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ON ALMOST PARACONTACT ALMOST PARACOMPLEX RIEMANNIAN MANIFOLDS *

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Abstract. Almost paracontact manifolds of odd dimension having an almost paracomplex structure on the paracontact distribution are studied. The components of the fundamental (0,3)-tensor, derived by the covariant derivative of the structure endomorphism and the metric on the considered manifolds in each of the basic classes are obtained. Then, the case of the lowest dimension 3 of these manifolds is considered. An associated tensor of the Nijenhuis tensor is introduced and the studied manifolds are characterized with respect to this pair of tensors. Moreover, a cases of paracontact and para-Sasakian types are commented. A family of examples is given.
Keywords: Paracontact manifold; Riemannian manifold; tensor; metric.

1. Introduction

In 1976, on a differentiable manifold of arbitrary dimension, I. Sato introduced in [10] the concept of (almost) paracontact structure compatible with a Riemannian metric as an analogue of almost contact Riemannian manifold. Then, he studied several properties of the considered manifolds. Later, a lot of geometers develop the differential geometry of these manifolds and in particular of paracontact Riemannian manifolds and para-Sasakian manifolds. In the beginning are the papers [11], [1], [12], [13] and [9] by I. Sato, T. Adati, T. Miyazawa, K. Matsumoto and S. Sasaki.

On an almost paracontact manifold can be considered two kinds of metrics compatible with the almost paracontact structure. If the structure endomorphism induces an isometry on the paracontact distribution of each tangent fibre, then the manifold has an almost paracontact Riemannian structure as in the papers mentioned above. In the case when the induced transformation is antiisometry, then the manifold has a structure of an almost paracontact metric manifold, where

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the metric is semi-Riemannian of type $(n+1, n)$. This case is studied by many geometers, see for example the papers [7], [14] of S. Zamkovoy and G. Nakova.

In 2001, M. Manev and M. Staikova give a classification in [6] of almost paracontact Riemannian manifold of type (n, n) according to the notion given by Sasaki in [9]. These manifolds are of dimension $2n+1$ and the induced almost product structure on the paracontact distribution is traceless, i.e. it is an almost paracomplex structure.

In the present paper, we continue investigations on these manifolds. The paper is organized as follows. In Sect. 2., we recall some facts about the almost paracontact Riemannian manifolds of the considered type and we make some additional comments. In Sect. 3., we reduce the basic classes of the considered manifolds in the case of the lowest dimension 3. In Sect. 4. and Sect. 5., we find the class of paracontact type and the class of normal type of the manifolds studied, respectively, and we obtain some related properties. In Sect. 6., we introduce an associated Nijenhuis tensor and we discuss relevant problems. In Sect. 7., we argue that the classes of the considered manifolds can be determined only by the pair of Nijenhuis tensors. Finally, in Sect. 8., we construct a family of Lie groups as examples of the manifolds of the studied type and we characterize them in relation with the above investigations.

2. Almost paracontact almost paracomplex Riemannian manifolds

Let $(\mathcal{M}, \phi, \xi, \eta)$ be an *almost paracontact manifold*, i.e. $\mathcal{M}$ is an m-dimensional real differentiable manifold with an *almost paracontact structure* (ϕ, ξ, η) if it admits a tensor field ϕ of type $(1,1)$ of the tangent bundle, a vector field ξ and a 1-form η, satisfying the following conditions:

$$\phi\xi = 0, \quad \phi^2 = I - \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

where I is the identity on the tangent bundle [10].

In [9], it is considered the so-called almost paracontact manifold of type (p, q), where p and q are the numbers of the multiplicity of the ϕ's eigenvalues $+1$ and -1, respectively. Moreover, ϕ has a simple eigenvalue 0. Therefore, we have $\mathrm{tr}\phi = p - q$.

Let us recall that an *almost product structure* P on an differentiable manifold of arbitrary dimension m is an endomorphism on the manifold such that $P^2 = I$. Then a manifold with such a structure is called an *almost product manifold*. In the case when the eigenvalues $+1$ and -1 of P have one and the same multiplicity n, the structure P is called an *almost paracomplex structure* and the manifold is known as an *almost paracomplex manifold* of dimension $2n$ [2]. Then $\mathrm{tr}P = 0$ follows.

Further we consider the case when the dimension of $\mathcal{M}$ is $m = 2n+1$. Then $\mathcal{H} = \ker(\eta)$ is the $2n$-dimensional paracontact distribution of the tangent bundle of $(\mathcal{M}, \phi, \xi, \eta)$, the endomorphism ϕ acts as an almost paracomplex structure on each fiber of $\mathcal{H}$ and the pair $(\mathcal{H}, \phi)$ induces a $2n$-dimensional almost paracomplex manifold. Then we give the following

Definition 2.1. A $(2n+1)$-dimensional differentiable manifold with a structure (ϕ, ξ, η) defined by (2.1) and $\mathrm{tr}\phi = 0$ is called *almost paracontact almost paracomplex manifold*. We denote it by $(\mathcal{M}, \phi, \xi, \eta)$.

Now we can introduce a metric on the considered manifold. It is known from [10] that $\mathcal{M}$ admits a Riemannian metric g which is compatible with the structure of the manifold by the following way:

$$g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y), \quad g(x, \xi) = \eta(x). \tag{2.2}$$

Here and further x, y, z will stand for arbitrary elements of the Lie algebra $\mathfrak{X}(\mathcal{M})$ of tangent vector fields on $\mathcal{M}$ or vectors in the tangent space $T_p\mathcal{M}$ at $p \in \mathcal{M}$.

In [11], an almost paracontact manifold of arbitrary dimension with a Riemannian metric g defined by (2.2) is called an *almost paracontact Riemannian manifold*.

It is easy to conclude that the requirement for a positive definiteness of the metric is not necessary, i.e g can be a pseudo-Riemannian metric. Then, since $g(\xi, \xi) = 1$ follows from (2.1) and (2.2), the signature of g has the form $(2k+1, 2n-2k)$, $k < n$. Since the signature of the metric is not crucial for our considerations, we suppose that g is Riemannian.

Definition 2.2. Let the manifold $(\mathcal{M}, \phi, \xi, \eta)$ be equipped with a Riemannian metric g satisfying (2.2). Then $(\mathcal{M}, \phi, \xi, \eta, g)$ is called an *almost paracontact almost paracomplex Riemannian manifold*.

The decomposition $x = \phi^2 x + \eta(x)\xi$ due to (2.1) generates the projectors h and v on any tangent space of $(\mathcal{M}, \phi, \xi, \eta)$. These projectors are determined by $hx = \phi^2 x$ and $vx = \eta(x)\xi$ and have the properties $h \circ h = h$, $v \circ v = v$, $h \circ v = v \circ h = 0$. Therefore, we have the orthogonal decomposition $T_p\mathcal{M} = h(T_p\mathcal{M}) \oplus v(T_p\mathcal{M})$. Obviously, it generates the corresponding orthogonal decomposition of the space $\mathcal{S}$ of the tensors S of type (0,2) over $(\mathcal{M}, \phi, \xi, \eta)$. This decomposition is invariant with respect to transformations preserving the structures of the manifold. Hereof, we use the following linear operators in $\mathcal{S}$:

$$\begin{aligned} &\ell_1(S)(x, y) = S(hx, hy), \qquad \ell_2(S)(x, y) = S(vx, vy), \\ &\ell_3(S)(x, y) = S(vx, hy) + S(hx, vy). \end{aligned} \tag{2.3}$$

Namely, we have the following decomposition:

$$\mathcal{S} = \ell_1(\mathcal{S}) \oplus \ell_2(\mathcal{S}) \oplus \ell_3(\mathcal{S}), \qquad \ell_i(\mathcal{S}) = \{S \in \mathcal{S} \mid S = \ell_i(S)\}, \quad i = 1, 2, 3.$$

The associated metric $\tilde{g}$ of g on $(\mathcal{M}, \phi, \xi, \eta, g)$ is defined by $\tilde{g}(x, y) = g(x, \phi y) + \eta(x)\eta(y)$. It is shown that $\tilde{g}$ is a compatible metric with $(\mathcal{M}, \phi, \xi, \eta)$ and it is a pseudo-Riemannian metric of signature $(n+1, n)$. Therefore, $(\mathcal{M}, \phi, \xi, \eta, \tilde{g})$ is also an almost paracontact almost paracomplex manifold but with a pseudo-Riemannian metric.

Since the metrics g and $\tilde{g}$ belong to $\mathcal{S}$, then they have corresponding components in the three orthogonal subspaces introduced above and we get them in the following form:

$$\begin{aligned} \ell_1(g) &= g(\phi\cdot,\phi\cdot) = g - \eta\otimes\eta, \qquad & \ell_2(g) &= \eta\otimes\eta, \qquad & \ell_3(g) &= 0,\\ \ell_1(\tilde{g}) &= g(\cdot,\phi\cdot) = \tilde{g} - \eta\otimes\eta, \qquad & \ell_2(\tilde{g}) &= \eta\otimes\eta, \qquad & \ell_3(\tilde{g}) &= 0. \end{aligned}$$

In the final part of the present section we recall the needed notions and results from [6].

In the cited paper, the manifolds under study are called almost paracontact Riemannian manifolds of type (n,n). The structure group of $(\mathcal{M},\phi,\xi,\eta,g)$ is $\mathcal{O}(n)\times\mathcal{O}(n)\times 1$, where $\mathcal{O}(n)$ is the group of the orthogonal matrices of size n.

The tensor F of type (0,3) plays a fundamental role in differential geometry of the considered manifolds. It is defined by:

$$F(x,y,z) = g\big((\nabla_x\phi)\,y,z\big), \tag{2.4}$$

where ∇ is the Levi-Civita connection of g. The basic properties of F with respect to the structure are the following:

$$\begin{aligned} F(x,y,z) &= F(x,z,y)\\ &= -F(x,\phi y,\phi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi). \end{aligned} \tag{2.5}$$

The relations of $\nabla\xi$ and $\nabla\eta$ with F are:

$$(\nabla_x\eta)y = g\left(\nabla_x\xi,y\right) = -F(x,\phi y,\xi). \tag{2.6}$$

If $\{\xi;e_i\}$ $(i=1,2,\dots,2n)$ is a basis of the tangent space $T_p\mathcal{M}$ at an arbitrary point $p\in\mathcal{M}$ and $\left(g^{ij}\right)$ is the inverse matrix of the matrix (g_{ij}) of g, then the following 1-forms are associated with F:

$$\theta(z) = g^{ij}F(e_i,e_j,z), \quad \theta^*(z) = g^{ij}F(e_i,\phi e_j,z), \quad \omega(z) = F(\xi,\xi,z). \tag{2.7}$$

These 1-forms are known also as the Lee forms of the considered manifolds. Obviously, the identities $\omega(\xi)=0$ and $\theta^*\circ\phi = -\theta\circ\phi^2$ are always valid.

There, it is made a classification of the almost paracontact almost paracomplex Riemannian manifolds with respect to F. The vector space $\mathbb{F}$ of all tensors F with the properties (2.5) is decomposed into 11 subspaces $\mathbb{F}_i$ $(i=1,2,\dots,11)$, which are orthogonal and invariant with respect to the structure group of the considered manifolds. This decomposition induces a classification of the manifolds under study. An almost paracontact almost paracomplex Riemannian manifold is said to be in the class $\mathcal{F}_i$ $(i=1,2,\dots,11)$, or briefly an $\mathcal{F}_i$-manifold, if the tensor F belongs to the subspace $\mathbb{F}_i$. Such a way, it is obtained that this classification consists of 11 basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, …, $\mathcal{F}_{11}$. The intersection of the basic classes is the special class $\mathcal{F}_0$ determined by the condition $F(x,y,z)=0$. Hence $\mathcal{F}_0$ is the class of the considered manifolds with ∇-parallel structures, i.e. $\nabla\phi=\nabla\xi=\nabla\eta=\nabla g=\nabla\tilde{g}=0$.

Moreover, it is given the conditions for F determining the basic classes $\mathcal{F}_i$ of $(\mathcal{M},\phi,\xi,\eta,g)$ and the components of F corresponding to $\mathcal{F}_i$. It is said that $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to the class $\mathcal{F}_i$ $(i=1,2,\dots,11)$ if and only if the equality $F=F_i$ is valid. In the last expression, F_i are the components of F in the subspaces $\mathbb{F}_i$ and they are given by the following equalities:

$$
\begin{aligned}
F_1(x,y,z)=&\tfrac{1}{2n}\big\{g(\phi x,\phi y)\theta(\phi^2 z)+g(\phi x,\phi z)\theta(\phi^2 y)\\
&\quad -g(x,\phi y)\theta(\phi z)-g(x,\phi z)\theta(\phi y)\big\},\\
F_2(x,y,z)=&\tfrac{1}{4}\big\{2F(\phi^2 x,\phi^2 y,\phi^2 z)+F(\phi^2 y,\phi^2 z,\phi^2 x)+F(\phi^2 z,\phi^2 x,\phi^2 y)\\
&\quad -F(\phi y,\phi z,\phi^2 x)-F(\phi z,\phi y,\phi^2 x)\big\}\\
&-\tfrac{1}{2n}\big\{g(\phi x,\phi y)\theta(\phi^2 z)+g(\phi x,\phi z)\theta(\phi^2 y)\\
&\quad -g(x,\phi y)\theta(\phi z)-g(x,\phi z)\theta(\phi y)\big\},\\
F_3(x,y,z)=&\tfrac{1}{4}\big\{2F(\phi^2 x,\phi^2 y,\phi^2 z)-F(\phi^2 y,\phi^2 z,\phi^2 x)-F(\phi^2 z,\phi^2 x,\phi^2 y)\\
&\quad +F(\phi y,\phi z,\phi^2 x)+F(\phi z,\phi y,\phi^2 x)\big\},\\
F_4(x,y,z)=&\tfrac{\theta(\xi)}{2n}\big\{g(\phi x,\phi y)\eta(z)+g(\phi x,\phi z)\eta(y)\big\},\\
F_5(x,y,z)=&\tfrac{\theta^*(\xi)}{2n}\big\{g(x,\phi y)\eta(z)+g(x,\phi z)\eta(y)\big\},\\
F_6(x,y,z)=&\tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi)+F(\phi^2 y,\phi^2 x,\xi)+F(\phi x,\phi y,\xi)\\
&\quad +F(\phi y,\phi x,\xi)]\eta(z)\\
&\quad +[F(\phi^2 x,\phi^2 z,\xi)+F(\phi^2 z,\phi^2 x,\xi)+F(\phi x,\phi z,\xi)\\
&\quad +F(\phi z,\phi x,\xi)]\eta(y)\big\}\\
&-\tfrac{\theta(\xi)}{2n}\big\{g(\phi x,\phi y)\eta(z)+g(\phi x,\phi z)\eta(y)\big\}\\
&-\tfrac{\theta^*(\xi)}{2n}\big\{g(x,\phi y)\eta(z)+g(x,\phi z)\eta(y)\big\},\\
F_7(x,y,z)=&\tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi)-F(\phi^2 y,\phi^2 x,\xi)+F(\phi x,\phi y,\xi)\\
&\quad -F(\phi y,\phi x,\xi)]\eta(z)\\
&\quad +[F(\phi^2 x,\phi^2 z,\xi)-F(\phi^2 z,\phi^2 x,\xi)+F(\phi x,\phi z,\xi)\\
&\quad -F(\phi z,\phi x,\xi)]\eta(y)\big\},\\
F_8(x,y,z)=&\tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi)+F(\phi^2 y,\phi^2 x,\xi)-F(\phi x,\phi y,\xi)\\
&\quad -F(\phi y,\phi x,\xi)]\eta(z)\\
&\quad +[F(\phi^2 x,\phi^2 z,\xi)+F(\phi^2 z,\phi^2 x,\xi)-F(\phi x,\phi z,\xi)\\
&\quad -F(\phi z,\phi x,\xi)]\eta(y)\big\},\\
F_9(x,y,z)=&\tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi)-F(\phi^2 y,\phi^2 x,\xi)-F(\phi x,\phi y,\xi)\\
&\quad +F(\phi y,\phi x,\xi)]\eta(z)\\
&\quad +[F(\phi^2 x,\phi^2 z,\xi)-F(\phi^2 z,\phi^2 x,\xi)-F(\phi x,\phi z,\xi)\\
&\quad +F(\phi z,\phi x,\xi)]\eta(y)\big\},\\
F_{10}(x,y,z)=&\eta(x)F(\xi,\phi^2 y,\phi^2 z),\\
F_{11}(x,y,z)=&\eta(x)\big\{\eta(y)\omega(z)+\eta(z)\omega(y)\big\}.
\end{aligned}
\tag{2.8}
$$

It is easy to conclude that a manifold of the considered type belongs to a direct sum of two or more basic classes, i.e. $(\mathcal{M},\phi,\xi,\eta,g)\in\mathcal{F}_i\oplus\mathcal{F}_j\oplus\cdots$, if and only if the fundamental tensor F on $(\mathcal{M},\phi,\xi,\eta,g)$ is the sum of the corresponding components F_i, F_j, … of F, i.e. the following condition is satisfied $F=F_i+F_j+\cdots$.

Finally in this section, we obtain immediately

Proposition 2.1. *The dimensions of the subspaces* $\mathbb{F}_i$ $(i = 1, 2, \dots, 11)$ *in the decomposition of the space* $\mathbb{F}$ *of the tensors* F *on* $(\mathcal{M}, \phi, \xi, \eta, g)$ *are the following:*

$$\begin{array}{lll} \dim \mathbb{F}_1 = 2n, & \dim \mathbb{F}_2 = n(n-1)(n+2), & \dim \mathbb{F}_3 = n^2(n-1), \\ \dim \mathbb{F}_4 = 1, & \dim \mathbb{F}_5 = 1, & \dim \mathbb{F}_6 = (n-1)(n+2), \\ \dim \mathbb{F}_7 = n(n-1), & \dim \mathbb{F}_8 = n^2, & \dim \mathbb{F}_9 = n^2, \\ \dim \mathbb{F}_{10} = n^2, & \dim \mathbb{F}_{11} = 2n. & \end{array}$$

Proof. Using the characteristic symmetries of the fundamental tensor F and the form of its components from (2.8) in each of $\mathcal{F}_i$ $(i = 1, 2, \dots, 11)$, we get the equalities in the statement. □

3. The components of the fundamental tensor for dimension 3

Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be the manifold under study with the lowest dimension, i.e. $\dim \mathcal{M} = 3$ (or $n = 1$) and let the system of three vectors $\{e_0 = \xi, e_1 = e, e_2 = \phi e\}$ be a ϕ-basis which satisfies the following conditions:

$$\begin{array}{c} g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = 1, \\ g(e_0, e_1) = g(e_1, e_2) = g(e_0, e_2) = 0. \end{array} \tag{3.1}$$

We denote the components of the tensors F, θ, θ^* and ω with respect to the ϕ-basis $\{e_0, e_1, e_2\}$ as follows $F_{ijk} = F(e_i, e_j, e_k)$, $\theta_k = \theta(e_k)$, $\theta^*_k = \theta^*(e_k)$ and $\omega_k = \omega(e_k)$. The properties (2.5) and (3.1) imply the equalities $F_{i12} = F_{i21} = 0$ and $F_{i11} = -F_{i22}$ for any i. Then, bearing in mind (2.7), we obtain for the Lee forms the following:

$$\begin{array}{lll} \theta_0 = F_{110} + F_{220}, & \theta_1 = F_{111} = -F_{122} = -\theta^*_2, & \omega_1 = F_{001}, \\ \theta^*_0 = F_{120} + F_{210}, & \theta_2 = F_{222} = -F_{211} = -\theta^*_1, & \omega_2 = F_{002}, \\ & & \omega_0 = 0. \end{array} \tag{3.2}$$

The arbitrary vectors x, y, z in $T_p\mathcal{M}$, $p \in \mathcal{M}$, have the expression $x = x^i e_i$, $y = y^i e_i$, $z = z^i e_i$ with respect to $\{e_0, e_1, e_2\}$.

Proposition 3.1. *The components* F_i $(i = 1, 2, \dots, 11)$ *of the fundamental tensor* F *for a 3-dimensional almost paracontact almost paracomplex Riemannian manifold are the following:*

$$\begin{array}{l} F_1(x,y,z) = \left(x^1\theta_1 - x^2\theta_2\right)\left(y^1z^1 - y^2z^2\right), \\ F_2(x,y,z) = F_3(x,y,z) = 0, \\ F_4(x,y,z) = \frac{\theta_0}{2}\left\{x^1\left(y^0z^1 + y^1z^0\right) + x^2\left(y^0z^2 + y^2z^0\right)\right\}, \\ F_5(x,y,z) = \frac{\theta^*_0}{2}\left\{x^1\left(y^0z^2 + y^2z^0\right) + x^2\left(y^0z^1 + y^1z^0\right)\right\}, \\ F_6(x,y,z) = F_7(x,y,z) = 0, \\ F_8(x,y,z) = \lambda\left\{x^1\left(y^0z^1 + y^1z^0\right) - x^2\left(y^0z^2 + y^2z^0\right)\right\}, \\ F_9(x,y,z) = \mu\left\{x^1\left(y^0z^2 + y^2z^0\right) - x^2\left(y^0z^1 + y^1z^0\right)\right\}, \\ F_{10}(x,y,z) = \nu\, x^0\left(y^1z^1 - y^2z^2\right), \\ F_{11}(x,y,z) = x^0\left\{\omega_1\left(y^0z^1 + y^1z^0\right) + \omega_2\left(y^0z^2 + y^2z^0\right)\right\}, \end{array} \tag{3.3}$$

where

$$\lambda = F_{110} = -F_{220}, \qquad \mu = F_{120} = -F_{210}, \qquad \nu = F_{011} = -F_{022}$$

and the components of the Lee forms are given in (3.2).

Proof. Using the expressions (2.8) of F_i for the corresponding classes $\mathcal{F}_i$ ($i = 1, \dots, 11$), the equalities (2.5), (3.1) and (3.2), we obtain the corresponding form of F_i for the lowest dimension of the considered manifold. □

As a result of Proposition 3.1, we establish the truthfulness of the following

Theorem 3.1. *The 3-dimensional almost paracontact almost paracomplex Riemannian manifolds belong to the basic classes $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_8$, $\mathcal{F}_9$, $\mathcal{F}_{10}$, $\mathcal{F}_{11}$ and to their direct sums.*

Let us remark that for the considered manifolds of dimension 3, the basic classes $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, $\mathcal{F}_7$ are restricted to the special class $\mathcal{F}_0$.

4. Paracontact almost paracomplex Riemannian manifolds

Let $(\mathcal{M}, \phi, \xi, \eta, g)$, $\dim \mathcal{M} = 2n + 1$, be an almost paracontact almost paracomplex Riemannian manifold such that the following condition is satisfied:

$$2g(x, \phi y) = (\mathfrak{L}_\xi g)(x, y), \tag{4.1}$$

where the Lie derivative $\mathfrak{L}$ of g along ξ has the following form in terms of $\nabla\eta$:

$$(\mathfrak{L}_\xi g)(x, y) = (\nabla_x \eta)y + (\nabla_y \eta)x. \tag{4.2}$$

Bearing in mind (2.6) and (4.2), $\mathfrak{L}_\xi g$ is expressed by F as follows:

$$(\mathfrak{L}_\xi g)(x, y) = -F(x, \phi y, \xi) - F(y, \phi x, \xi). \tag{4.3}$$

In [11], it is said that an m-dimensional almost paracontact Riemannian manifold endowed with the property $2g(x, \phi y) = (\nabla_x \eta)y + (\nabla_y \eta)x$ is a *paracontact Riemannian manifold*.

Definition 4.1. An almost paracontact almost paracomplex Riemannian manifold satisfied (4.1) is called *paracontact almost paracomplex Riemannian manifold*.

Now we determine the class of paracontact almost paracomplex Riemannian manifolds with respect to the basic classes $\mathcal{F}_i$. Firstly, we compute $\mathfrak{L}_\xi g$ on each $\mathcal{F}_i$-manifold using (4.3) and (2.8). Then we obtain

Proposition 4.1. *Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) $(\mathfrak{L}_\xi g)(x,y) = 0$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$*;*

b) $(\mathfrak{L}_\xi g)(x,y) = -\frac{1}{n}\theta(\xi)g(x,\phi y)$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_4$*;*

c) $(\mathfrak{L}_\xi g)(x,y) = -\frac{1}{n}\theta^*(\xi)g(\phi x,\phi y)$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_5$*;*

d) $(\mathfrak{L}_\xi g)(x,y) = 2\big(\nabla_x\eta\big)y$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_6 \oplus \mathcal{F}_9$*;*

e) $(\mathfrak{L}_\xi g)(x,y) = -\eta(x)\omega(\phi y) - \eta(y)\omega(\phi x)$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_{11}$*.*

It is known that ξ is a Killing vector field when $\mathfrak{L}_\xi g = 0$. Therefore, the latter proposition implies

Corollary 4.1. *An almost paracontact almost paracomplex Riemannian manifold* $(\mathcal{M},\phi,\xi,\eta,g)$ *has a Killing vector field* ξ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_i$ $(i = 1,2,3,7,8,10)$ *or to their direct sums.*

We denote by ${\mathcal{F}_4}'$ the subclass of $\mathcal{F}_4$ determined by $\theta(\xi) = -2n$, i.e.

(4.4) $$ {\mathcal{F}_4}' = \big\{\mathcal{F}_4 \,|\, \theta(\xi) = -2n\big\}. $$

Then, the component ${F_4}'$ of F corresponding to the subclass ${\mathcal{F}_4}'$ is

(4.5) $$ {F_4}'(x,y,z) = -g(\phi x,\phi y)\eta(z) - g(\phi x,\phi z)\eta(y). $$

Theorem 4.1. *Paracontact almost paracomplex Riemannian manifolds belong to* ${\mathcal{F}_4}'$ *or to its direct sums with* $\mathcal{F}_1$*,* $\mathcal{F}_2$*,* $\mathcal{F}_3$*,* $\mathcal{F}_7$*,* $\mathcal{F}_8$ *and* $\mathcal{F}_{10}$*.*

Proof. Let us consider an arbitrary almost paracontact almost paracomplex Riemannian manifold, i.e. $F = F_1 + \ldots + F_{11}$. Using the expressions (2.8) of F_i for the corresponding classes $\mathcal{F}_i$ $(i = 1,\ldots,11)$ and the condition (4.1), we obtain

(4.6) $$ F = F_1 + F_2 + F_3 + {F_4}' + F_7 + F_8 + F_{10}, $$

where ${F_4}'$ is determined by (4.5).

Vice versa, if (4.6) holds true, then it implies (4.1) by (4.3), i.e. $(\mathcal{M},\phi,\xi,\eta,g)$ is a paracontact almost paracomplex Riemannian manifold. Supposing that $(\mathcal{M}, \phi,\xi,\eta,g)$ belongs to some of $\mathcal{F}_i$ $(i = 1,2,3,7,8,10)$ or their direct sum, it follows that g is degenerate. Therefore, the component ${F_4}'$ is indispensable and we get the statement. □

Let us remark that ${\mathcal{F}_4}'$ and $\mathcal{F}_0$ are subclasses of $\mathcal{F}_4$ without common elements.

Moreover, bearing in mind Corollary 4.1 and Theorem 4.1, we conclude that paracontact almost paracomplex Riemannian manifolds with a Killing vector field ξ do not exist, i.e. for the manifolds studied, there is no analogue of a K-contact manifold.

In [13], it is introduced the notion of a para-Sasakian Riemannian manifold of an arbitrary dimension by the condition $\phi x = \nabla_x \xi$. The same condition determines a special kind of paracontact almost paracomplex Riemannian manifolds. These manifolds we call para-Sasakian paracomplex Riemannian manifolds. Then, using (4.4), we obtain the truthfulness of the following

Theorem 4.2. *The class of the para-Sasakian paracomplex Riemannian manifolds is* $\mathcal{F}_4{}'$.

5. The Nijenhuis tensor

5.1. Introduction of the Nijenhuis tensor

Let us consider the product manifold $\check{\mathcal{M}}$ of an almost paracontact almost paracomplex manifold $(\mathcal{M}, \phi, \xi, \eta)$ and the real line $\mathbb{R}$, i.e. $\check{\mathcal{M}} = \mathcal{M} \times \mathbb{R}$. We denote a vector field on $\check{\mathcal{M}}$ by $\left(x, a\frac{\mathrm{d}}{\mathrm{d}r}\right)$, where x is tangent to $\check{\mathcal{M}}$, r is the coordinate on $\mathbb{R}$ and a is a function on $\mathcal{M} \times \mathbb{R}$. Further, we use the denotation $\partial_r = \frac{\mathrm{d}}{\mathrm{d}r}$ for brevity. Following [10], we define an almost paracomplex structure $\check{P}$ on $\check{\mathcal{M}}$ by:

$$\check{P}\,(x,\, a\partial_r) = \left(\phi x + \frac{a}{r}\xi,\, r\eta(x)\partial_r\right) \tag{5.1}$$

that implies

$$\check{P}x = \phi x, \qquad \check{P}\xi = r\partial_r, \qquad \check{P}\partial_r = \frac{1}{r}\xi.$$

Further, we use the setting $\zeta = r\partial_r$. It easy to check that $\check{P}^2 = I$ and $\mathrm{tr}\check{P} = 0$. In the case when $\check{P}$ is integrable, it is said that the almost paracontact structure (ϕ, ξ, η) is *normal*.

It is known, the vanishing of the Nijenhuis torsion $\left[\check{P}, \check{P}\right]$ of $\check{P}$ is a necessary and sufficient condition for integrability of $\check{P}$. According to [10], the condition of normality is equivalent to vanishing of the following four tensors:

$$\begin{aligned} &N^{(1)}(x,y) = [\phi,\phi](x,y) - \mathrm{d}\eta(x,y)\xi,\\ &N^{(2)}(x,y) = (\mathfrak{L}_{\phi x}\eta)(y) - (\mathfrak{L}_{\phi y}\eta)(x),\\ &N^{(3)}(x) = (\mathfrak{L}_{\xi}\phi)(x),\\ &N^{(4)}(x) = (\mathfrak{L}_{\xi}\eta)(x), \end{aligned} \tag{5.2}$$

where the Nijenhuis torsion of ϕ is determined by:

$$[\phi,\phi](x,y) = [\phi x, \phi y] + \phi^2[x,y] - \phi[\phi x, y] - \phi[x, \phi y] \tag{5.3}$$

and $\mathrm{d}\eta$ is the exterior derivative of η given by:

$$\mathrm{d}\eta(x,y) = (\nabla_x\eta)y - (\nabla_y\eta)x. \tag{5.4}$$

According to (2.6) and (5.4), $\mathrm{d}\eta$ is expressed by F as follows:

$$\mathrm{d}\eta(x,y) = -F(x,\phi y,\xi) + F(y,\phi x,\xi). \tag{5.5}$$

Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost paracontact almost paracomplex Riemannian manifold.

In [10], it is proved that the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$, $N^{(4)}$. Then $N^{(1)}$ is denoted simply by N, i.e.

$$N(x, y) = [\phi, \phi](x, y) - \mathrm{d}\eta(x, y)\xi, \tag{5.6}$$

and it is called the *Nijenhuis tensor of the structure* (ϕ, ξ, η). Therefore, an almost paracontact structure (ϕ, ξ, η) is normal if and only if its Nijenhuis tensor is zero.

Obviously, N is an antisymmetric tensor, i.e. $N(x, y) = -N(y, x)$. According to (5.3), (5.4) and (5.6), the tensor N has the following form in terms of the covariant derivatives of ϕ and η with respect to ∇:

$$N(x, y) = (\nabla_{\phi x}\phi)y - (\nabla_{\phi y}\phi)x - \phi(\nabla_x\phi)y + \phi(\nabla_y\phi)x - (\nabla_x\eta)y\,\xi + (\nabla_y\eta)x\,\xi.$$

The corresponding tensor of type (0,3) of the Nijenhuis tensor on $(\mathcal{M}, \phi, \xi, \eta, g)$ is defined by equality $N(x, y, z) = g\left(N(x, y), z\right)$. Then, using (2.4) and (2.6), we express N in terms of the fundamental tensor F as follows:

$$\begin{aligned} N(x, y, z) = F(\phi x, y, z) - F(\phi y, x, z) - F(x, y, \phi z) + F(y, x, \phi z) \\ + \eta(z)\left\{F(x, \phi y, \xi) - F(y, \phi x, \xi)\right\}. \end{aligned} \tag{5.7}$$

Proposition 5.1. *The Nijenhuis tensor on an almost paracontact almost paracomplex Riemannian manifold has the following properties:*

$$\begin{aligned} &N(\phi^2 x, \phi y, \phi z) = -N(\phi^2 x, \phi^2 y, \phi^2 z), && N(\phi^2 x, \phi^2 y, \phi^2 z) = N(\phi x, \phi y, \phi^2 z),\\ &N(x, \phi^2 y, \phi^2 z) = -N(x, \phi y, \phi z), && N(\phi^2 x, \phi^2 y, z) = N(\phi x, \phi y, z),\\ &N(\xi, \phi y, \phi z) = -N(\xi, \phi^2 y, \phi^2 z), && N(\phi x, \phi y, \xi) = N(\phi^2 x, \phi^2 y, \xi). \end{aligned}$$

Proof. The equalities from the above follow by direct computations from the properties (2.5) and the expression (5.7). □

In [10], there are given the following relations between the tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$:

$$\begin{aligned} &N^{(2)}(x, y) = -\eta\left(N^{(1)}(x, \phi y)\right) - \eta\left(N^{(1)}(\phi x, \xi)\right)\eta(y),\\ &N^{(3)}(x) = -N^{(1)}(\phi x, \xi),\\ &N^{(4)}(x) = -N^{(2)}(\phi x, \xi), \qquad N^{(4)}(x) = -\eta\left(N^{(3)}(\phi x)\right). \end{aligned} \tag{5.8}$$

Applying the expression (5.7) to equalities (5.8), we obtain the form of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ in terms of the fundamental tensor F:

$$\begin{aligned} &N^{(2)}(x, y) = -F(x, y, \xi) + F(y, x, \xi) - F(\phi x, \phi y, \xi) + F(\phi y, \phi x, \xi),\\ &N^{(3)}(x, y) = F(\xi, x, y) - F(x, y, \xi) + F(\phi x, \phi y, \xi),\\ &N^{(4)}(x) = -F(\xi, \xi, \phi x), \end{aligned} \tag{5.9}$$

where it is used the denotation $N^{(3)}(x, y) = g\left(N^{(3)}(x), y\right)$.

Table 5.1: Nijenhuis tensors

	$N^{(1)}(x,y,z)$	$N^{(2)}(x,y)$	$N^{(3)}(x,y)$	$N^{(4)}(x)$
$\mathcal{F}_1$	0	0	0	0
$\mathcal{F}_2$	0	0	0	0
$\mathcal{F}_3$	$-2\Big\{F(\phi x,\phi y,\phi z)+F(\phi^2 x,\phi^2 y,\phi z)\Big\}$	0	0	0
$\mathcal{F}_4$	0	0	0	0
$\mathcal{F}_5$	0	0	0	0
$\mathcal{F}_6$	0	0	0	0
$\mathcal{F}_7$	$4F(x,\phi y,\xi)\eta(z)$	$-4F(x,y,\xi)$	0	0
$\mathcal{F}_8$	$2\Big\{\eta(x)F(y,\phi z,\xi)-\eta(y)F(x,\phi z,\xi)\Big\}$	0	$-2F(x,y,\xi)$	0
$\mathcal{F}_9$	$2\Big\{\eta(x)F(y,\phi z,\xi)-\eta(y)F(x,\phi z,\xi)\Big\}$	0	$-2F(x,y,\xi)$	0
$\mathcal{F}_{10}$	$-\eta(x)F(\xi,y,\phi z)+\eta(y)F(\xi,x,\phi z)$	0	$F(\xi,x,y)$	0
$\mathcal{F}_{11}$	$\eta(z)\Big\{\eta(x)\omega(\phi y)-\eta(y)\omega(\phi x)\Big\}$	$\eta(y)\omega(x)-\eta(x)\omega(y)$	$\eta(y)\omega(x)$	$-\omega(\phi x)$

Proposition 5.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an $\mathcal{F}_i$-manifold ($i=1,2,\dots,11$). Then the four tensors $N^{(k)}$ ($k=1,2,3,4$) on this manifold have the form in the respective cases, given in Table 5.1.*

Proof. We apply direct computations, using (2.8), (5.7) and (5.9). □

By virtue Proposition 5.2, we have the following

Theorem 5.1. *An almost paracontact almost paracomplex Riemannian manifold $(\mathcal{M},\phi,\xi,\eta,g)$ has:*

a) vanishing $N^{(1)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$ or to their direct sums;

b) vanishing $N^{(2)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_6$, $\mathcal{F}_8$, $\mathcal{F}_9$, $\mathcal{F}_{10}$ or to their direct sums;

c) vanishing $N^{(3)}$ if and only if it belongs to the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_7$ or to some of their direct sums;

d) vanishing $N^{(4)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_{10}$ or to their direct sums.

Bearing in mind Theorem 5.1, we conclude the following

Corollary 5.1. *The class of normal almost paracontact almost paracomplex Riemannian manifolds is $\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6$.*

5.2. The exterior derivative of the structure 1-form

According to (2.3), the 2-form $\mathrm{d}\eta$ on $(\mathcal{M},\phi,\xi,\eta,g)$ can be decomposed as follows:

$$\mathrm{d}\eta=\ell_1(\mathrm{d}\eta)+\ell_3(\mathrm{d}\eta),$$

$$(5.10)\qquad \begin{array}{l} \ell_1(\mathrm{d}\eta)(x,y)=\mathrm{d}\eta(hx,hy), \qquad \ell_2(\mathrm{d}\eta)(x,y)=0,\\ \ell_3(\mathrm{d}\eta)(x,y)=\mathrm{d}\eta(vx,hy)+\mathrm{d}\eta(hx,vy). \end{array}$$

The next proposition gives geometric conditions for vanishing the components of $\mathrm{d}\eta$.

Proposition 5.3. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) the paracontact distribution $\mathcal{H}$ of $(\mathcal{M},\phi,\xi,\eta,g)$ is involutive if and only if $\ell_1(\mathrm{d}\eta)=0$;

b) the integral curves of ξ are geodesics on $(\mathcal{M},\phi,\xi,\eta,g)$ if and only if $\ell_3(\mathrm{d}\eta)=0$.

Proof. It is said that $\mathcal{H}$ is an involutive distribution when $[x,y]$ belongs to $\mathcal{H}$ for $x,y\in\mathcal{H}$, i.e. $\eta([hx,hy])=0$ holds for arbitrary x and y. By virtue of the identity $\eta([hx,hy])=-\mathrm{d}\eta(hx,hy)$ and (5.10), we have the equality $\eta([hx,hy])=-\ell_1(\mathrm{d}\eta)(x,y)$. This accomplishes the proof of a).

As it is known, the integral curves of ξ are geodesics on $(\mathcal{M},\phi,\xi,\eta,g)$ if and only if $\nabla_\xi\xi$ vanishes. The equality (5.10) implies that $\ell_3(\mathrm{d}\eta)=0$ is valid if and only if $\mathrm{d}\eta(x,\xi)=0$ holds. Applying (5.4) and (2.6), we obtain the equality $\mathrm{d}\eta(x,\xi)=-g(\nabla_\xi\xi,x)$. Then, it is clear that b) holds true. □

Next, we compute $\mathrm{d}\eta$ on the considered manifold belonging to each of the basic classes and obtain the following

Proposition 5.4. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) $\mathrm{d}\eta(x,y)=0$ if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_i$ $(i=1,\dots,6,9,10)$ or to their direct sums;

b) $\mathrm{d}\eta(x,y)=\ell_1(\mathrm{d}\eta)(x,y)=2\big(\nabla_x\eta\big)y$ if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_7$, $\mathcal{F}_8$ or $\mathcal{F}_7\oplus\mathcal{F}_8$;

c) $\mathrm{d}\eta(x,y)=\ell_3(\mathrm{d}\eta)(x,y)=-\eta(x)\omega(\phi y)+\eta(y)\omega(\phi x)$ if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_{11}$.

By Proposition 5.3 and Proposition 5.4, we get the following theorem, which gives a geometric characteristic of the manifolds of some classes with respect to the form of $\mathrm{d}\eta$.

Theorem 5.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) the structure 1-form η is closed if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_i$ $(i=1,\dots,6,9,10)$ or to their direct sums;

b) the paracontact distribution $\mathcal{H}$ of $(\mathcal{M},\phi,\xi,\eta,g)$ is involutive if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_i$ $(i=1,\dots,6,9,10,11)$ or to their direct sums;

c) the integral curves of the structure vector field ξ *are geodesics on* $(\mathcal{M}, \phi, \xi, \eta, g)$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_i$ $(i = 1, \ldots, 10)$ *or to their direct sums.*

5.3. The Nijenhuis torsion of the structure endomorphism of the paracontact distribution

Proposition 5.5. *Let* $(\mathcal{M}, \phi, \xi, \eta, g)$ *be an almost paracontact almost paracomplex Riemannian manifold. Then for the Nijenhuis torsion of* ϕ *we have:*

a) $[\phi, \phi](x, y) = 0$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_i$ $(i = 1, 2, 4, 5, 6, 11)$ *or to their direct sums;*

b) $[\phi, \phi](x, y) = -2\left\{\phi\left(\nabla_{\phi x}\phi\right)\phi y + \phi\left(\nabla_{\phi^2 x}\phi\right)\phi^2 y\right\}$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_3$*;*

c) $[\phi, \phi](x, y) = -2\left(\nabla_x\eta\right)(y)\,\xi$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_7$*;*

d) $[\phi, \phi](x, y) = -2\left\{\eta(x)\nabla_y\xi - \eta(y)\nabla_x\xi - \left(\nabla_x\eta\right)(y)\,\xi\right\}$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_8$*;*

e) $[\phi, \phi](x, y) = -2\left\{\eta(x)\nabla_y\xi - \eta(y)\nabla_x\xi\right\}$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_9$*;*

f) $[\phi, \phi](x, y) = -\eta(x)\,\phi\left(\nabla_\xi\phi\right)y + \eta(y)\,\phi\left(\nabla_\xi\phi\right)x$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_{10}$*.*

Proof. Using (5.6) and the forms of the Nijenhuis tensor N and the 2-form $\mathrm{d}\eta$, given in Proposition 5.2 and Proposition 5.4, respectively, we get the statements from the above by direct computations. □

Now, we specialize the form of $[\phi, \phi]$ for the class of paracontact almost paracomplex Riemannian manifolds and we find its subclasses of manifolds whose almost paracomplex structure ϕ on $\mathcal{H}$ is integrable.

Theorem 5.3. *Let* $(\mathcal{M}, \phi, \xi, \eta, g)$ *be a paracontact almost paracomplex Riemannian manifold. Then it has:*

a) an integrable almost paracomplex structure ϕ*, i.e.* $[\phi, \phi] = 0$*, if and only if the manifold belongs to* $\mathcal{F}_4{}'$ *or to its direct sums with* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*;*

b) an nonintegrable almost paracomplex structure ϕ*, i.e.* $[\phi, \phi] \neq 0$ *if and only if the manifold belongs to the rest of the classes, given in Theorem 4.1.*

Proof. We establish the truthfulness of the statements using Theorem 4.1 and Proposition 5.5. □

Bearing in mind Theorem 5.3 a), the manifolds from the classes $\mathcal{F}_4{}'$, $\mathcal{F}_1 \oplus \mathcal{F}_4{}'$, $\mathcal{F}_2 \oplus \mathcal{F}_4{}'$ and $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4{}'$ we call *paracontact paracomplex Riemannian manifolds.* In the other case, the manifolds from the rest of the classes, given in Theorem 4.1, we call *paracontact almost paracomplex Riemannian manifolds.*

6. The associated Nijenhuis tensor

By analogy with the skew-symmetric Lie bracket (the commutator), determined by $[x,y]=\nabla_x y-\nabla_y x$, let us consider the symmetric braces (the anticommutator), defined by $\{x,y\}=\nabla_x y+\nabla_y x$ as in [5]. Bearing in mind the definition of the Nijenhuis torsion $\big[\check{P},\check{P}\big]$ of an almost paracomplex structure $\check{P}$ on $\check{\mathcal{M}}$, we give a definition of a tensor $\big\{\check{P},\check{P}\big\}$ of type $(1,2)$ as follows:

$$\big\{\check{P},\check{P}\big\}(\check{x},\check{y})=\{\check{x},\check{y}\}+\big\{\check{P}\check{x},\check{P}\check{y}\big\}-\check{P}\big\{\check{P}\check{x},\check{y}\big\}-\check{P}\big\{\check{x},\check{P}\check{y}\big\},$$

where the action of $\check{P}$ is given in (5.1) and the anticommutator on the tangent bundle of $\check{\mathcal{M}}$ is determined by:

$$\big\{(x,a\partial_r),(y,b\partial_r)\big\}=\big(\{x,y\},(x(b)+y(a))\,\partial_r\big).$$

We call $\big\{\check{P},\check{P}\big\}$ an *associated Nijenhuis tensor of the almost paracomplex manifold* $\big(\check{\mathcal{M}},\check{P}\big)$. Obviously, this tensor is symmetric with respect to its arguments, i.e. $\big\{\check{P},\check{P}\big\}(\check{x},\check{y})=\big\{\check{P},\check{P}\big\}(\check{y},\check{x})$.

Since the almost paracomplex manifold $\big(\check{\mathcal{M}},\check{P}\big)$ is generated from the almost paracontact almost paracomplex manifold $(\mathcal{M},\phi,\xi,\eta)$, we seek to express the associated Nijenhuis tensor $\big\{\check{P},\check{P}\big\}$ by tensors for the structure (ϕ,ξ,η). Since $\big\{\check{P},\check{P}\big\}$ is a tensor field of type (1,2) on $\check{\mathcal{M}}$, it suffices to compute the following two expressions:

$$\begin{aligned}\big\{\check{P},\check{P}\big\}\big((x,0),(y,0)\big)&=\big\{(x,0),(y,0)\big\}+\big\{\check{P}(x,0),\check{P}(y,0)\big\}-\check{P}\big\{\check{P}(x,0),(y,0)\big\}\\&\qquad-\check{P}\big\{(x,0),\check{P}(y,0)\big\}\\&=\big(\{x,y\},0\big)+\big\{(\phi x,\eta(x)\zeta),(\phi y,\eta(y)\zeta)\big\}\\&\quad-\check{P}\big\{(\phi x,\eta(x)\zeta),(y,0)\big\}-\check{P}\big\{(x,0),(\phi y,\eta(y)\zeta)\big\}\\&=\big(\{\phi,\phi\}(x,y)-(\mathfrak{L}_\xi g)(x,y)\xi,\\&\qquad((\mathfrak{L}_\xi g)(\phi x,y)+(\mathfrak{L}_\xi g)(x,\phi y))\,\zeta\big)\end{aligned}$$

and

$$\begin{aligned}\big\{\check{P},\check{P}\big\}\big((x,0),(0,\zeta)\big)&=\big\{(x,0),(0,\zeta)\big\}+\big\{\check{P}(x,0),\check{P}(0,\zeta)\big\}-\check{P}\big\{\check{P}(x,0),(0,\zeta)\big\}\\&\qquad-\check{P}\big\{(x,0),\check{P}(0,\zeta)\big\}\\&=\big\{(\phi x,\eta(x)\zeta),(\xi,0)\big\}-\check{P}\big\{(\phi x,\eta(x)\zeta),(0,\zeta)\big\}\\&\qquad-\check{P}\big\{(x,0),(\xi,0)\big\}\\&=(\{\phi x,\xi\}-\phi\{x,\xi\},(\mathfrak{L}_\xi g)(x,\xi)\zeta)\,.\end{aligned}$$

In the latter expressions, we use the Lie derivative $(\mathfrak{L}_\xi g)(x,y)$, determined by (4.2), of the Riemannian metric g of $(\mathcal{M},\phi,\xi,\eta,g)$.

Then, we define the following four tensors $\widehat{N}^{(k)}$ $(k=1,2,3,4)$ of type (1,2), (0,2), (1,1), (0,1), respectively:

$$\begin{aligned}&\widehat{N}^{(1)}(x,y)=\{\phi,\phi\}(x,y)-(\mathfrak{L}_\xi g)(x,y)\xi,\\&\widehat{N}^{(2)}(x,y)=(\mathfrak{L}_\xi g)(\phi x,y)+(\mathfrak{L}_\xi g)(x,\phi y),\\&\widehat{N}^{(3)}(x)=\{\phi x,\xi\}-\phi\{x,\xi\},\\&\widehat{N}^{(4)}(x)=(\mathfrak{L}_\xi g)(x,\xi),\end{aligned}\tag{6.1}$$

where $\{\phi,\phi\}$ is the symmetric tensor of type (1,2) determined by:

$$\{\phi,\phi\}(x,y)=\{\phi x,\phi y\}+\phi^2\{x,y\}-\phi\{\phi x,y\}-\phi\{x,\phi y\}. \tag{6.2}$$

By direct converting their definitions, we find relations between the four tensors $\widehat{N}^{(k)}$ as follows:

$$\begin{aligned}
&\widehat{N}^{(2)}(x,y)=-\eta\left(\widehat{N}^{(1)}(x,\phi y)\right)-\eta\left(\widehat{N}^{(1)}(\phi x,\xi)\right)\eta(y),\\
&\widehat{N}^{(3)}(x)=\widehat{N}^{(1)}(\phi x,\xi)-\eta(x)\phi\widehat{N}^{(1)}(\xi,\xi)\\
&\widehat{N}^{(4)}(x)=-\eta\left(\widehat{N}^{(1)}(x,\xi)\right)=\tfrac{1}{2}g\left(\widehat{N}^{(1)}(\xi,\xi),x\right),\\
&\widehat{N}^{(4)}(x)=\widehat{N}^{(2)}(\phi x,\xi),\qquad \widehat{N}^{(4)}(x)=-\eta\left(\widehat{N}^{(3)}(\phi x)\right).
\end{aligned} \tag{6.3}$$

Theorem 6.1. *For an almost paracontact almost paracomplex Riemannian manifold we have:*

a) if $\widehat{N}^{(1)}$ vanishes, then all the other tensors $\widehat{N}^{(2)}$, $\widehat{N}^{(3)}$ and $\widehat{N}^{(4)}$ vanish;

b) if any one of $\widehat{N}^{(2)}$ and $\widehat{N}^{(3)}$ vanishes, then $\widehat{N}^{(4)}$ vanishes.

Proof. The statements above are consequences of the relations (6.3) between $\widehat{N}^{(k)}$ $(k=1,2,3,4)$. □

Therefore, $\widehat{N}^{(1)}$ plays a main role between them and we denote it simply by $\widehat{N}$, i.e.

$$\widehat{N}(x,y)=\{\phi,\phi\}(x,y)-(\mathfrak{L}_\xi g)(x,y)\xi \tag{6.4}$$

and we call it an *associated Nijenhuis tensor of the structure* (ϕ,ξ,η,g). Obviously, $\widehat{N}$ is symmetric, i.e. $\widehat{N}(x,y)=\widehat{N}(y,x)$. Applying the expressions (6.2), (4.2) and (6.4), the associated Nijenhuis tensor has the following form in terms of $\nabla\phi$ and $\nabla\eta$:

$$\widehat{N}(x,y)=(\nabla_{\phi x}\phi)y+(\nabla_{\phi y}\phi)x-\phi(\nabla_x\phi)y-\phi(\nabla_y\phi)x-(\nabla_x\eta)y\,\xi-(\nabla_y\eta)x\,\xi.$$

The corresponding tensor of type (0,3) is defined by $\widehat{N}(x,y,z)=g\left(\widehat{N}(x,y),z\right)$. According to (2.4) and (2.6), we express $\widehat{N}$ in terms of the fundamental tensor F as follows:

$$\begin{aligned}\widehat{N}(x,y,z)=&F(\phi x,y,z)+F(\phi y,x,z)-F(x,y,\phi z)-F(y,x,\phi z)\\&+\eta(z)\left\{F(x,\phi y,\xi)+F(y,\phi x,\xi)\right\}.\end{aligned} \tag{6.5}$$

Proposition 6.1. *The associated Nijenhuis tensor on an almost paracontact almost paracomplex Riemannian manifold has the following properties:*

$$\begin{aligned}
&\widehat{N}(\phi^2x,\phi y,\phi z)=-\widehat{N}(\phi^2x,\phi^2y,\phi^2z), &&\widehat{N}(\phi^2x,\phi^2y,\phi^2z)=\widehat{N}(\phi x,\phi y,\phi^2z),\\
&\widehat{N}(x,\phi^2y,\phi^2z)=-\widehat{N}(x,\phi y,\phi z), &&\widehat{N}(\phi^2x,\phi^2y,z)=\widehat{N}(\phi x,\phi y,z),\\
&\widehat{N}(\xi,\phi y,\phi z)=-\widehat{N}(\xi,\phi^2y,\phi^2z), &&\widehat{N}(\phi x,\phi y,\xi)=\widehat{N}(\phi^2x,\phi^2y,\xi).
\end{aligned}$$

Table 6.1: Associated Nijenhuis tensors

	$\widehat{N}^{(1)}(x,y,z)$	$\widehat{N}^{(2)}(x,y)$	$\widehat{N}^{(3)}(x,y)$	$\widehat{N}^{(4)}(x)$
$\mathcal{F}_1$	$\frac{2}{n}\Big\{g(x,\phi y)\theta(\phi^2 z)-g(\phi x,\phi y)\theta(\phi z)\Big\}$	0	0	0
$\mathcal{F}_2$	$-2\Big\{F(\phi x,\phi y,\phi z)+F(\phi^2 x,\phi^2 y,\phi z)\Big\}$	0	0	0
$\mathcal{F}_3$	0	0	0	0
$\mathcal{F}_4$	$\frac{2}{n}\theta(\xi)g(x,\phi y)\eta(z)$	$-\frac{2}{n}\theta(\xi)g(\phi x,\phi y)$	0	0
$\mathcal{F}_5$	$\frac{2}{n}\theta^*(\xi)g(\phi x,\phi y)\eta(z)$	$-\frac{2}{n}\theta^*(\xi)g(x,\phi y)$	0	0
$\mathcal{F}_6$	$4F(x,\phi y,\xi)\eta(z)$	$-4F(x,y,\xi)$	0	0
$\mathcal{F}_7$	0	0	0	0
$\mathcal{F}_8$	$-2\Big\{\eta(x)F(y,\phi z,\xi)+\eta(y)F(x,\phi z,\xi)\Big\}$	0	$2F(x,y,\xi)$	0
$\mathcal{F}_9$	$-2\Big\{\eta(x)F(y,\phi z,\xi)+\eta(y)F(x,\phi z,\xi)\Big\}$	0	$2F(x,y,\xi)$	0
$\mathcal{F}_{10}$	$-\eta(x)F(\xi,y,\phi z)-\eta(y)F(\xi,x,\phi z)$	0	$F(\xi,x,y)$	0
$\mathcal{F}_{11}$	$\eta(z)\Big\{\eta(x)\omega(\phi y)+\eta(y)\omega(\phi x)\Big\}$ $-2\eta(x)\eta(y)\omega(\phi z)$	$-\eta(x)\omega(y)-\eta(y)\omega(x)$	$\omega(x)\eta(y)+2\eta(x)\omega(y)$	$-\omega(\phi x)$

Proof. The results follow form the properties (2.5) of F and the expression (6.5). □

Applying (6.5) to (6.3), we give the form of $\widehat{N}^{(2)}$, $\widehat{N}^{(3)}$ and $\widehat{N}^{(4)}$ in terms of F:

$$
\begin{aligned}
&\widehat{N}^{(2)}(x,y)=-F(x,y,\xi)-F(y,x,\xi)-F(\phi x,\phi y,\xi)-F(\phi y,\phi x,\xi),\\
&\widehat{N}^{(3)}(x,y)=F(\xi,x,y)+F(x,y,\xi)-F(\phi x,\phi y,\xi),\\
&\widehat{N}^{(4)}(x)=-F(\xi,\phi x,\xi),
\end{aligned}
\tag{6.6}
$$

where we use the denotation $\widehat{N}^{(3)}(x,y)=g\left(\widehat{N}^{(3)}(x),y\right)$.

Proposition 6.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an $\mathcal{F}_i$-manifold ($i=1,2,\dots,11$). Then the four tensors $\widehat{N}^{(k)}$ ($k=1,2,3,4$) on this manifold have the form in the respective cases, given in Table 6.1.*

Proof. The calculations are made, using (6.5), (6.6) and the expression (2.8) of each of F_i for the corresponding class $\mathcal{F}_i$. □

As a result of Proposition 6.2, we establish the truthfulness of the following

Theorem 6.2. *An almost paracontact almost paracomplex Riemannian manifold $(\mathcal{M},\phi,\xi,\eta,g)$ has:*

a) vanishing $\widehat{N}^{(1)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_3$, $\mathcal{F}_7$ or to their direct sum;

b) vanishing $\widehat{N}^{(2)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_7$, …, $\mathcal{F}_{10}$ or to their direct sums;

c) vanishing $\widehat{N}^{(3)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_7$ or to their direct sums;

d) vanishing $\widehat{N}^{(4)}$ *if and only if it belongs to some of the basic classes* $\mathcal{F}_1, \dots, \mathcal{F}_{10}$ *or to their direct sums.*

By virtue of Theorem 6.2, we obtain the following

Corollary 6.1. *The class of almost paracontact almost paracomplex Riemannian manifolds with a vanishing associated Nijenhuis tensor* $\widehat{N}$ *is* $\mathcal{F}_3 \oplus \mathcal{F}_7$.

7. The pair of Nijenhuis tensors and the classification of the considered manifolds

In the previous two sections, by (5.7) and (6.5), we give the expressions of the Nijenhuis tensor N and its associated $\widehat{N}$ by the tensor F, respectively. Here, we find how the fundamental tensor F is determined by the pair of Nijenhuis tensors. Since F is used for classifying the manifolds studied, we can expressed the classes $\mathcal{F}_i$ $i = (1, 2, \dots, 11)$ only by the pair $(N, \widehat{N})$.

Theorem 7.1. *Let* $(\mathcal{M}, \phi, \xi, \eta, g)$ *be an almost paracontact almost paracomplex Riemannian manifold. Then its fundamental tensor is expressed by* N *and* $\widehat{N}$ *by the formula:*

$$\begin{aligned} F(x,y,z) = {}& \tfrac{1}{4}\big[N(\phi x,y,z) + N(\phi x,z,y) + \widehat{N}(\phi x,y,z) + \widehat{N}(\phi x,z,y)\big] \\ & - \tfrac{1}{2}\eta(x)\big[N(\xi,y,\phi z) + \widehat{N}(\xi,y,\phi z) + \eta(z)\widehat{N}(\xi,\xi,\phi y)\big]. \end{aligned} \tag{7.1}$$

Proof. Taking the sum of (5.7) and (6.5), we obtain:

$$F(\phi x,y,z) - F(x,y,\phi z) = \frac{1}{2}\big[N(x,y,z) + \widehat{N}(x,y,z)\big] - \eta(z)F(x,\phi y,\xi). \tag{7.2}$$

The identities (2.5) together with (2.1) imply:

$$F(x,y,\phi z) + F(x,z,\phi y) = \eta(z)F(x,\phi y,\xi) + \eta(y)F(x,\phi z,\xi). \tag{7.3}$$

A suitable combination of (7.2) and (7.3) yields:

$$F(\phi x,y,z) = \frac{1}{4}\big[N(x,y,z) + N(x,z,y) + \widehat{N}(x,y,z) + \widehat{N}(x,z,y)\big]. \tag{7.4}$$

Applying (2.1), we obtain from (7.4) the following:

$$\begin{aligned} F(x,y,z) = {}& \tfrac{1}{4}\big[N(\phi x,y,z) + N(\phi x,z,y) + \widehat{N}(\phi x,y,z) + \widehat{N}(\phi x,z,y)\big] \\ & + \eta(x)F(\xi,y,z). \end{aligned} \tag{7.5}$$

Set $x = \xi$ and $z \to \phi z$ into (7.2) and use (2.1) to get:

$$F(\xi,y,z) = -\frac{1}{2}\big[N(\xi,y,\phi z) + \widehat{N}(\xi,y,\phi z)\big] + \eta(z)\omega(y). \tag{7.6}$$

Finally, using (6.5) and the general identities $\omega(\xi) = 0$, we obtain:

$$\omega(z) = -\frac{1}{2}\widehat{N}(\xi,\xi,\phi z). \tag{7.7}$$

Substitute (7.7)into (7.6) and the obtained identity insert into (7.5) to get (7.1). □

Corollary 7.1. *The class of almost paracontact almost paracomplex Riemannian manifolds with vanishing tensors N and $\widehat{N}$ is the special class $\mathcal{F}_0$.*

8. A family of Lie groups as manifolds of the studied type

Let $\mathcal{L}$ be a $(2n+1)$-dimensional real connected Lie group and let its associated Lie algebra with a global basis $\{E_0, E_1, \dots, E_{2n}\}$ of left invariant vector fields on $\mathcal{L}$ be defined by:

$$[E_0, E_i] = -a_i E_i - a_{n+i} E_{n+i}, \qquad [E_0, E_{n+i}] = -a_{n+i} E_i + a_i E_{n+i}, \tag{8.1}$$

where $a_1, \dots, a_{2n}$ are real constants and $[E_j, E_k] = 0$ in other cases.

Let (ϕ, ξ, η) be an almost paracontact almost paracomplex structure determined for any $i \in \{1, \dots, n\}$ by:

$$\begin{array}{lll} \phi E_0 = 0, & \phi E_i = E_{n+i}, & \phi E_{n+i} = E_i, \\ \xi = E_0, & \eta(E_0) = 1, & \eta(E_i) = \eta(E_{n+i}) = 0. \end{array} \tag{8.2}$$

Let g be a Riemannian metric defined by:

$$\begin{array}{l} g(E_0, E_0) = g(E_i, E_i) = g(E_{n+i}, E_{n+i}) = 1, \\ g(E_0, E_j) = g(E_j, E_k) = 0, \end{array} \tag{8.3}$$

where $i \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, 2n\}$, $j \neq k$. Thus, since (2.1) is satisfied, the induced $(2n+1)$-dimensional manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ is an almost paracontact almost paracomplex Riemannian manifold.

Let us remark that in [8] the same Lie group is considered with an appropriate almost contact structure and a compatible Riemannian metric. Then, the generated almost cosymplectic manifold is studied. On the other hand, in [3], the same Lie group is equipped with an almost contact structure and B-metric. Then, the obtained manifold is characterized. Moreover, in [4], the case of the lowest dimension is considered and properties of the constructed manifold are determined.

Let us consider the constructed almost paracontact almost paracomplex Riemannian manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ of dimension 3, i.e. for $n = 1$.

According to (8.1) and (8.3) for $n = 1$, by the Koszul equality

$$2g\left(\nabla_{E_i} E_j, E_k\right) = g\left([E_i, E_j], E_k\right) + g\left([E_k, E_i], E_j\right) + g\left([E_k, E_j], E_i\right)$$

for the Levi-Civita connection ∇ of g, we obtain:

$$\begin{array}{ll} \nabla_{E_1} E_0 = a_1 E_1 + a_2 E_2, & \nabla_{E_2} E_0 = a_2 E_1 - a_1 E_2, \\ \nabla_{E_1} E_1 = -\nabla_{E_2} E_2 = -a_1 E_0, & \nabla_{E_1} E_2 = \nabla_{E_2} E_1 = -a_2 E_0, \end{array} \tag{8.4}$$

and the others $\nabla_{E_i} E_j$ are zero.

Then, using (8.4), (8.2), (2.4) and (3.3), we get the following components $F_{ijk} = F(E_i, E_j, E_k)$ of the fundamental tensor:

$$F_{101} = F_{110} = F_{202} = F_{220} = -a_2, \qquad F_{102} = F_{120} = -F_{201} = -F_{210} = -a_1,$$

and the other components of F are zero. Thus, we have the expression of F for arbitrary vectors $x = x^i E_i$, $y = y^i E_i$, $z = z^i E_i$ as follows:

$$F(x,y,z) = -a_2 \left\{x^1 \left(y^0 z^1 + y^1 z^0\right) + x^2 \left(y^0 z^2 + y^2 z^0\right)\right\} \\ -a_1 \left\{x^1 \left(y^0 z^2 + y^2 z^0\right) - x^2 \left(y^0 z^1 + y^1 z^0\right)\right\}. \tag{8.5}$$

Bearing in mind the latter equality, we obtain that F has the following form:

$$F(x,y,z) = F_4(x,y,z) + F_9(x,y,z),$$

by virtue of (3.3) for $\mu = -a_1$, $\theta_0 = -2a_2$. Therefore, we have proved the following

Proposition 8.1. *The constructed 3-dimensional almost paracontact almost paracomplex Riemannian manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ belongs to:*

a) $\mathcal{F}_4 \oplus \mathcal{F}_9$ if and only if $a_1 \neq 0$, $a_2 \neq 0$;

b) $\mathcal{F}_4$ if and only if $a_1 = 0$, $a_2 \neq 0$;

c) $\mathcal{F}_9$ if and only if $a_1 \neq 0$, $a_2 = 0$;

d) $\mathcal{F}_0$ if and only if $a_1 = 0$, $a_2 = 0$.

Finally, we get the following

Proposition 8.2. *The constructed 3-dimensional almost paracontact almost paracomplex Riemannian manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ has the following properties:*

a) It has vanishing $N^{(4)}$ and $\widehat{N}^{(4)}$;

b) It is a normal almost paracontact almost paracomplex Riemannian manifold with vanishing $\widehat{N}^{(3)}$ if and only if $a_1 = 0$ and arbitrary a_2;

c) It is a para-Sasakian paracomplex Riemannian manifold if and only if $a_1 = 0$, $a_2 = 1$;

d) It has vanishing $\widehat{N}^{(2)}$ if and only if $a_2 = 0$ and arbitrary a_1.

Proof. According to (8.5), (3.3) and Proposition 5.2, we find the following form of the Nijenhuis tensor of $(\mathcal{L}, \phi, \xi, \eta, g)$:

$$N(x,y,z) = -2a_1 \left\{\left(x^1 y^2 - x^2 y^1\right) z^0 + \left(x^0 y^1 - x^1 y^0\right) z^1 - \left(x^0 y^2 - x^2 y^0\right) z^2\right\}.$$

From the latter equality and (5.8) (or alternatively from (8.5) and (5.9)), we have:

$$N^{(2)}(x,y) = 2a_1(x^1 y^1 - x^2 y^2), \quad N^{(3)}(x,y) = 2a_1(x^1 y^2 - x^2 y^1), \quad N^{(4)}(x) = 0.$$

Similarly, for the associated Nijenhuis tensor of $(\mathcal{L}, \phi, \xi, \eta, g)$ we obtain:

$$\widehat{N}(x,y,z) = -4a_2 \left(x^1 y^2 + x^2 y^1\right) z^0 + 2a_1 \left\{\left(x^0 y^1 + x^1 y^0\right) z^1 - \left(x^0 y^2 + x^2 y^0\right) z^2\right\}.$$

By virtue of (6.3) and the equality from above (or in other way by (8.5) and (6.6)), we get:

$$\widehat{N}^{(2)}(x,y) = 4a_2\left(x^1y^1 + x^2y^2\right), \quad \widehat{N}^{(3)}(x,y) = -2a_1\left(x^1y^2 - x^2y^1\right), \quad \widehat{N}^{(4)}(x) = 0.$$

As a conclusion, the obtained results imply the propositions in a), b) and d). Moreover, the case of the ${\mathcal{F}_4}'$-manifold, i.e. the proposition in c), follows from Proposition 8.1 b). □

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