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HILBERT MATRIX AND DIFFERENCE OPERATOR OF ORDER m

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Abstract. In this paper, some applications of the Hilbert matrix in image processing and cryptology are mentioned and an algorithm related to the Hilbert view of a digital image is given. New matrix domains are constructed and some of their properties are investigated. Furthermore, dual spaces of new matrix domains are computed and matrix transformations are characterized. Finally, examples of transformations of new spaces are given.

Keywords. Hilbert matrix; cryptology; image processing; matrix domains.

1. Introduction

1.1. Hilbert Matrix and Applications

We consider the matrices H and H_n as follows:

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots & \\ 1/3 & 1/4 & \cdots & & \\ 1/4 & \cdots & & & \\ \vdots & \vdots & & & \end{bmatrix}$$

$$H_n = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & & \\ 1/3 & 1/4 & \cdots & & & \\ \vdots & \vdots & & & & \\ 1/n & \cdots 1/(2n-1) & & & & \end{bmatrix}$$

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It is well known that these matrices are called the infinite Hilbert matrix and the $n \times n$ Hilbert matrix, respectively. A famous inequality of Hilbert ([8], Section 9) asserts that the matrix H determines a bounded linear operator on the Hilbert space of square summable complex sequences. Also, $n \times n$ Hilbert matrices are well-known examples of extremely ill-conditioned matrices.

Frequently, Hilbert matrices are used in both mathematics and computational science. For example, in image processing, Hilbert matrices are commonly used. Any $2D$ array of natural numbers in the range $[0, n]$ for all $n \in \mathbb{N}$ can be viewed as a greyscale digital image.

We take the Hilbert matrix $H_n(n \times n$ matrix). If we use the Mathematica, then we can write

$$hilbert = HilbertMatrix[5]//MatrixForm$$

and we can obtain

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

Now, we use **MatrixPlot** to obtain the image shown in Fig. 1..1

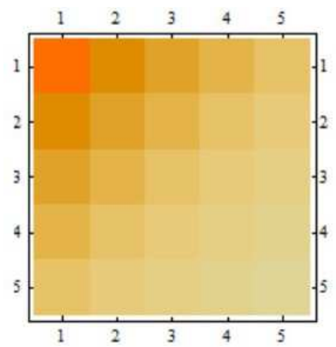

Fig. 1..1: 2D Plot

With the following algorithm that used the Mathematica script[16], we can obtain the Hilbert view of a digital image:

i. Extract an $n \times n$ subimage h from your favorite greyscale digital image.

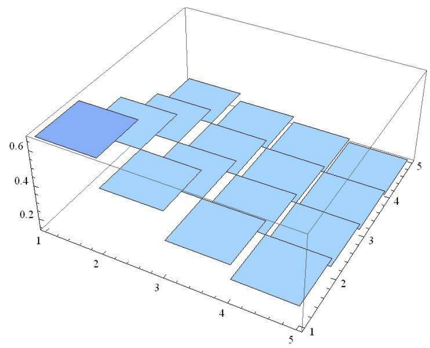

FIG. 1..2: 3D Plot

ii. Multiply the subimage h by the corresponding Hilbert $n \times n$ matrix. Let $hilbertim = h. * HilbertMatrix[n]$.

iii. Produce a $2D$ MatrixPlot for *hilbertim* like the one in Fig. 1..1

iv. Use the following scrip to produce a $3D$ plot for *hilbertim* like the one in Fig. 1..2

$$ListPlot3D[HilbertMatrix[n], \to 0, Mesh \to None].$$

Again, cryptography is an example of Hilbert matrix applications. Cryptography is a science of using mathematics to encrypt and decrypt data. A classical crypt-analysis involves an interesting combination of analytical reasoning, application of mathematical tools and pattern finding. In some studies related to cryptographic methods, the Hilbert matrix is used for authentication and confidentiality[18]. It is well known that the Hilbert matrix is very unstable [15] and so it can be used in security systems.

2. The Hilbert matrix, difference operator and new spaces

Let ω, ℓ_∞, c, c_0, ϕ denote sets of all complex, bounded, convergent, null convergent and finite sequences, respectively. Also, for the sets of convergent, bounded and absolutely convergent series, we denote cs, bs and ℓ_1.

Let $A = (a_{nk})$, $(n, k \in \mathbb{N})$ be an infinite matrix of complex numbers and X, Y be subsets of ω. We write $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)$ for all $n \in \mathbb{N}$. All the series $A_n x$ converge. The set $X_A = \{x \in \omega : Ax \in X\}$ is called the matrix domain of A in X. We write $(X : Y)$ for the space of those matrices which send the whole of sequence space X into the sequence space Y in this sense.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. For the triangle matrices A, B and a sequence x, $A(Bx) = (AB)x$ holds. We remark that a triangle matrix A uniquely has an inverse $A^{-1} = B$ and the matrix B is also a triangle.

Let X be a normed sequence space. If X contains a sequence (b_n) with the property that for every $x \in X$, then there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} \alpha_k b_k \right\| = 0.$$

Thus, (b_n) is called Schauder basis for X.

In [11], new sequence spaces are defined and some topological and structural properties are investigated. A flow chart of the stages of the newly constructed sequence spaces and the algorithms of the workings at each step are given by Kirisci [11].

The difference operator was first used in the sequence spaces by Kızmaz[12]. The idea of difference sequence spaces of Kızmaz was generalized by Çolak and Et[6, 7]. The difference matrix $\Delta = \delta_{nk}$ defined by

$$\delta_{nk} := \begin{cases} (-1)^{n-k} & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (0 < n-1 \text{ or } n > k) \end{cases}$$

The difference operator order of m is defined by $\Delta^m : \omega \to \omega$, $(\Delta^1 x)_k = (x_k - x_{k-1})$ and $\Delta^m x = (\Delta^1 x)_k \circ (\Delta^{m-1} x)_k$ for $m \geq 2$.

The triangle matrix $\Delta^{(m)} = \delta_{nk}^{(m)}$ defined by

$$\delta_{nk} := \begin{cases} (-1)^{n-k}\binom{m}{n-k} & , \quad (\max\{0, n-m\} \leq k \leq n) \\ 0 & , \quad (0 \leq k < \max\{0, n-m\} \text{ or } n > k) \end{cases}$$

for all $k, n \in \mathbb{N}$ and for any fixed $m \in \mathbb{N}$.

We can also mention Fibonacci matrices as an example of difference matrices[9].

The Hilbert matrix is defined by $H_n = [h_{ij}] = [\frac{1}{i+j-1}]_{i,j=1}^{n}$ for each $n \in \mathbb{N}$. The inverse of Hilbert matrix is defined by

$$H_n^{-1} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-1}{i-1}^2 \tag{2.1}$$

for all $k, i, j, n \in \mathbb{N}$[5]. Polat [17], has defined new spaces by using the Hilbert matrix. Let h_c, h_0, h_∞ be convergent Hilbert, null convergent Hilbert and bounded Hilbert spaces, respectively. Then, we have:

$$X = \{x \in \omega : Hx \in Y\}$$

where $X = \{h_c, h_0, h_\infty\}$ and $Y = \{c, c_0, \ell_\infty\}$.

Now, we will give new difference Hilbert sequence spaces as below:

$$\begin{aligned} h_c\left(\Delta^{(m)}\right) &= \left\{x \in \omega : \Delta^{(m)}x \in h_c\right\} \\ h_0\left(\Delta^{(m)}\right) &= \left\{x \in \omega : \Delta^{(m)}x \in h_0\right\} \\ h_\infty\left(\Delta^{(m)}\right) &= \left\{x \in \omega : \Delta^{(m)}x \in h_\infty\right\}. \end{aligned}$$

These new spaces, as the set of all sequences whose $\Delta^{(m)}-$transforms are in the Hilbert sequences spaces which are defined by Polat[17].

We also define the $H\Delta^{(m)}-$transform of a sequence, as below:

$$y_n = \left(H\Delta^{(m)}x\right)_n = \sum_{k=1}^{n}\left[\sum_{i=k}^{n}\frac{1}{n+i-1}(-1)^{i-k}\binom{m}{i-k}\right]x_k \tag{2.2}$$

for each $m, n \in \mathbb{N}$. Here and after by $H^{(m)}$, we denote the matrix $H^{(m)} = H\Delta^{(m)} = [h_{nk}]$ defined by

$$h_{nk} = \sum_{i=k}^{n}\frac{1}{n+i-1}(-1)^{i-k}\binom{m}{i-k}$$

for each $k, m, n \in \mathbb{N}$.

Theorem 2.1. *The Hilbert sequences spaces derived by the difference operator of m are isomorphic copies of the convergent, null convergent and bounded sequence spaces.*

Proof. We will only prove that the null convergent Hilbert sequence space is an isomorphic copy of the null convergent sequence space. To prove the fact $h_0(\Delta^{(m)}) \cong c_0$, we should show the existence of a linear bijection between the spaces $h_0(\Delta^{(m)})$ and c_0. Consider the transformation T, defined with the notation (2.2) from $h_0(\Delta^{(m)})$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective. Let $y \in c_0$ and define the sequence $x = (x_n)$ by

$$x_n = \sum_{k=1}^{n}\left[\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h_{ik}^{-1}\right]y_k \tag{2.3}$$

where h_{ik}^{-1} is defined by (2.1). Then,

$$\begin{aligned}\lim_{n\to\infty}\left(H\Delta^{(m)}x\right)_k &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\Delta^{(m)}x_k\\ &= \sum_{k=1}^{n}\frac{1}{n+k-1}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}\\ &= \sum_{k=1}^{n}\left[\sum_{i=k}^{n}\frac{1}{n+i-1}(-1)^{i-k}\binom{m}{i-k}\right]x_k=\lim_{n\to\infty}y_n=0.\end{aligned}$$

Thus, we have that $x\in h_0(\Delta^{(m)})$. Consequently, T is surjective and is norm preserving. Hence, T is linear bijection which implies that the null convergent Hilbert sequence space is an isomorphic copy of the null convergent sequence space. □

It is well known that the spaces c, c_0 and ℓ_∞ are $BK-$spaces. Considering the fact that $\Delta^{(m)}$ is a triangle, we can say that the Hilbert sequences spaces derived by the difference operator of m are $BK-$spaces with the norm

$$\|x\|_\Delta=\|H\Delta^{(m)}x\|_\infty=\sup_n\left|\sum_{k=1}^{n}\frac{1}{n+k-1}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}\right|. \tag{2.4}$$

In the theory of matrix domain, it is well known that the matrix domain X_A of a normed sequence space X has a basis if and only if X has a basis whenever $A=(a_{nk})$ is a triangle. Then, we have:

Corollary 2.2. *Define the sequence $b^{(k)}=(b_n^{(k)}(\Delta^{(m)}))_{n\in\mathbb{N}}$ by*

$$b_n^{(k)}(\Delta^{(m)}):=\begin{cases}\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h_{ik}^{-1} & , \quad (n\geq k)\\ 0 & , \quad (n<k)\end{cases}$$

for every fixed $k\in\mathbb{N}$. The following statements hold:

i. The sequence $b^{(k)}(\Delta^{(m)})=(b_n^{(k)}(\Delta^{(m)}))_{n\in\mathbb{N}}$ is a basis for the null convergent Hilbert sequence space, and for any $x\in h_0(\Delta^{(m)})$ has a unique representation of the form

$$x=\sum_k\left(H\Delta^{(m)}x\right)_k b^{(k)}.$$

ii. The set $\{t,b^{(1)},b^{(2)},\cdots\}$ is a basis for the convergent Hilbert sequence space, and for any $x\in h_c(\Delta^{(m)})$ has a unique representation of form

$$x=st+\sum_k\left[\left(H\Delta^{(m)}x\right)_k-s\right]b^{(k)}$$

where $t=t_n(\Delta^{(m)})=\sum_{k=1}^{n}\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h_{ik}^{-1}$ for all $k\in\mathbb{N}$ and $s=\lim_{k\to\infty}(H\Delta^{(m)}x)_k$.

If we take into consideration the fact that a space which has a Schauder basis is separable, then we can give the following corollary:

Corollary 2.3. *The convergent Hilbert and null convergent Hilbert sequence spaces are separable.*

3. Dual Spaces and Matrix Transformations

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space* of X and Y. In the special cases of $Y = \{\ell_1, cs, bs\}$, we write $x^{\alpha} = x^{-1} * \ell_1$, $x^{\beta} = x^{-1} * cs$, $x^{\gamma} = x^{-1} * bs$ and $X^{\alpha} = M(X, \ell_1)$, $X^{\beta} = M(X, cs)$, $X^{\gamma} = M(X, bs)$ for the $\alpha-$dual, $\beta-$dual, $\gamma-$dual of X. By $A_n = (a_{nk})_{k=0}^{\infty}$ we denote the sequence in the $n-$th row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ $n = (0, 1, ...)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n.

Lemma 3.1. *[4, Lemma 5.3] Let X, Y be any two sequence spaces. $A \in (X : Y_T)$ if and only if $TA \in (X : Y)$, where A an infinite matrix and T a triangle matrix.*

Lemma 3.2. *[1, Theorem 3.1] Let $U = (u_{nk})$, be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$. Let $B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and inverse of the triangle matrix $U = (u_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} a_j v_{jk}$$

for all $k, n \in \mathbb{N}$. Then,

$$X_U^{\beta} = \{a = (a_k) \in \omega : B^U \in (X : c)\}.$$

and

$$X_U^{\gamma} = \{a = (a_k) \in \omega : B^U \in (X : \ell_{\infty})\}.$$

Now, we list the following useful conditions.

Table 3.1:

To → From ↓	ℓ_∞	c	bs	cs
c_0	**1.**	**2.**	**3.**	**4.**
c	**1.**	**5.**	**3.**	**6.**
ℓ_∞	**1.**	**7.**	**3.**	**8.**

$$\sup_n \sum_k |a_{nk}| < \infty \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0 \tag{3.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} \quad \textit{exists} \tag{3.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left|\lim_{n\to\infty} a_{nk}\right| \tag{3.4}$$

$$\lim_n a_{nk} = 0 \quad \text{for all k} \tag{3.5}$$

$$\sup_m \sum_k \left|\sum_{n=0}^{m}\right| < \infty \tag{3.6}$$

$$\sum_n a_{nk} \quad \textit{convergent for all } k \tag{3.7}$$

$$\sum_n \sum_k a_{nk} \quad \textit{convergent} \tag{3.8}$$

$$\lim_n a_{nk} \quad \text{exists for all k} \tag{3.9}$$

$$\lim_m \sum_k \left|\sum_{n=m}^{\infty} a_{nk}\right| = 0 \tag{3.10}$$

Lemma 3.3. *For the characterization of the class $(X : Y)$ with $X = \{c_0, c, \ell_\infty\}$ and $Y = \{\ell_\infty, c, cs, bs\}$, we can give the necessary and sufficient conditions from Table 3.1, where*

1. *(3.1)*	***2.*** *(3.1), (3.9)*	***3.*** *(3.6)*	***4.*** *(3.6), (3.7)*
5. *(3.1), (3.9), (3.3)*	***6.*** *(3.6), (3.7), (3.8)*	***7.*** *(3.9), (3.4)*	***8.*** *(3.10)*

Let h_{nk}^{-1} is defined by (2.1). For the proof of Theorem 3.4, we define the matrix $V = (v_{nk})$ as below:

$$v_{nk} = \left[\sum_{i=k}^{n} \binom{m+n-i-1}{n-i} h_{ik}^{-1} a_n\right] \tag{3.11}$$

Theorem 3.4. *The $\beta-$ and $\gamma-$ duals of the Hilbert sequence spaces derived by the difference operator of m defined by*

$$\left[h_{c_0}\left(\Delta^{(m)}\right)\right]^{\beta} = \{a = (a_k) \in \omega : V \in (c_0 : c)\}$$
$$\left[h_{c}\left(\Delta^{(m)}\right)\right]^{\beta} = \{a = (a_k) \in \omega : V \in (c : c)\}$$
$$\left[h_{\infty}\left(\Delta^{(m)}\right)\right]^{\beta} = \{a = (a_k) \in \omega : V \in (\ell_\infty : c)\}$$
$$\left[h_{c_0}\left(\Delta^{(m)}\right)\right]^{\gamma} = \{a = (a_k) \in \omega : V \in (c_0 : \ell_\infty)\}$$
$$\left[h_{c}\left(\Delta^{(m)}\right)\right]^{\gamma} = \{a = (a_k) \in \omega : V \in (c : \ell_\infty)\}$$
$$\left[h_{\infty}\left(\Delta^{(m)}\right)\right]^{\gamma} = \{a = (a_k) \in \omega : V \in (\ell_\infty : \ell_\infty)\}$$

Proof. We will only show the $\beta-$ and $\gamma-$ duals of the null convergent Hilbert sequence spaces derived by difference operator of m. Let $a = (a_k) \in \omega$. We begin the equality

$$\begin{aligned} \sum_{k=1}^{n} a_k x_k &= \sum_{k=1}^{n}\sum_{i=1}^{k}\left[\sum_{j=i}^{k}\binom{m+k-j-1}{k-j} h_{ij}^{-1}\right] a_k y_i \qquad (3.12)\\ &= \sum_{k=1}^{n}\left\{\sum_{i=1}^{k}\left[\sum_{j=i}^{k}\binom{m+k-j-1}{k-j} h_{ij}^{-1}\right] a_i\right\} y_k \\ &= (Vy)_n \end{aligned}$$

where $V = (v_{nk})$ is defined by (3.11). Using (3.12), we can see that $ax = (a_k x_k) \in cs$ or bs whenever $x = (x_k) \in h_{c_0}(\Delta^{(m)})$ if and only if $Vy \in c$ or ℓ_∞ whenever $y = (y_k) \in c_0$. Then, from Lemma 3.1 and Lemma 3.2, we obtain the result that $a = (a_k) \in \left(h_{c_0}(\Delta^{(m)})\right)^{\beta}$ or $a = (a_k) \in \left(h_{c_0}(\Delta^{(m)})\right)^{\gamma}$ if and only if $V \in (c_0 : c)$ or $V \in (c_0 : \ell_\infty)$, which is what we wished to prove. □

Therefore, the $\beta-$ and $\gamma-$ duals of new spaces will help us in the characterization of the matrix transformations.

Let X and Y be arbitrary subsets of ω. We shall show that the characterizations of the classes (X, Y_T) and (X_T, Y) can be reduced to that of (X, Y), where T is a triangle.

It is well known that if $h_{c_0}(\Delta^{(m)}) \cong c_0$, then the equivalence

$$x \in h_{c_0}(\Delta^{(m)}) \Leftrightarrow y \in c_0$$

holds. Then, the following theorems will be proved and given some corollaries which can be obtained in a way similar to that of Theorems 3.5 and 3.6. Then, using Lemmas 3.1 and 3.2, we have:

Theorem 3.5. *Consider the infinite matrices $A = (a_{nk})$ and $D = (d_{nk})$. These matrices get associated with each other the following relations:*

$$d_{nk} = \sum_{j=k}^{\infty} \binom{m+n-j-1}{n-j} h_{jk}^{-1} a_{nj} \tag{3.13}$$

for all $k, m, n \in \mathbb{N}$. Then, the following statements are true:
i. *$A \in (h_{c_0}(\Delta^{(m)}) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [h_{c_0}(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and $D \in (c_0 : Y)$, where Y be any sequence space.*
ii. *$A \in (h_c(\Delta^{(m)}) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [h_c(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and $D \in (c : Y)$, where Y be any sequence space.*
iii. *$A \in (h_\infty(\Delta^{(m)}) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [h_\infty(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and $D \in (\ell_\infty : Y)$, where Y be any sequence space.*

Proof. We assume that the (3.13) holds between the entries of $A = (a_{nk})$ and $D = (d_{nk})$. Let us remember that from Theorem 2.1, the spaces $h_{c_0}(\Delta^{(m)})$ and c_0 are linearly isomorphic. Firstly, we choose any $y = (y_k) \in c_0$ and consider $A \in (h_{c_0}(\Delta^{(m)}) : Y)$. Then, we obtain that $DH\Delta^{(m)}$ exists and $\{a_{nk}\} \in \left(h_{c_0}\Delta^{(m)}\right)^\beta$ for all $k \in \mathbb{N}$. Therefore, the necessity of (3.13) yields and $\{d_{nk}\} \in c_0^\beta$ for all $k, n \in \mathbb{N}$. Hence, Dy exists for each $y \in c_0$. Thus, if we take $m \to \infty$ in the equality

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m} \left[\sum_{i=1}^{k} \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} h_{ij}^{-1} \right] a_{nk} = \sum_{k} d_{nk}y_k$$

for all $m, n \in \mathbb{N}$, then, we understand that $Dy = Ax$. So, we obtain that $D \in (c_0 : Y)$.

Now, we consider that $\{a_{nk}\}_{k\in\mathbb{N}} \in (h_{c_0}\Delta^{(m)})^\beta$ for all $n \in \mathbb{N}$ and $D \in (c_0 : Y)$. We take any $x = (x_k) \in h_{c_0}\Delta^{(m)}$. Then, we can see that Ax exists. Therefore, for $m \to \infty$, from the equality

$$\sum_{k=1}^{m} d_{nk}y_k = \sum_{k=1}^{m} a_{nk}x_k$$

for all $n \in \mathbb{N}$, we obtain that $Ax = Dy$. Therefore, this shows that $A \in (h_{c_0}(\Delta^{(m)} : Y)$. $\square$

Theorem 3.6. *Consider that the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ with*

$$e_{nk} := \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{n+j-1} (-1)^{j-k} \binom{m}{j-k} a_{jk}. \tag{3.14}$$

Then, the following statements are true:
i. *$A = (a_{nk}) \in (X : h_{c_0}(\Delta^{(m)})$ if and only if $E \in (X : c_0)$*

ii. *$A = (a_{nk}) \in (X : h_c(\Delta^{(m)})$ if and only if $E \in (X : c)$*
iii. *$A = (a_{nk}) \in (X : h_\infty(\Delta^{(m)})$ if and only if $E \in (X : \ell_\infty)$*

Proof. We take any $z = (z_k) \in X$. Using the (3.14), we have

$$\sum_{k=1}^{m} e_{nk} z_k = \sum_{k=1}^{m} \left[\sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{n+j-1} (-1)^{j-k} \binom{m}{j-k} a_{jk} \right] z_k \tag{3.15}$$

for all $m, n \in \mathbb{N}$. Then, for $m \to \infty$, equation (3.15) gives us that $(Ez)_n = \{H\Delta^{(m)}(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in h_{c_0}(\Delta^{(m)}$ whenever $z \in X$ if and only if $Ez \in c_0$ whenever $z \in X$. Thus, the proof is completed. □

4. Examples

If we choose any sequence spaces X and Y in Theorem 3.5 and 3.6 in the previous section, then we can find several consequences in every choice. For example, if we take the space ℓ_∞ and the spaces which are isomorphic to ℓ_∞ instead of Y in Theorem 3.5, we obtain the following examples:

Example 4.1. *The Euler sequence space e^r_∞ is defined by ([3] and [2])*

$$e^r_\infty = \{x \in \omega : \sup_{n \in \mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k| < \infty\}.$$

We consider the infinite matrix $A = (a_{nk})$ and define the matrix $C = (c_{nk})$ by

$$c_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \qquad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(h_{c_0}(\Delta^{(m)}) : e^r_\infty)$ in Theorem 3.5, then, we replace the entries of the matrix A by those of the matrix C.

Example 4.2. *Let $T_n = \sum_{k=0}^{n} t_k$ and $A = (a_{nk})$ be an infinite matrix. We define the matrix $G = (g_{nk})$ by*

$$g_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk} \qquad (k, n \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A belongs to the class $(h_{c_0}(\Delta^{(m)}) : r^t_\infty)$ are obtained from in Theorem 3.5 by replacing the entries of the matrix A by those of the matrix G; where r^t_∞ is the space of all sequences whose R^t-transforms is in the space ℓ_∞ [14].

Example 4.3. *In the space r_∞^t, if we take $t = e$, then this space becomes a Cesaro sequence space of non-absolute type X_∞ [13]. As a special case, example 4.2 includes the characterization of class $(h_{c_0}(\Delta^{(m)}) : r_\infty^t)$.*

Example 4.4. *The Taylor sequence space t_∞^r is defined by ([10])*

$$t_\infty^r = \{x \in \omega : \sup_{n\in\mathbb{N}} |\sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k| < \infty\}.$$

We consider the infinite matrix $A = (a_{nk})$ and define the matrix $P = (p_{nk})$ by

$$p_{nk} = \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} a_{jk} \qquad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(h_{c_0}(\Delta^{(m)}) : t_\infty^r)$ in Theorem 3.5, then, we replace the entries of the matrix A by those of the matrix P.

If we take the spaces c, cs and bs instead of X in Theorem 3.6, or Y in Theorem 3.5 we can write the following examples. Firstly, we give some conditions and the following lemmas:

$$\lim_k a_{nk} = 0 \quad \text{for all n}, \tag{4.1}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0, \tag{4.2}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0, \tag{4.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = 0, \tag{4.4}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty \tag{4.5}$$

$$\lim_k (a_{nk} - a_{n,k+1}) \text{ exists for all k} \tag{4.6}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = \sum_k \left|\lim_{n\to\infty} (a_{nk} - a_{n,k+1})\right| \tag{4.7}$$

$$\sup_n \left|\lim_k a_{nk}\right| < \infty \tag{4.8}$$

Lemma 4.5. *Consider that the $X \in \{\ell_\infty, c, bs, cs\}$ and $Y \in \{c_0\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be read as the following from Table 4.1:*

9. *(4.3)*	***10.*** *(3.1), (3.5), (4.2)*	***11.*** *(4.1), (4.4)*	***12.*** *(3.5), (4.5)*
13. *(4.1), (4.6), (4.7)*	***14.*** *(4.5), (3.9)*	***15.*** *(4.1), (4.5)*	***16.*** *(4.5), (4.8)*

Table 4.1:

From →	ℓ_∞	c	bs	cs
To ↓				
c_0	**9.**	**10.**	**11.**	**12.**
c	**7.**	**5.**	**13.**	**14.**
ℓ_∞	**1.**	**1.**	**15.**	**16.**

5. Conclusion

In 1894, Hilbert introduced the Hilbert matrix. Hilbert matrices are notable examples of poorly conditioned (ill-conditioned) matrices, making them notoriously difficult to use in numerical computation. In other words, Hilbert matrices whose entries are specified as machine-precision numbers are difficult to invert using numerical techniques. That is why we offered some examples related to the usage of the Hilbert matrix such as image processing and cryptography. We also provided an algorithm. Further, we constructed new spaces with the Hilbert matrix and difference operator of order m. We calculated dual spaces of new spaces and characterized some matrix classes. In the last section, we gave some examples of matrix classes. Images of new spaces can be plotted using the Mathematica as a continuation of this study. Again, different applications of cryptography can be investigated.

Conflict of Interests

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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