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SOME RESULTS ON $(k, \mu)'$-ALMOST KENMOTSU MANIFOLDS *

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Abstract. In this paper, we study the quasi-conformal curvature tensor $\tilde{C}$ and projective curvature tensor P on a $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1} of dimension greater than 3. We obtain that if M^{2n+1} is non-Kenmotsu and satisfies $R \cdot \tilde{C} = 0$ or $P \cdot P = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Keywords: Almost Kenmotsu manifold, $(k, \mu)'$-nullity condition, quasi-conformal curvature tensor, projective curvature tensor.

1. Introduction

In 1972, K. Kenmotsu introduced a new class of almost contact metric manifolds, nowadays known as Kenmotsu manifolds [8]. The concept of almost Kenmotsu manifolds, regarded as a generalization of Kenmotsu manifolds, was studied by Janssens and Vanhecke (see [4]). In 2007, Pitiş [7] published a book containing many systematic studies related to Kenmotsu manifolds. Some geometric properties and fundamental formulas of almost Kenmotsu manifolds were obtained by Kim and Pak [11] and Pastore et al. [5, 6]. Several authors studied almost Kenmotsu manifolds considering some curvature conditions (see [12, 13, 14]). Recently, some curvature properties of some types of almost Kenmotsu manifolds were obtained by Wang and Liu in [15, 16, 17, 18].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat (see [2]). For $n \geq 1$, M is locally projectively flat if and

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only if the projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)U = R(X,Y)U - \frac{1}{2n}[S(Y,U)X - S(X,U)Y] \tag{1.1}$$

for any vector fields $X, Y, U \in \mathfrak{X}(M)$, where S is the Ricci tensor of M.

The Weyl conformal curvature tensor C on a $(2n+1)$-dimensional manifold M is defined by [20]

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y] \\ & - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \end{aligned} \tag{1.2}$$

for any vector fields X, Y, Z on M, where S, Q and r denote the Ricci curvature tensor, the Ricci operator with respect to the metric g and the scalar curvature, respectively. Note that the Weyl conformal curvature tensor on any three dimension Riemannian manifold vanishes.

For a $(2n+1)$-dimensional manifold M, the quasi-conformal curvature tensor $\tilde{C}$ is defined by [21]

$$\begin{aligned} \tilde{C}(X,Y)Z =& aR(X,Y)Z - \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y] \\ & + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.3}$$

where a and b are two constants. If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to the Weyl conformal curvature tensor.

In this paper, we aim to extend some known results regarding the projective and quasi-conformal curvature tensor on Kenmotsu manifolds (see [1, 2, 9, 10]) to a class of almost Kenmotsu manifolds. In Section 2, we recall some basic formulas and properties of almost Kenmotsu manifolds and the notion of $(k,\mu)'$-almost Kenmotsu manifolds. In Section 3, we introduce some properties of such manifolds used to prove our main results. In Section 4 and 5, we classify almost Kenmotsu manifolds satisfying $R \cdot \tilde{C} = 0$ and $P \cdot P = 0$, respectively.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost contact metric manifold of dimension $2n+1$, equipped with an almost contact metric structure (ϕ, ξ, η, g) (see [3]) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$, where ϕ, ξ, η, g and $\mathfrak{X}(M)$ denote a $(1,1)$-tensor field, a vector field, a 1-form, the Riemannian metric and the Lie algebra of all differentiable vector fields on M^{2n+1}, respectively.

The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any fields $X, Y \in \mathfrak{X}(M)$. M^{2n+1} is called an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The almost contact metric manifold is said to be normal if the Nijenhuis tensor of ϕ is given by $[\phi,\phi] = -2d\eta \otimes \xi$, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold [4].

On an almost Kenmotsu manifold M^{2n+1}, the two $(1,1)$-type tensor fields $l = R(\cdot,\xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi\phi$ are symmetric, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. Then we get

$$h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0. \tag{2.3}$$

We also have the following formulas presented in [5, 6]:

$$\nabla_X\xi = -\phi^2X - \phi hX (\Rightarrow \nabla_\xi\xi = 0), \tag{2.4}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$trl = S(\xi,\xi) = g(Q\xi,\xi) = -2n - trh^2, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X \tag{2.7}$$

for any $X, Y \in \mathfrak{X}(M)$, where $h' = h \circ \phi$ and S, Q, ∇, $\mathfrak{X}(M)$ denote the Ricci tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the Lie algebra of all vector fields on M^{2n+1}, respectively.

3. Some properties of $(k,\mu)'$-almost Kenmotsu manifolds

If the characteristic vector field ξ of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfies the $(k,\mu)'$-nullity condition (see [6]), then it is called a $(k,\mu)'$-almost Kenmotsu manifold. The $(k,\mu)'$-nullity condition is defined as follows:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y] \tag{3.1}$$

for any vector fields X, Y, where both k and μ are constant on M^{2n+1}. M^{2n+1} is said to be a (k,μ)-almost manifold Kenmotsu manifold if there holds $R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$ for any vector fields X, Y and $k, \mu \in \mathbb{R}$. A (k,μ)-almost Kenmotsu manifold satisfies $k = -1$ and $h = 0$ (see [6]). A (k,μ)-almost Kenmotsu manifold is a special case of $(k,\mu)'$-almost Kenmotsu manifolds. Following [6], on any $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1}, we have

$$h'^2X = -(k+1)X + (k+1)\eta(X)\xi \tag{3.2}$$

for any vector field $X \in \mathfrak{X}(M)$ and $\mu = -2$. From (3.2), we know that $h' = 0$ is equivalent to $k = -1$ and $h' \neq 0$ everywhere if and only if $k < -1$. Furthermore, by (3.1) and the symmetry of the Riemannian curvature tensor R, it is easy to see that

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] - 2[g(h'X,Y)\xi - \eta(Y)h'X] \tag{3.3}$$

for any $X, Y \in \mathfrak{X}(M)$. In case of $k < -1$, we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces of h' corresponding two eigenvalues $\lambda > 0$ and $-\lambda$, respectively. Obviously, by (3.2), we have

$$\lambda = \sqrt{-k-1} > 0. \tag{3.4}$$

Before presenting one of our main results, we give the following two lemmas.

Lemma 3.1. [6, Proposition 4.2] *Let M^{2n+1} be a $(k,\mu)'$-almost Kenmotsu manifold such that $h' = 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \tag{3.5}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0, \tag{3.6}$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \tag{3.7}$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \tag{3.8}$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \tag{3.9}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]. \tag{3.10}$$

Lemma 3.2. [18, Lemma 3.2] *Let M^{2n+1} be a $(k,\mu)'$-almost Kenmotsu manifold such that $h' \neq 0$. Then the Ricci operator of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.11}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Proof. See the proof of [19, Lemma 3.2]. □

4. $(k,\mu)'$-almost Kenmotsu manifolds satisfying $R(X,Y)\cdot\tilde{C} = 0$

In this section, we consider a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} satisfying the condition

$$R(X,Y)\cdot\tilde{C} = 0, \tag{4.1}$$

or equivalently

$$\begin{aligned}(R(X,Y)\cdot\tilde{C})(U,V)W &= R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U,V)W \\ &\quad - \tilde{C}(U,R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W \\ &= 0\end{aligned} \tag{4.2}$$

for any $X, Y, U, V, W \in \mathfrak{X}(M)$.

From the definition of $\tilde{C}$ (see (1.3)), we have

$$\begin{aligned}\tilde{C}(\xi, Y)Z = &[ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]g(Y, Z)\xi \\ &- [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]\eta(Z)Y \\ &- (-a\mu + 2nb)g(h'Y, Z)\xi + (-a\mu + 2nb)\eta(Z)h'Y,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\tilde{C}(\xi, Y)\xi = &[ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]\eta(Y)\xi \\ &- [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]Y \\ &+ (-a\mu + 2nb)h'Y,\end{aligned} \tag{4.4}$$

where r, a and b denote the scalar curvature and two constants, respectively. Let us denote by $A = [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]$, $B = -A$, $D = (-a\mu + 2nb)$ and $E = -D$.

Substituting $X = U = \xi$ in (4.2) we have

$$\begin{aligned}(R(\xi, Y) \cdot \tilde{C})(\xi, V)W = &R(\xi, Y)\tilde{C}(\xi, V)W - \tilde{C}(R(\xi, Y)\xi, V)W \\ &- \tilde{C}(\xi, R(\xi, Y)V)W - \tilde{C}(\xi, V)R(\xi, Y)W \\ = &0\end{aligned} \tag{4.5}$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Making use of (3.3), (4.3) and (4.4) we calculate every term in equation (4.5) straightly. Then we have

$$\begin{aligned}&R(\xi, Y)\tilde{C}(\xi, V)W \\ =&k[g(Y, \tilde{C}(\xi, V)W)\xi - \eta(\tilde{C}(\xi, V)W))Y] \\ &+ \mu[g(h'Y, \tilde{C}(\xi, V)W)\xi - \eta(\tilde{C}(\xi, V)W))h'Y] \\ =&k\{A[\eta(Y)g(V, W)\xi - \eta(W)g(Y, V)\xi] \\ &+ E[\eta(Y)g(h'V, W)\xi - \eta(W)g(Y, h'V)\xi]\} \\ &- k\{A[g(V, W)Y - \eta(W)\eta(V)Y] + Eg(h'V, W)Y\} \\ &+ \mu\{-A\eta(W)g(h'Y, V)\xi - E\eta(W)g(h'Y, h'V)\xi\} \\ &- \mu\{A[g(V, W)h'Y - \eta(W)\eta(V)h'Y] + Eg(h'V, W)h'Y\}.\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\tilde{C}(R(\xi, Y)\xi, V)W \\ =&k\eta(Y)\tilde{C}(\xi, V)W - k\tilde{C}(Y, V)W - \mu\tilde{C}(h'Y, V)W \\ =&k\{A[\eta(Y)g(V, W)\xi - \eta(W)\eta(Y)V] + E[\eta(Y)g(h'V, W)\xi \\ &- \eta(W)\eta(Y)h'V]\} - k\tilde{C}(Y, V)W - \mu\tilde{C}(h'Y, V)W.\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&\tilde{C}(\xi, R(\xi, Y)V)W \\
=&kg(Y,V)\tilde{C}(\xi,\xi)W - k\eta(V)\tilde{C}(\xi,Y)W \\
&+ \mu g(h'Y,V)\tilde{C}(\xi,\xi)W - \mu\eta(V)\tilde{C}(\xi,h'Y)W \\
=&-k\{A[\eta(V)g(Y,W)\xi - \eta(W)\eta(V)Y] \\
&+ E[\eta(V)g(h'Y,W)\xi - \eta(W)\eta(V)h'Y]\} \\
&- \mu\{A[\eta(V)g(h'Y,W)\xi - \eta(W)\eta(V)h'Y]\} \\
&+ E[\eta(V)g(h'^2Y,W)\xi - \eta(W)\eta(V)h'^2Y]\}.
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
&\tilde{C}(\xi, V)R(\xi, Y)W \\
=&kg(Y,W)\tilde{C}(\xi,V)\xi - k\eta(W)\tilde{C}(\xi,V)Y \\
&+ \mu g(h'Y,W)\tilde{C}(\xi,V)\xi - \mu\eta(W)\tilde{C}(\xi,V)h'Y \\
=&k\{A[g(Y,W)\eta(V)\xi - g(Y,W)V] + Dg(Y,W)h'V\} \\
&- k\{A[\eta(W)g(V,Y)\xi - \eta(Y)\eta(W)V] \\
&+ E[\eta(W)g(h'V,Y)\xi - \eta(Y)\eta(W)h'V]\} \\
&+ \mu\{A[g(h'Y,W)\eta(V)\xi - g(h'Y,W)V] + Dg(h'Y,W)h'V\} \\
&- \mu\{A\eta(W)g(V,h'Y)\xi + E\eta(W)g(h'V,h'Y)\xi\}
\end{aligned} \tag{4.9}
$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Substituting (4.6)-(4.9) into (4.5) and using (3.2) gives

$$
\begin{aligned}
&k\tilde{C}(Y,V)W + \mu\tilde{C}(h'Y,V)W - kAg(V,W)Y \\
&-kEg(h'V,W)Y - \mu Ag(V,W)h'Y - \mu Eg(h'V,W)h'Y \\
&+kE\eta(V)g(h'Y,W)\xi - kE\eta(W)\eta(V)h'Y - \mu E(k+1)\eta(V)g(Y,W)\xi \\
&+\mu E(k+1)\eta(V)\eta(W)Y + kAg(Y,W)V + kEg(Y,W)h'V \\
&+\mu Ag(h'Y,W)V + \mu Eg(h'Y,W)h'V = 0
\end{aligned} \tag{4.10}
$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Substituting $Y = h'Y$ in (4.10) and using (3.2) we obtain

$$
\begin{aligned}
&k\tilde{C}(h'Y,V)W - \mu(k+1)\tilde{C}(Y,V)W - kAg(V,W)h'Y \\
&-kEg(h'V,W)h'Y + \mu A(k+1)g(V,W)Y + \mu E(k+1)g(h'V,W)Y \\
&-kE(k+1)\eta(V)g(Y,W)\xi + kE(k+1)\eta(V)\eta(W)Y \\
&-\mu E(k+1)\eta(V)g(h'Y,W)\xi + \mu E(k+1)\eta(V)\eta(W)h'Y + kAg(h'Y,W)V \\
&+kEg(h'Y,W)h'V - \mu A(k+1)g(Y,W)V - \mu E(k+1)g(Y,W)h'V = 0
\end{aligned} \tag{4.11}
$$

for any $Y, V, W \in \mathfrak{X}(M)$. Subtracting μ multiple of (4.11) from k multiple of (4.10) and using $\mu = -2$ implies

$$
\begin{aligned}
&(k+2)^2\tilde{C}(Y,V)W - (k+2)^2\{Ag(V,W)Y + Eg(h'V,W)Y \\
&-E\eta(V)g(h'Y,W)\xi + E\eta(V)\eta(W)h'Y - Ag(Y,W)V - Eg(Y,W)h'V\} = 0
\end{aligned} \tag{4.12}
$$

for any $Y,V,W\in\mathfrak{X}(M)$. Next, we assume that $Y=V=W\in[-\lambda]'$ in (1.3), where $[-\lambda]'$ is eigenspace of h' corresponding eigenvalue $-\lambda$. Thus, by applying Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned}&\tilde{C}(Y,V)W\\ =&[a(k+2\lambda)-\frac{r}{2n+1}(\frac{a}{2n}+2b)+4nb(\lambda-1)][g(V,W)Y-g(Y,W)V]\end{aligned}\tag{4.13}$$

for any $Y,V,W\in\mathfrak{X}(M)$.

With the help of (4.13) and assuming $Y=V=W\in[-\lambda]'$, from (4.12) we get

$$2nb(k+2)^2(\lambda-1-k)[g(V,W)Y-g(Y,W)V]=0.\tag{4.14}$$

Putting (3.4) into (4.14) we have

$$\lambda(\lambda-1)^2(\lambda+1)^3=0.\tag{4.15}$$

In view of the fact $\lambda>0$, we obtain $\lambda=1$ and hence $k=-2$. From [6, Corollary 4.2] and [5, Theorem 6], we know that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

Therefore we have the following:

Theorem 4.1. *If a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} of dimension greater than 3 satisfies $R\cdot\tilde{C}=0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Since quasi-conformally symmetric manifold ($\nabla\tilde{C}=0$) implies $R\cdot\tilde{C}=0$, therefore from Theorem 4.1 we state the following:

Corollary 4.1. *A quasi-conformally symmetric non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold $M^{2n+1}(n>1)$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Since $R\cdot R$ implies $R\cdot\tilde{C}=0$, we get the following:

Corollary 4.2. *A semisymmetric non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold $M^{2n+1}(n>1)$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

The above corollary has been proved by Wang and Liu [15].

5. $(k,\mu)'$-almost Kenmotsu manifolds satisfying $P(X,Y)\cdot P=0$

In this section, we consider a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1} satisfying the condition

$$P(X,Y)\cdot P=0,\tag{5.1}$$

which implies

$$
\begin{aligned}
&(P(X,Y)\cdot P)(U,V)W\\
=&P(X,Y)P(U,V)W-P(P(X,Y)U,V)W\\
&-P(U,P(X,Y)V)W-P(U,V)P(X,Y)W\\
=&0
\end{aligned}
\tag{5.2}
$$

for any $X,Y,U,V,W\in\mathfrak{X}(M)$.

Making use of (1.1), we get

$$
\begin{aligned}
&P(X,Y)P(U,V)W\\
&=R(X,Y)R(U,V)W-\frac{1}{2n}S(V,W)R(X,Y)U+\frac{1}{2n}S(U,W)R(X,Y)V\\
&-\frac{1}{2n}\{S(Y,R(U,V)W)X-\frac{1}{2n}S(V,W)S(Y,U)X+\frac{1}{2n}S(U,W)S(Y,V)X\}\\
&+\frac{1}{2n}\{S(X,R(U,V)W)Y-\frac{1}{2n}S(V,W)S(X,U)Y+\frac{1}{2n}S(U,W)S(X,V)Y\},
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&P(P(X,Y)U,V)W\\
&=R(R(X,Y)U,V)W-\frac{1}{2n}S(Y,U)R(X,V)W+\frac{1}{2n}S(X,U)R(Y,V)W\\
&-\frac{1}{2n}\{S(V,W)R(X,Y)U-\frac{1}{2n}S(V,W)S(Y,U)X+\frac{1}{2n}S(V,W)S(X,U)Y\}\\
&+\frac{1}{2n}\{S(R(X,Y)U,W)V-\frac{1}{2n}S(Y,U)S(X,W)V+\frac{1}{2n}S(X,U)S(Y,W)V\},
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
&P(U,P(X,Y)V)W\\
&=R(U,R(X,Y)V)W-\frac{1}{2n}S(Y,V)R(U,X)W+\frac{1}{2n}S(X,V)R(U,Y)W\\
&-\frac{1}{2n}\{S(R(X,Y)V,W)U-\frac{1}{2n}S(Y,V)S(X,W)U+\frac{1}{2n}S(X,V)S(Y,W)U\}\\
&+\frac{1}{2n}\{S(U,W)R(X,Y)V-\frac{1}{2n}S(U,W)S(Y,V)X+\frac{1}{2n}S(U,W)S(X,V)Y\},
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
&P(U,V)P(X,Y)W\\
&=R(U,V)R(X,Y)W-\frac{1}{2n}S(Y,W)R(U,V)X+\frac{1}{2n}S(X,W)R(U,V)Y\\
&-\frac{1}{2n}\{S(V,R(X,Y)W)U-\frac{1}{2n}S(Y,W)S(V,X)U+\frac{1}{2n}S(X,W)S(V,Y)U\}\\
&+\frac{1}{2n}\{S(U,R(X,Y)W)V-\frac{1}{2n}S(Y,W)S(U,X)V+\frac{1}{2n}S(X,W)S(U,Y)V\}.
\end{aligned}
\tag{5.6}
$$

Substituting (5.3)-(5.6) into (5.2), we have

$$
\begin{aligned}
&(R(X,Y)\cdot R)(U,V)W-\frac{1}{2n}\{S(Y,R(U,V)W)X-S(X,R(U,V)W)Y\}\\
&+\frac{1}{2n}\{S(Y,U)R(X,V)W-S(X,U)R(Y,V)W-S(R(X,Y)U,W)V\}\\
&+\frac{1}{2n}\{S(Y,V)R(U,X)W-S(X,V)R(U,Y)W+S(R(X,Y)V,W)U\}\\
&+\frac{1}{2n}\{S(Y,W)R(U,V)X-S(X,W)R(U,V)Y+S(V,R(X,Y)W)U\\
&-S(U,R(X,Y)W)V\}=0
\end{aligned}
\tag{5.7}
$$

for any vector fields $X,Y,U,V,W\in\mathfrak{X}(M)$. If (5.1) holds, putting $Y=U=\xi$ into (5.7), we obtain

$$
\begin{aligned}
&(R(X,\xi)\cdot R)(\xi,V)W-\frac{1}{2n}\{S(\xi,R(\xi,V)W)X-S(X,R(\xi,V)W)\xi\}\\
&+\frac{1}{2n}\{S(\xi,\xi)R(X,V)W-S(X,\xi)R(\xi,V)W-S(R(X,\xi)\xi,W)V\}\\
&+\frac{1}{2n}\{S(\xi,V)R(\xi,X)W-S(X,V)R(\xi,\xi)W+S(R(X,\xi)V,W)\xi\}\\
&+\frac{1}{2n}\{S(\xi,W)R(\xi,V)X-S(X,W)R(\xi,V)\xi+S(V,R(X,\xi)W)\xi\\
&-S(\xi,R(X,\xi)W)V\}=0
\end{aligned}
\tag{5.8}
$$

for any vector fields $X,V,W\in\mathfrak{X}(M)$. In Section 4, we know that $S(\xi,V)=2nk\eta(V)$, using the equation and (3.1), we have

$$
\begin{aligned}
&S(R(\xi,X)Y,Z)\\
=&2n\{\eta(Z)[k^2g(X,Y)-2kg(h'X,Y)]\\
&+\eta(Y)[kg(X,Z)-k(k+1)\eta(Z)\eta(X)\\
&+kg(X,h'Z)-2g(h'X,Z)-2g(h'X,h'Z)]\}
\end{aligned}
\tag{5.9}
$$

for any vector fields $X,Y,Z\in\mathfrak{X}(M)$. Combining (5.9) with (5.8) and assuming that $X\in[\lambda]$ and $V=W\in[-\lambda]$ in (5.8) are eigenvector fields of h' corresponding two eigenvalues λ and $-\lambda$, respectively. Thus, by applying Lemma 3.1, we obtian

$$
(R(X,\xi)\cdot R)(\xi,V)W=[k^2+2k\lambda+k(k+2)]g(V,W)X. \tag{5.10}
$$

On the other hand, by a straightforward computation and applying Lemma 3.1, Wang and Liu [15, Theorem 1.1] obtained the following relation (one can check it by a direct calculation).

$$
\begin{aligned}
&(R(X,\xi)\cdot R)(\xi,V)W\\
=&R(X,\xi)R(\xi,V)W-R(R(X,\xi)\xi,V)W\\
&-R(\xi,R(X,\xi)V)W-R(\xi,V)R(X,\xi)W\\
=&[(k-2\lambda)(k+2)-k^2+4\lambda^2]g(V,W)X.
\end{aligned}
\tag{5.11}
$$

From (5.10) and (5.11), we get $\lambda^2(\lambda - 1) = 0$. In view of the fact $\lambda > 0$, we obtain $\lambda = 1$ and hence $k = -2$. From [6, Corollary 4.2] and [5, Theorem 6] we can know that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Consequently, we have the following theorem:

Theorem 5.1. *If a non-Kenmotsu $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1} satisfies $P \cdot P = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

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