(X, \xi)Z = \tilde{C}(X, Y)\xi = 0.$$

Then, making use of (5.8), (5.9) and the above values, we obtain from (5.1) that

$$\tilde{C}_e(X, Y)Z = 0.$$

Hence the theorem is proved.

Theorem 5.2. *An almost Kenmotsu manifold with ξ belonging to the (k, μ) -nullity distribution is extended ξ -quasi-conformally flat if and only if the manifold is Einstein.*

Proof: Suppose $\tilde{C}_e(X, Y)\xi = 0$ and putting $Y = \xi$, we get from (5.1)

$$(5.10) \quad aR(X, \xi)\xi + b[S(\xi, \xi)X - S(X, \xi)\xi + QX - \eta(X)Q\xi] + (a + 4nb)(X - \eta(X)\xi) - \eta(X)\tilde{C}(\xi, \xi)\xi - \tilde{C}(X, \xi)\xi - \tilde{C}(X, \xi)\xi = 0.$$

Now, using (3.4) and (3.5) the above equation reduces to

$$(5.11) \quad bQX = -2nbX + 2\tilde{C}(X, \xi)\xi.$$

Now, Using (3.2), (3.4) and (3.5) we obtain

$$(5.12) \quad \tilde{C}(X, \xi)\xi = 2nbX + bQX.$$

Putting the value of $\tilde{C}(X, \xi)\xi$ in (5.11) we get

$$(5.13) \quad QX = -2nX,$$

which implies that the manifold is Einstein.

Conversely, if the manifold is Einstein then obviously $\tilde{C}_e(X, Y)\xi = 0$. Hence the theorem is established.

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