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COMMON FIXED POINT THEOREMS USING (ψ_1, ψ_2, ϕ)-WEAK CONTRACTION IN PARTIAL ORDERED METRIC SPACES

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Abstract. The purpose of this note is twofold: (i) to improve the results of Abkar and Choudhury ([1]), and (ii) to obtain common fixed point theorems in partially ordered metric spaces for three self-maps under the weaker condition of the commutativity in line of Banach Fixed Point Theorem. The contraction condition used in our results is weaker than the Banach Contraction Condition. We have supported our result by providing illustrative examples.

Keywords: Common fixed points,(ψ_1, ψ_2, ϕ)-weak contractions, weakly commuting maps, partially ordered metric spaces.

1. Introduction and Preliminaries

The most fundamental result in fixed point theory is the Banach Fixed Point Theorem ([19]) which states that an operator $T : X \to X$ satisfying the inequality

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$, where k is a some constant in $(0, 1)$ and X is a complete metric space, has a unique fixed point which can be obtained by using the Picard iteration scheme. After almost four decades Edelstein ([12]) established a fixed point theorem in a complete metric space (X, d) by using the following inequality for $x \neq y \in X$:

$$d(Tx, Ty) < d(x, y). \tag{1.2}$$

After him, Kannan ([16]) obtained a fixed point theorem in a complete metric space (X, d) by using the following inequality for all $x, y \in X$:

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \tag{1.3}$$

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where $\beta \in [0, \frac{1}{2})$.

It is interesting to note here that inequality (1.3) implies that the self-mapping T is discontinuous. After the result of Kannan([16]) a spate of results appeared in the literature on Metric Fixed Point Theory and its applications.
In 1984, Khan, Swalesh and Sessa ([11]) addressed a new category of fixed point problems involving a control function which they called an altering distance function in the setting of Rakotch ([17]) and Boyd and Wong([5]).

Definition 1.2.([11]) (Altering distance function) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if it satisfies the following conditions:

1. ψ is a monotonically increasing and continuous function;
2. $\psi(t) = 0$ if and only if $t = 0$.

In the meantime, after Banach's Fixed Point Theorem, it was quite natural to ask if there existed any condition that was weaker than the contraction condition. It was Rhoades([2] who, inspired by Alber and Gurerre-Delabriere ([24], extended the same into a complete metric space. Rhoades ([2]) established a fixed point theorem in a complete metric space by using the following contraction condition:

A mapping $T : X \to X$ satisfying the condition

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1.4}$$

where $x, y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark: In the above condition if $\varphi(t) = (1-k)t$ where $k \in (0,1)$, then we have the condition (1.1) of Banach.

So, in view of (1.1), condition (1.4) is a weaker condition.

In recent years, fixed points are also obtained in partially ordered metric spaces endowed with a partial ordering (refer([9], [8], [14]).
Throughout this paper (X, d) stands for a complete metric space and $' \preceq'$ is a partial order on X.

A mapping $T : X \to X$ is said to be non-decreasing if $T(x) \preceq T(y)$, whenever $x \preceq y$.

In this paper we shall assume that X has the following property:

$$\textit{If a non-decreasing sequence } \{x_n\} \textit{ converges to } z \in X, \textit{ then } x_n \preceq z \textit{ for each } n \geq 0. \tag{1.5}$$

Abkar and Choudhury ([1]) proved the following theorem in a partially ordered metric space using (ϕ, ψ) contraction condition initially introduced by Doric ([4]).

Here we list the statements of theorems 2.2, 2.3 and 3.1 as given in ([1]):

Theorem 1.1. Let (X, d) be a complete partially ordered metric space with a partial order '$\preceq$' and the property described in (1.3). Let $T, S : X \to X$ be two self-mappings such that for all comparable x, y $\in X$ with

$$\psi_1(d(Tx, Sy))) \leq \psi_2(M(x, y)) - \phi(M(x, y)) \tag{1.6}$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Sy)), \tfrac{1}{2}(d(y, Tx) + d(x, Sy))\},$$

$\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ are continuous monotone non-decreasing functions and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function which satisfies $\psi_1(t) - \psi_2(t) + \phi(t) > 0$. If X has the property (1.5) and if there exists a point $x_0 \in X$ satisfying $x_0 \preceq Sx_0 \preceq TSx_0 \preceq S(TS)x_0 \preceq (TS)^2x_0 \preceq ....$ Then there exists a point $u \in X$ such that $Su = Tu = u$.

Theorem 1.2. Let (X, d) be a complete metric space with a partial order'$\preceq$' and the property described in (1.3). Let $T : X \to X$ be a self-mapping which is non-decreasing and satisfies the following inequality:

$$\psi_1(d(Tx, Ty))) \leq \psi_2(M(x, y)) - \phi(M(x, y)) \tag{1.7}$$

for all comparable x, y $\in X$, where,

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)), \tfrac{1}{2}(d(y, Tx) + d(x, Ty))\}$$

and $\psi_1, \psi_2, \phi : [0, \infty) \to [0, \infty)$ are such that $\psi_1 and \psi_2$ are continuous, ϕ is lower semi-continuous and $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for all $t > 0$. If X has the property described in (1.5) and there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Theorem 1.3. Let(X, d) be a complete metric space with a partial order '$\preceq$' and $T : X \to X$ be a self mapping which is non-decreasing and satisfies the following inequality:

$$\psi_1(d(Tx, Ty))) \leq \psi_2(N(x, y)) - h(Q(x, y)) \tag{1.8}$$

for $x, y \in X$, where x and y are comparable, $x \neq y$,

$$N(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)), \tfrac{1}{2}(d(y, Tx) + d(x, Ty))\},$$

$$Q(x, y) = min\{d(x, y), d(x, Tx), d(y, Ty)), \tfrac{1}{2}(d(y, Tx) + d(x, Ty))\},$$

$\psi_1, \psi_2, h : [0,\infty) \to [0,\infty)$ are such that ψ_1 and ψ_2 are continuous, $h : [0,\infty) \to [0,\infty)$ monotone decreasing in $(0,\infty)$, lower semi-continuous in $(0,\infty)$ with $h(t) > 0$ for all $t > 0$ and

$$\psi_1(s) - \psi_2(s) + h(s) > 0, s > 0.$$

Further, for all $x \in X$, we assume

$$d(x, T^2x) \geq 2d(Tx, T^2x) \tag{1.9}$$

If X has the property described in (1.5) and if there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Remark: In Theorem 3.1 of ([1]), we observe that condition $h(0) = 0$ if and only if $t = 0$ is missing on page 5 and equation (3.5) (page 5) is not used anywhere in the proof. Similarly, on page 9 case 5 of the example should be written as $x \geq 3$, $y = 0$. Based on those observations, we rewrite the statement under conditions that are weaker than those of ([1]). So, our theorem improves the results given in ([1]).
Now we rewrite the theorem presented in the next section as a corollary. Also, we generalize Theorem 2.2 and Theorem 3.1 of Abkar and Choudhury ([1]) in partially ordered metric spaces using the weakly commuting property for three self-mappings and more control functions.

2. Section(Fixed Point Theorems for Three Self Mappings)

Definition 2.1. ([15]) Let T, S and f are self-maps of a complete metric space (X, d) with a partially ordered relation '$\preceq$' having the property described in (1.5) are said to be weakly commuting with respect to S if and only if

$$d(fTfx, Sfy) \leq d(Tffx, Sfy),$$

for all $x \in X$

Theorem 2.2. Let(X, d) be a complete ordered metric space with a partial order '$\preceq$' and $T, S, f : X \to X$ be three self-mappings which are non-decreasing and satisfy the following inequality:

$$\psi_1(d(Tfx, Sfy))) \leq \psi_2(M(x, y)) - \phi(M(x, y)) \tag{2.1}$$

for $x, y \in X$, where x and y are comparable, $x \neq y$, where

$$M(x, y) = max\{d(x, y), d(x, Tfx), d(y, Sfy)), \frac{1}{2}(d(y, Tfx) + d(x, Sfy))\}, \tag{2.2}$$

$\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ are continuous monotone non-decreasing functions and $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with

$$\psi_1(t) - \psi_2(t) + \phi(t) > 0, \quad s > 0.$$

If X has the property described in (1.5) and if there exists $x_0 \in X$ such that

$$x_0 \preceq Sfx_0 \preceq Tf(Sf(x_0)) \preceq Sf(Tf(Sfx_0))) \preceq (Tf(Sf))^2 x_0 \preceq \dots$$

and if T, S and f are a weakly commuting pair of maps with respect to S. then there exists a point $z \in X$ such that $Sz = Tz = fz = z$.

Proof. Let $x_0 \in X$ be any arbitrary point and let us define

$$x_1 = Sf(x_0) \text{ and } x_2 = Tf(x_1) = Tf(Sf(x_0)).$$

Inductively, we obtain

$$x_{2n+1} = Sf(x_{2n}) \; and \; x_{2n+2} = Tf(x_{2n+1}) \tag{2.3}$$

Then from the condition of the theorem, it follows that

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq ...x_n \preceq x_{n+1} \preceq .....$$

We have to prove that the following:

1. $lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$
2. the sequence $\{x_n\}$ is a cauchy sequence, so that $x_n \to z$ for some $z \in X$.
3. $Sz = Tz = fz = z$.
4. The common fixed point z is unique.

For this : putting $x = x_{2n+1}$, $y = x_{2n}$ in (2.1) and (2.2), we have

$$\psi_1(d(x_{2n+2}, x_{2n+1})) \leq \psi_2(d(M(x_{2n+1}, x_{2n})) - \phi(M(x_{2n+1}, x_{2n})) \tag{2.4}$$

where

$$\begin{aligned} M(x_{2n+1}, x_{2n}) = max\{&d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Tfx_{2n+1}), d(x_{2n}, Tfx_{2n})), \\ &\tfrac{1}{2}(d(x_{2n}, Tfx_{2n+1}) + d(x_{2n+1}, Sfx_{2n}))\} \\ = max\{&d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1})), \\ &\tfrac{1}{2}(d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}))\} \\ = max\{&d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \tfrac{1}{2}(d(x_{2n}, x_{2n+2}))\}. \end{aligned}$$

If possible, for some n let,

$$d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2}), \tag{2.5}$$

By using a triangular inequality, we can write

$$\begin{aligned}\frac{d(x_{2n},x_{2n+2})}{2} &\le \frac{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}{2}\\ &< d(x_{2n+1},x_{2n+2}),\end{aligned} \tag{2.6}$$

By the above inequalities (2.5) and (2.6), we get

$$M(x_{2n+1},x_{2n}) = d(x_{2n+1},x_{2n+2}). \tag{2.7}$$

By (2.4) and (2.7) we get,

$$\psi_1(d(x_{2n+2},x_{2n+1})) \le \psi_2(d(x_{2n+1},x_{2n+2})) - \phi(d(x_{2n+1},x_{2n+2})),$$

We arrive at a contradiction. Thus

$$d(x_{2n+1},x_{2n+2}) \le d(x_{2n},x_{2n+1}). \tag{2.8}$$

Using the above inequality (2.8), we shall obtain the following

$$M(x_{2n+1},x_{2n}) = d(x_{2n},x_{2n+1}). \tag{2.9}$$

We get

$$\psi_1(d(x_{2n+2},x_{2n+1})) \le \psi_2(d(x_{2n},x_{2n+1})) - \phi(d(x_{2n},x_{2n+1})). \tag{2.10}$$

Again, from (2.8), the sequence $d(x_{2n},x_{2n+1})$ is a monotonic, decreasing sequence of a non-negative real number, so there exists $r > 0$ such that

$$lim_{n\to\infty} d(x_{2n},x_{2n+1}) = r > 0. \tag{2.11}$$

Taking the limit $n \to \infty$ in the inequality in (2.10), we shall obtain

$$\psi_1(r) \le \psi_2(r) - \phi(r),$$

which is a contradiction. Hence $r = 0$, that is

$$lim_{n\to\infty} d(x_{2n},x_{2n+1}) = 0. \tag{2.12}$$

Similarly, by replacing $x = x_{2n+2}$ and $y = x_{2n+1}$ in (2.1) and (2.2), we have

$$lim_{n\to\infty} d(x_{2n+1},x_{2n+2}) = 0.$$

Thus we can write for all $n \ge 0$,

$$lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.13}$$

Next we shall prove that $\{x_n\}$ is a cauchy sequence. It will be enough to show that the subsequence $\{x_{2n}\}$ is a cauchy sequence. If not, we suppose $\{x_{2n}\}$ is not a cauchy sequence so there exist $\epsilon > 0$ and a sequence of natural number $\{2n(k)\}$ and $\{2m(k)\}$ such that for every natural number k, $2n(k) > 2m(k) > 2k$ and

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon \tag{2.14}$$

corresponding to $2m(k)$. We can choose $2n(k)$ to be the smallest integer such that (2.14) is satisfied. Then we have

$$d(x_{2m(k)}, x_{2n(k)-1}) < \epsilon \tag{2.15}$$

$$d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).$$

Taking $k \to \infty$ and using (2.13)-(2.15), we obtain

$$lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{2.16}$$

Similarly, we can have

$$d(x_{2m(k)-1}, x_{2n(k)-1}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2n(k)}),$$

$$d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).$$

Letting the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)-1}) = \epsilon. \tag{2.17}$$

Now again,

$$d(x_{2m(k)-1}, x_{2n(k)}) \leq d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)}),$$

$$d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}).$$

Letting the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \epsilon. \tag{2.18}$$

Now putting, $x = x_{2m(k)-1}$ and $y = x_{2n(k)-1}$ in (2.1) and (2.2), we obtain

$$\begin{aligned} \psi_1(d(Tfx_{2m(k)-1}, Sfx_{2n(k)-1})) &\leq \psi_1(d(x_{2m(k)}, x_{2n(k)})) \\ &\leq \psi_2(d(M(x_{2m(k)-1}, x_{2n(k)-1})) - \phi(M(x_{2m(k)-1}, x_{2n(k)-1})) \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M(x_{2m(k)-1}, x_{2n(k)-1}) = max\{&d(x_{2m(k)-1}, x_{2n(k)-1}), d(x_{2m(k-1}, x_{2m(k)}), \\ &d(x_{2n(k)-1}, x_{2n(k)})), \tfrac{1}{2}(d(x_{2n(k)-1}, x_{2m(k)}) + d(x_{2m(k)-1}, x_{2n(k)}))\}. \end{aligned}$$

Letting the limit $k \to \infty$ in (2.19) and using (2.13)-(2.18), we get

$$\psi_1(\epsilon) \leq \psi_2(\epsilon) - \phi(\epsilon)$$

which is a contradiction. Hence the sequence $\{x_n\}$ is a cauchy sequence. Since (X, d) is a complete metric space, therefore the sequence$\{x_n\}$ is a convergent sequence and converges to a point say z.
Assume $x_n \to z$ as $n \to \infty$. Since $\{x_n\}$ is a monotonically increasing sequence, we have $x_n \preceq z$ for all $n \geq 0$.

Next we shall obtain fixed points:
Step 1: Let $d(z, Sfz) \neq 0$ and by putting $x = x_{2n+1}$ and $y = z$ in (2.1), we get

$$\psi_1(d(Tfx_{2n+1}, Sfz)) \leq \psi_2(M(x_{2n+1}, z)) - \phi(M(x_{2n+1}, z)).$$

Taking $n \to \infty$, we get

$$\psi_1(z, Sfz)) \leq \psi_2(M(z, z)) - \phi(M(z, z)) \tag{2.20}$$

$M(x_{2n+1}, z) = max\{d(x_{2n+1}, z), d(x_{2n+1}, x_{2n+2}), d(z, Sfz), \frac{1}{2}(d(z, x_{2n+2}) + d(x_{2n+1}, Sfz))\}$

$$lim_{n\to\infty} M(x_{2n+1}, z) = d(z, Sfz). \tag{2.21}$$

Using(2.20) and (2.21), we get

$$\psi_1(d(z, Sfz)) \leq \psi_2(d(z, Sfz)) - \phi(d(z, Sfz))$$

which is a contradiction and $d(z, Sfz) = 0$ implies $z = Sfz$.

Step 2: Let $d(z, T, fz) \neq 0$ and by putting $x = z$ and $y = x_{2n}$ in (2.1), we get,

$$\psi_1(d(Tfz, Sfx_{2n})) \leq \psi_2(M(z, x_{2n})) - \phi(M(z, x_{2n}).$$

Taking $n \to \infty$ we shall get

$$\psi_1 d(z, Sfz)) \leq \psi_2(M(z, z)) - \phi(M(z, z)) \tag{2.22}$$

$M(z, x_{2n}) = max\{d(z, x_{2n}), d(z, Tfz), d(x_{2n}, Sfx_{2n}), \frac{1}{2}(d(x_{2n}, Tfz) + d(z, Sfx_{2n}))\}.$

Using $Sfz = z$, we obtain

$$lim_{n\to\infty} M(z, x_{2n}) = M(z, z) = d(z, Tfz). \tag{2.23}$$

By (2.22) and (2.23), we get

$$\psi_1(d(Tfz, z)) \leq \psi_2(d(z, Tfz)) - \phi(d(z, Tfz))$$

which is a contradiction. Hence, we have $d(z, Tfz) = 0 \Rightarrow z = Tfz$.

Now we shall assume T, S and f are weakly commuting maps with respect to S. Let $d(fz, z) \neq 0$ and by putting $x = fz$ and $y = z$ in (2.1) and (2.2), since ψ_1 is monotonic nondecreasing,

$$\psi_1 d(fz, z)) = \psi_1 d(fTfz, Sfz)) \leq \psi_1 d(Tffz, Sfz)) \leq \psi_2 M(fz, z)) - \phi(M(fz, z)) \tag{2.24}$$

where

$$M(fz, z) = max\{d(fz, z), d(fz, Tf(fz)), d(z, Sfz), \tfrac{1}{2}(d(z, Tf(fz)) + d(fz, Sfz))\}$$

$$M(fz, z) = max\{d(fz, z), d(fz, fz), d(z, z), \tfrac{1}{2}(d(z, fz) + d(fz, z))\} = d(z, fz).$$

By (2.24) and the above equation, it is implied that

$$\psi_1(d(fz, z)) \leq \psi_2((d(z, fz)) - \phi(d(z, fz))$$

which is a contradiction. Hence $d(z, fz) = 0 \Rightarrow fz = z$.
Now $Tfz = z \Rightarrow Tz = z$ and $Sfz = z \Rightarrow Sz = z$.
Hence $Tz = Sz = fz = z$.
□

Remark: When we take $f = Identitymap$ of X in Theorem 2.1 then we get Theorem 2.2 of Abkar and Choudhury ([1]). When we take $T = S$, and f= identity map in Theorem 2.1 we get Theorem 2.3 of Abkar and Choudhury ([1]).

Now we generalize Theorem 3.1 of Abkar and Choudhury ([1]) in a partially ordered metric space under the weakly commuting property for three self-mappings.

Theorem 2.3. Let(X, d) be a complete metric space with a partial order '$\preceq$' and $T, S, f : X \to X$ self-mappings which are non-decreasing and satisfy the following inequality:

$$\psi_1(d(Tfx, Sfy))) \leq \psi_2(M(x, y)) - \phi(N(x, y)) \tag{2.25}$$

for $x, y \in X$, where x and y are comparable for $x \neq y$ where,

$$M(x, y) = max\{d(x, y), d(x, Tfx), d(y, Sfy)), \tfrac{1}{2}(d(y, Tfx) + d(x, Sfy))\} \text{ and}$$

$$N(x, y) = min\{d(x, y), d(x, Tfx), d(y, Sfy)), \tfrac{1}{2}(d(y, Tfx) + d(x, Sfy))\}$$

$\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ are continuous, monotone non-decreasing functions and $\phi : [0, \infty) \to [0, \infty)$ is monotone decreasing in $(0, \infty)$, lower semi-continuous in $(0, \infty)$ with $\phi(t) > 0$ for all $t > 0$, $\phi(t) = 0$ if and only if $t = 0$ and

$$\psi_1(s) - \psi_2(s) + \phi(s) > 0, \quad s > 0.$$

If X has the property described in(1.5) and if there exists $x_0 \in X$ such that $x_0 \preceq Sfx_0 \preceq Tf(Sf(x_{0)}) \preceq Sf(Tf(Sfx_0))) \preceq (Tf(Sf))^2x_0 \preceq \ldots$ and if T, S and f are a weakly commuting pair of maps with respect to S.
then there exists a point $u \in X$ such that $Su = Tu = fu = u$.

Proof. Let $x_0 \in X$ be any arbitrary point and let us define

$$x_1 = Sf(x_0) \text{ and } x_2 = Tf(x_1) = Tf(Sf(x_0)).$$

Define

$$x_{2n+1} = Sf(x_{2n}) \; and \; x_{2n+2} = Tf(x_{2n+1}) for \; n = 0, 1, 2, ...$$

then from the condition of the theorem, it follows that

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq ...x_n \preceq x_{n+1} \preceq ....$$

We have to prove that the following:

1. $lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$
2. The sequence $\{x_n\}$ is a Cauchy sequence, so that $x_n \to z$ for some $z \in X$.
3. $Sz = Tz = fz = z$.
4. The common fixed point z is unique.

For this : by putting $x = x_{2n+1}$, $y = x_{2n}$ in (2.25), we get

$$\psi_1(d(x_{2n+2}, x_{2n+1})) \leq \psi_2(d(M(x_{2n+1}, x_{2n})) - \Phi(N(x_{2n+1}, x_{2n})) \tag{2.26}$$

where

$$\begin{aligned} M(x_{2n+1}, x_{2n}) = max\{&d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Tfx_{2n+1}), d(x_{2n}, Tfx_{2n})), \\ &\tfrac{1}{2}(d(x_{2n}, Tfx_{2n+1}) + d(x_{2n+1}, Sfx_{2n}))\} \\ = max\{&d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1})), \\ &\tfrac{1}{2}(d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}))\} \text{ and} \end{aligned}$$

$$N(x_{2n+1}, x_{2n}) = min\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \tfrac{1}{2}(d(x_{2n}, x_{2n+2}))\}.$$

If possible, for some n, let

$$d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2}) \tag{2.27}$$

By using the triangular inequality we have

$$\frac{d(x_{2n}, x_{2n+2})}{2} \leq \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2} < d(x_{2n+1}, x_{2n+2}). \tag{2.28}$$

By (2.27) and (2.28), we can write

$$M(x_{2n+1}, x_{2n}) = d(x_{2n+1}, x_{2n+2}). \tag{2.29}$$

Using the triangle inequality, we have

(2.30)

$$N(x_{2n+1}, x_{2n}) \leq min\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2}(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}))\}$$

Using (2.27), we get

$$d(x_{2n}, x_{2n+1}) < \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2} \tag{2.31}$$

Using (2.27), (2.30) and (2.31), we can write

$$0 < N(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2}). \tag{2.32}$$

By (2.26), (2.29),(2.32) and by the monotone decreasing property of ϕ function, we obtain

$$\psi_1(d(x_{2n+1}, x_{2n+2})) \leq \psi_2(d(x_{2n+1}, x_{2n+2})) - \phi(d(x_{2n+1}, x_{2n+2})).$$

The above inequality implies that $d(x_{2n+1}, x_{2n+2}) = 0$, which is a contradiction. Hence

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) \tag{2.33}$$

for all $n \geq 0$, then we have

$$M(x_{2n+1}, x_{2n}) = d(x_{2n}, x_{2n+1}) \tag{2.34}$$

and

$$N(x_{2n+1}, x_{2n}) \leq min\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2}(d(x_{2n}, x_{2n+2}))\} \leq d(x_{2n}, x_{2n+1}). \tag{2.35}$$

Using the monotone decreasing property of ϕ and for all $n \geq 0$, we have

$$\psi_1(d(x_{2n+1}, x_{2n+2})) \leq \psi_2(d(x_{2n}, x_{2n+1})) - \phi(d(x_{2n}, x_{2n+1})) \tag{2.36}$$

2.33 implies that the sequence $d(x_{2n}, x_{2n+1})$ is a monotone decreasing sequence of non-negative real numbers, so there exists $r > 0$ such that

$$lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = r > 0. \tag{2.37}$$

Let if possible $r \neq 0$, taking the limit $n \to \infty$ in (2.36), employing the continuity of ψ_1 and ψ_2 and lower semi-continuity of ϕ, so we have

$$\psi_1(r) \leq \psi_2(r) - \phi(r),$$

We get a contradiction.

Hence

$$lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0.$$

Repeating the above process by putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (2.25), we have

$$lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0.$$

Hence we can write, for all $n \geq 0$, we obtain

$$lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.38}$$

Next, to prove that $\{x_n\}$ is a Cauchy sequence, it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. For the contrary, suppose $\{x_{2n}\}$ is not a Cauchy sequence, so there exists $\epsilon > 0$ and the sequence of natural number $\{2n(k)\}$ and $\{2m(k)\}$ such that for every natural number k, $2n(k) > 2m(k) > 2k$ and

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon \tag{2.39}$$

Corresponding to $2m(k)$, we can choose $2n(k)$ is the smallest positive integer satisfying (2.39). Then we have

$$d(x_{2m(k)}, x_{2n(k)-1}) < \epsilon \tag{2.40}$$

for $k > 0$,

$$d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).$$

Taking $k \to \infty$ in the above inequality, then we have

$$lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{2.41}$$

Similarly, we have

$$d(x_{2m(k)-1}, x_{2n(k)-1}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2n(k)}),$$

$$d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}).$$

Letting the limit $k \to \infty$, we obtain,

$$lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)-1}) = \epsilon \tag{2.42}$$

Now again,

$$d(x_{2m(k)-1}, x_{2n(k)}) \leq d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)}),$$

$$d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}).$$

Taking the limit $k \to \infty$, we obtain,

$$lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \epsilon. \tag{2.43}$$

Now putting, $x = x_{2m(k)-1}$ and $y = x_{2n(k)-1}$ in (2.25), we have

$$\psi_1(d(Tfx_{2m(k)-1}, Sfx_{2n(k)-1})) \leq \psi_1(d(x_{2m(k)}, x_{2n(k)})) \tag{2.44}$$

$$\leq \psi_2(d(M(x_{2m(k)-1}, x_{2n(k)-1})) - \Phi(N(x_{2m(k)-1}, x_{2n(k)-1}))$$

where

$$M(x_{2m(k)-1}, x_{2n(k)-1}) = max\{d(x_{2m(k)-1}, x_{2n(k)-1}), d(x_{2m(k-1}, x_{2m(k)}), \tag{2.45}$$

$$d(x_{2n(k)-1}, x_{2n(k)})), \tfrac{1}{2}(d(x_{2n(k)-1}, x_{2m(k)}) + d(x_{2m(k)-1}, x_{2n(k)}))\}$$

and

$$N(x_{2m(k)-1}, x_{2n(k)-1}) = min\{d(x_{2m(k)-1}, x_{2n(k)-1}), d(x_{2m(k-1}, x_{2m(k)}), \tag{2.46}$$

$$d(x_{2n(k)-1}, x_{2n(k)})), \tfrac{1}{2}(d(x_{2n(k)-1}, x_{2m(k)}) + d(x_{2m(k)-1}, x_{2n(k)}))\}$$

Taking the limit $n \to \infty$, we get

$$lim_{n\to\infty} M(x_{2m(k)-1}, x_{2n(k)-1}) = \epsilon, \tag{2.47}$$

$$lim_{n\to\infty} N(x_{2m(k)-1}, x_{2n(k)-1}) = 0. \tag{2.48}$$

By employing the property of ϕ,

$$liminf_{k\to\infty} \phi(N(x_{m(k)-1}, x_{n(k)-1})) = c > 0 \tag{2.49}$$

Further, taking the limit inf as $k \to \infty$ (2.44), and using (2.47), (2.48) and continuity of ψ, we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - c$$

Next, we see that the constructions of (2.42) and (2.43) are valid whenever ϵ is replaced by a smaller value. This is because of the fact that for any $p \in X$, $\{x : d(p, x) < \epsilon\} \subseteq \{x : d(p, x) < \epsilon'\}$ whenever $\epsilon' < \epsilon$. Hence (2.49) is valid if ϵ is replaced by a smaller value.

Then taking $\epsilon \to 0$ in (2.49) we obtain that $c \leq 0$, which is a contradiction. Hence sequence $\{x_n\}$ is a cauchy sequence.Let $x_n \to z$ as $n \to \infty$ again, $\{x_n\}$ is monotone increasing sequence. Hence by the property of (1.1) we have $x_n \leq z$ for all $n \geq 0$. $\{x_n\}$ is a Cauchy sequence and the subsequence also converges to z.

Now we are ready to find the fixed point:

Step 1: Suppose $d(z, Sfz) \neq 0$ and put $x = x_{2n+1}$ and $y = z$, then by using (2.25), we have

$$\psi_1(d(Tfx_{2n+1}, Sfz)) \leq \psi_2(M(x_{2n+1}, z)) - \phi(M(x_{2n+1}, z)) \tag{2.50}$$

$$M(x_{2n+1}, z) = max\{d(x_{2n+1}, z), d(x_{2n+1}, x_{2n+2}), d(z, Sfz), \frac{1}{2}(d(z, x_{2n+2}) + d(x_{2n+1}, Sfz))\}$$

$$lim_{n\to\infty} M(x_{2n+1}, z) = d(z, Sfz) and \tag{2.51}$$

$$N(x_{2n+1}, z) = min\{d(x_{2n+1}, z), d(x_{2n+1}, x_{2n+2}), d(z, Sfz), \frac{1}{2}(d(z, x_{2n+2}) + d(x_{2n+1}, Sfz))\}$$

$$lim_{n\to\infty} N(x_{2n+1}, z) = \frac{1}{2} d(z, Sfz). \tag{2.52}$$

Taking the limit $n \to \infty$ in (2.50), we get

$$\psi_1(d(z, Sfz)) \leq \psi_2(d(z, Sfz)) - \phi(\tfrac{1}{2} d(z, Sfz)).$$

Since ϕ is a monotone decreasing sequence,

$$\psi_1(d(z, Sfz) \leq \psi_2(d(z, Sfz)) - \phi(d(z, Sfz))$$

which contradicts our assumption. Hence $d(z, Sfz) = 0$, this implies that $z = Sfz$.

Step 2: Let $d(z, Tfz) \neq 0$ and let us put $x = z$ and $y = x_{2n}$; then by (2.25), we obtain

$$\psi_1(d(Tfz, Sfx_{2n})) \leq \psi_2(M(z, x_{2n})) - \phi(N(z, x_{2n}) \tag{2.53}$$

$M(z, x_{2n}) = max\{d(z, x_{2n}), d(z, Tfz), d(x_{2n}, Sfx_{2n}), \frac{1}{2}(d(x_{2n}, Tfz) + d(z, Sfx_{2n}))\}$
and

$$N(z, x_{2n}) = min\{d(z, x_{2n}), d(z, Tfz), d(x_{2n}, Sfx_{2n}), \tfrac{1}{2}(d(x_{2n}, Tfz) + d(z, Sfx_{2n}))\}$$

By using $Sfz = z$ we get,

$$lim_{n\to\infty} M(z, x_{2n}) = M(z, z) = d(z, Tfz), \tag{2.54}$$

$$lim_{n\to\infty} N(z, x_{2n}) = N(z, z) = \frac{1}{2} d(z, Tfz). \tag{2.55}$$

Taking the limit$n \to \infty$ in (2.53), we get

$$\psi_1(d(Tfz,z)) \leq \psi_2(d(z,Tfz)) - \phi(\tfrac{1}{2}d(z,Tfz)).$$

Since ϕ is a monotone decreasing sequence,

$$\psi_1(d(z,Tfz) \leq \psi_2(d(z,Tfz)) - \phi(d(z,Tfz))$$

which is a contradiction. Thus $d(z,Tfz)=0$. This implies that $z=Tfz$.

Since T,S and f are a weakly commuting pair with respect to S. Let $d(fz,z)\neq 0$ and let us put $x=fz$ and $y=z$ in (2.25) since ψ_1 is monotonic nondecreasing,

$$\begin{aligned}\psi_1(d(fz,z)) = \psi_1(d(fTfz,Sfz)) &\leq \psi_1(d(Tffz,Sfz))\\ &\leq \psi_2(M(fz,z)) - \phi(N(fz,z)),\end{aligned} \tag{2.56}$$

where
$M(fz,z) = max\{d(fz,z), d(fz,Tf(fz)), d(z,Sfz),$

$$\tfrac{1}{2}(d(z,Tf(fz)) + d(fz,Sfz))\} = d(z,fz)$$

and

$N(fz,z) = min\{d(fz,z), d(fz,Tf(fz)), d(z,Sfz),$

$$\tfrac{1}{2}(d(z,Tf(fz)) + d(fz,Sfz))\} = 0.$$

Using the above in (2.56), we get,

$$\psi_1 d(fz,z)) \leq \psi_2((d(z,fz)) - \phi(0)).$$

Since ϕ is a monotone decreasing sequence, we can write

$$\psi_1(d(z,fz)) \leq \psi_2(d(z,fz)) - \phi(d(z,fz)$$

This is a contradiction. Thus $d(z,fz)=0 \Rightarrow fz=z$.
Therefore $Tfz=z \Rightarrow Tz=z$ and $Sfz=z \Rightarrow Sz=z$. Hence $Tz=Sz=fz=z$.

When we take $f=I_X$ we shall have the following theorem for two self-maps.

Theorem 2.4. Let (X,d) be a complete partially ordered metric space with a partial order '$\preceq$' and let $T,S : X \to X$ be a self-mapping which is non-decreasing and satisfies the following inequality:

$$\psi_1(d(Tx,Sy))) \leq \psi_2(M(x,y)) - \phi(N(x,y)) \tag{2.57}$$

for $x,y \in X$, where x and y are comparable, $x \neq y$ where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Sy)), \tfrac{1}{2}(d(y,Tx) + d(x,Sy))\}$$

and

$$N(x,y) = min\{d(x,y), d(x,Tx), d(y,Sy)), \tfrac{1}{2}d(y,Tx) + d(x,Sy))\},$$

$\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ are continuous monotone non-decreasing functions and $\phi : [0,\infty) \to [0,\infty)$ is monotonic decreasing in $(0,\infty)$, lower semi-continuous in $(0,\infty)$ with $\phi(t) > 0$ for all $t > 0$, $\phi(t) = 0$ if and only if $t = 0$ and

$$\psi_1(s) - \psi_2(s) + \phi(s) > 0, s > 0.$$

If X has the property described in (1.5) and if there exists $x_0 \in X$ such that $x_0 \preceq Sx_0 \preceq T(S(x_{0)}) \preceq S(T(Sx_0))) \preceq (T(S))^2 x_0 \preceq ....$ then there exists a point $z \in X$ such that $Sz = Tz = fz = z$.

When we take $S = T$ and $f = I_X$, then we have a modified version Theorem 3.1 due to Abkar and Choudhary ([1]).

Theorem 2.5. Let (X,d) be a complete metric space with a partial order' $\preceq$ '. Also, let $T : X \to X$ be a self-mapping which is non-decreasing and satisfies the following inequality:

$$\psi_1(d(Tx,Ty))) \leq \psi_2(N(x,y)) - \phi(Q(x,y)) \tag{2.58}$$

for $x, y \in X$, where x and y are comparable, $x \neq y$ where,

$$N(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)), \tfrac{1}{2}(d(y,Tx) + d(x,Ty))\}$$

$$Q(x,y) = min\{d(x,y), d(x,Tx), d(y,Ty)), \tfrac{1}{2}(d(y,Tx) + d(x,Ty))\}$$

$\psi_1, \psi_2, \phi : [0,\infty) \to [0,\infty)$ are such that ψ_1 and ψ_2 are continuous, $\phi : [0,\infty) \to [0,\infty)$, monotone decreasing in $(0,\infty)$, lower semi-continuous in $(0,\infty)$ with $\phi(t) > 0$ for all $t > 0$, $\phi(t) = 0$ iff $t = 0$, Also

$$\psi_1(t) - \psi_2(t) + \phi(t) > 0, \quad t > 0.$$

If X has the property described in (1.5) and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,, then T has a fixed point. □

3. Section(Example)

Example 3.1. Let $X = \{0, 1, 2, 3, 4, 5, ...\}$ and define

$$d(x,y) = \begin{cases} x + y & : \text{if } x \neq y; \\ 0 & : \text{if } x = y. \end{cases}$$

Then X is a complete metric space. Suppose the partial order $' \preceq'$ is defined on X as $x \preceq y$ whenever $y \geq x$. Also, define T, S and $f : X \to X$ by:

$$T(x) = \begin{cases} x-1 & : \text{if } x \neq 0; \\ 0 & : \text{if } x = 0; \end{cases} \quad S(x) = \begin{cases} \frac{x}{3} & : \text{if } x \neq 0; \\ 0 & : \text{if } x = 0; \end{cases}$$

and
$$f(x) = \begin{cases} 2x-1 & : \text{if } x \neq 0; \\ 0 & : \text{if } x = 0. \end{cases}$$

Now we shall verify that T, S and f are weakly commuting maps with respect to S.
$fTf(x) = fT(2x-1) = f(2x-2) = 2(2x-2)-1 = 4x-5$, $Sfy = S(2y-1) = \frac{2y-1}{3}$ and $Tffx = Tf(2x-1) = T(2(2x-1)-1) = T(4x-3) = 4x-4$.
Therefore,

$$d(fTfx, Sfy) = 4x + \frac{2y}{3} - \frac{16}{3} \leq 4x + \frac{2y}{3} - \frac{13}{3} = d(Tffx, Sfy)$$

The above implies that for all $x, y \in X$, T, S and f are weakly commuting maps with respect to S.

Let $\psi_1(t) = t$ for all $t \geq 0$ and

$$\psi_2(t) = \begin{cases} 2t & : \text{if } 0 \leq t \leq 1; \\ t + \frac{1}{t} & : \text{if } t > 1; \end{cases} \quad \textit{and} \quad \phi(t) = \begin{cases} 1 & : \text{if } t > 0; \\ 0 & : \text{if } t = 0. \end{cases}$$

We can see that $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for each $t > 0$.
Now we shall verify the inequality of Theorem 2.2.
Without loss of generality we may assume that $x > y$.

$$\psi_1(d(Tfx, Sfy))) \leq \psi_2(M(x, y)) - \phi(M(x, y))$$

where,
$M(x, y) = max\{d(x, y), d(x, Tfx), d(y, Sfy)), 1/2(d(y, Tfx) + d(x, Sfy))\}$.

Then the following cases arise:
Case 1 : If $x = 1$ and $y = 0$;
$\psi_1(d(Tfx, Sfy)) = \psi_1(Tfx + Sfy) = (2x + \frac{2y}{3} - \frac{7}{3}) = -\frac{1}{3}$
$M(x, y) = max\{x + y, 3x - 2, \frac{5y-1}{3}, \frac{1}{2}(3x + \frac{5y}{3} - \frac{7}{3})\} = 1$
$\psi_2(M(x, y)) - \phi(M(x, y)) = \psi_2(1) - \phi(1) = 1 \geq \frac{-1}{3} = \psi_1 d(Tfx, Sfy)$.
The inequality is true in this case.

Case 2 : If $x = 2$ and $y = 1$;
$\psi_1(d(Tfx, Sfy)) = \psi_1(2x + \frac{2y}{3} - \frac{7}{3}) = \frac{7}{3}$
$M(x, y) = max\{x + y, 3x - 2, \frac{5y-1}{3}, \frac{1}{2}(3x + \frac{5y}{3} - \frac{7}{3})\} = max\{3, 4, \frac{2}{3}, \frac{16}{6}\} = 4$
$\psi_2(M(x, y)) - \phi(M(x, y)) = \psi_2(4) - \phi(4) = 4 + \frac{1}{4} - 1 = 3.25 > \frac{7}{3} = \psi_1(d(Tfx, Sfy))$.
The inequality is true in this case as well.

Case 3 : If $x = 2$ and $y = 0$;
$\psi_1(d(Tfx, Sfy)] = \psi_1(2x + \frac{2y}{3} - \frac{7}{3}) = \frac{5}{3}$
$M(x, y) = max\{x + y, 3x - 2, \frac{5y-1}{3}, \frac{1}{2}(3x + \frac{5y}{3} - \frac{7}{3})\} = max\{2, 4, \frac{-1}{3}, \frac{11}{6}\} = 4$
$\psi_2(M(x, y)) - \phi(M(x, y)0 = \psi_2(4) - \phi(4) = 4 + \frac{1}{4} - 1 = 3.25 > \frac{5}{3} = \psi_1(d(Tfx, Sfy))$.
The inequality is true in this case, too.

Case 4 : If $x \geq 3$ and $y = 0$;
$\psi_1(d(Tfx, Sfy)) = \psi_1(2x + \frac{2y}{3} - \frac{7}{3}) = 2x - \frac{7}{3}$
$M(x, y) = max\{x + y, 3x - 2, \frac{5y-1}{3}, \frac{1}{2}(3x + \frac{5y}{3} - \frac{7}{3})\} = 3x - 2$
$\psi_2(M(x, y)0 - \phi(M(x, y)) = \psi_2(3x - 20 - \phi(3x - 2) = (3x - 2) - \frac{1}{3x-2} - 1 \geq (2x - \frac{7}{3}) = \psi_1(d(Tfx, Sfy))$.
The inequality is true in this case as well.

Case 5 : If $x \geq 3$ and $y > 0$;
$\psi_1(d(Tfx, Sfy)) = \psi_1(2x + \frac{2y}{3} - \frac{7}{3}) = 2x + \frac{2y}{3} + \frac{7}{3}$
$M(x, y) = max\{x + y, 3x - 2, \frac{5y-1}{3}, \frac{1}{2}(3x + \frac{5y}{3} - \frac{7}{3})\} = 3x - 2$
$\psi_2(M(x, y)) - \phi(M(x, y)0 = \psi_2(3x - 2) - \phi(3x - 2) = (3x - 2) - \frac{1}{3x-2} - 1 \geq [2x + \frac{2y}{3} - \frac{7}{3}] = \psi_1(d(Tfx, Sfy))$.
The inequality is true in this case, too.
Hence we can see that $x = 0$ is the only common fixed point of T, S and f in X.

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