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ON GENERALIZED ϕ-RECURRENT AND GENERALIZED CONCIRCULARLY ϕ-RECURRENT $N(\kappa)$-PARACONTACT METRIC MANIFOLDS

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Abstract. The purpose of the present paper is to study generalized ϕ-recurrent, generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifolds and generalized ϕ-recurrent paracontact metric manifolds of constant curvature.
Keywords:generalized ϕ-recurrent, generalized concirculary ϕ -recurrent, $N(\kappa)$-paracontact metric manifold.

1. Introduction

Almost paracontact metric structures are the natural odd-dimensional analogue to almost paraHermitian structures, just like almost contact metric structures correspond to the almost Hermitian ones. The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [6] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy, [10]. An important class among paracontact metric manifolds is that of the κ-spaces, which satisfy the nullity condition [2]. This class includes the para-Sasakian manifolds [6, 10], the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all X, Y vector fields on the manifold [11], etc.

Let M be an $2n+1$-dimensional connected semi-Riemannian manifold with semi-Riemannian metric g and Levi-Civita connection ∇. M is called *locally symmetric* if its curvature tensor is parallel with respect to ∇. The notion of locally symmetric manifold has been weakend such as *recurrent manifold* by Walker [9], in 1977 Takahashi [8] introduced the notion of *local ϕ -symmetry* on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry, De et al. [3] introduced and studied the notion of *ϕ-recurrent* Sasakian manifold. Then in [4] and [7], De and Gazi and Peyghan et al. studied ϕ-recurrent $N(\kappa)$-contact metric manifolds. Dubey [5] introduced the notion of *generalized recurrent* manifold.

Motivated by these considerations, the author make the first contribution to study generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifolds (which includes both the

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notion of local ϕ-symmetry and also ϕ-recurrence) and generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifolds.

The paper is organized as follows:

Section 2 is preliminary section, where we recall basic facts which we will need throughout the paper. In Section 3, we prove that a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold for $\kappa \neq -1, 0$. We show that in a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold, the characteristic vector field ξ and the vector field $\rho_1\kappa + \rho_2$ associated to the 1-form $A\kappa + B$ are co-directional. We find the relation between associated 1-forms A and B for a three dimensional generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold. In Section 4, we mainly give the relation between associated 1-forms A and B in a generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold (M^{2n+1}, g) of constant curvature $c \neq 0$. In Section 5, we prove that a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold for $\kappa \neq -1, 0$. We give the relation between associated 1-forms A and B for a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold and we show that in a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold, the characteristic vector field ξ and the vector field $\rho_1 c + \rho_2$ associated to the 1-form $Ac + B$ are co-directional. Finally, we show that for a three dimensional generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold, r is not necessarily be a constant.

2. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold and ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a one-form on M. Then (ϕ, ξ, η) is called an *almost paracontact structure* on M if

(i) $\phi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1,$

(ii) the tensor field ϕ induces an almost paracomplex structure on the distribution $D = \ker\, \eta$, that is the eigendistributions $D^{\pm}$, corresponding to the eigenvalues ± 1, have equal dimensions, $dim\, D^+ = dim\, D^- = n$.

The manifold M is said to be an *almost paracontact manifold* if it is endowed with an almost paracontact structure [10].

Let M be an almost paracontact manifold. M will be called an *almost paracontact metric manifold* if it is additionally endowed with a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.1}$$

For such manifold, we have

$$\eta(X) = g(X, \xi), \phi(\xi) = 0, \eta \circ \phi = 0. \tag{2.2}$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.3}$$

usually called *fundamental form*.

For an almost paracontact manifold, there exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a *ϕ-basis.*

On an almost paracontact manifold, one defines the $(1,2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X,Y) = [\phi,\phi](X,Y) - 2d\eta(X,Y)\xi, \tag{2.4}$$

where $[\phi,\phi]$ is the *Nijenhuis torsion* of ϕ

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be *normal* [10]. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$

$$J(X, \lambda\frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt}),$$

is integrable.

If $d\eta(X,Y) = g(X, \phi Y)$, then (M, ϕ, ξ, η, g) is said to be *paracontact metric manifold.* In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [10] that h anti-commutes with ϕ and satisfies

$$i)h\xi = 0, \quad ii)trh = trh\phi = 0, \quad iii)\nabla\xi = -\phi + \phi h, \tag{2.5}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).

Moreover $h = 0$ if and only if ξ is a Killing vector field. In this case (M, ϕ, ξ, η, g) is said to be a *K-paracontact manifold.* Similarly as in the class of almost contact metric manifolds [1], a normal almost paracontact metric manifold will be called *para-Sasakian* if $\Phi = d\eta$.

On an almost paracontact metric manifold M, if the Ricci operator satisfies

$$Q = \alpha id + \beta\eta \otimes \xi,$$

where both α and β are smooth functions, then the manifold is said to be an *η-Einstein manifold.* An η-Einstein manifold with β vanishing and α a constant is obviously an Einstein manifold.

The *κ-nullity distribution* $N(\kappa)$ of a semi-Riemannian manifold M is defined by

$$N(\kappa) : p \to N_p(\kappa) = \{Z \in T_pM \mid R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)\}, \tag{2.6}$$

for some real constant κ. If the characteristic vector field ξ belongs to $N(\kappa)$, then we call a paracontact metric manifold an *$N(\kappa)$-paracontact metric manifold.* For a $N(\kappa)$-paracontact metric manifold [2] we have,

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y), \tag{2.7}$$
$$S(X,\xi) = 2n\kappa\eta(X), \tag{2.8}$$
$$h^2 = (1+\kappa)\phi^2. \tag{2.9}$$

for all X, Y vector fields on M, where κ is constant and S is the Ricci tensor.

Lemma 2.1. *[2]In any $(2n+1)$-dimensional paracontact (κ,μ)-manifold (M,ϕ,ξ,η,g) such that $\kappa\neq -1$, the Ricci operator Q is given by*

$$Q=(2(1-n)+n\mu)I+(2(n-1)+\mu)h+(2(n-1)+n(2\kappa-\mu))\eta\otimes\xi. \tag{2.10}$$

Using (2.10), we have

$$S(\phi X,\phi Y)=S(X,Y)-4(1-n)g(X,Y)+(4(1-n)-2n\kappa)\eta(X)\eta(Y). \tag{2.11}$$

Definition 2.1. A $N(\kappa)$-paracontact metric manifold is said to be a generalized ϕ-recurrent if its curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z+B(W)(g(Y,Z)X-g(X,Z)Y), \tag{2.12}$$

where A and B are two 1-forms, B is non zero and they are defined by

$$A(X)=g(X,\rho_1),\quad B(X)=g(X,\rho_2), \tag{2.13}$$

where ρ_1 and ρ_2 are vector fields associated with 1-forms A,B respectively.

Definition 2.2. A (2n+1)-dimensional $N(\kappa)$-paracontact metric manifold is called a generalized concircular ϕ-recurrent if its concircular curvature tensor C

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y], \tag{2.14}$$

satisfies the condition

$$\phi^2((\nabla_W C)(X,Y)Z)=A(W)C(X,Y)Z+B(W)(g(Y,Z)X-g(X,Z)Y), \tag{2.15}$$

where A and B are defined as (2.13) and $r=tr(S)$ is the scalar curvature.

In the above definitions, X,Y,Z,W are arbitrary vector fields and not necessarily orthogonal to ξ.

Remark 2.1. A flat manifold satisfies $R=0$ and $\nabla R=0$, so flat manifolds are trivial examples of generalized ϕ-recurrent paracontact metric manifolds.

3. Generalized ϕ-recurrent $N(\kappa)$ -paracontact metric manifolds

Theorem 3.1. *For $\kappa\neq -1,0$, a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1},g) is an η-Einstein manifold.*

Proof. In view of (2.12), we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z-\eta((\nabla_W R)(X,Y)Z)\xi\\ =A(W)R(X,Y)Z+B(W)(g(Y,Z)X-g(X,Z)Y).\end{aligned} \tag{3.1}$$

Taking the inner product on both sides of (3.1) with U, we obtain

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) &= A(W)g(R(X,Y)Z,U) \\ &\quad +B(W)(g(Y,Z)g(X,U) \\ &\quad -g(X,Z)g(Y,U)). \end{aligned} \tag{3.2}$$

Let e_i, $1 \leq i \leq 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and getting the summation over i, one can get

$$\begin{aligned} &(\nabla_W S)(Y,Z) - \sum_{i=1}^{2n+1} \varepsilon_i \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ &= A(W)S(Y,Z) + 2nB(W)g(Y,Z). \end{aligned} \tag{3.3}$$

Now, let calculate the second term of the left hand side of the above equation by replacing Z by ξ. Using (2.6) and the fact that $(\nabla_W g) = 0$, we get

$$\varepsilon_i g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{3.4}$$

Putting $Z = \xi$ in (3.3) and using (2.8) and (3.4), we obtain

$$(\nabla_W S)(Y,\xi) = 2n\eta(Y)(\kappa A(W) + B(W)). \tag{3.5}$$

Using the property (iii) of (2.5) and (2.8) in $(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi)$, we have

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= 2n\kappa(\nabla_W \eta)(Y) + S(Y,\phi W - \phi hW) \\ &= 2n\kappa g(-\phi W + \phi hW, Y) + S(Y, \phi W - \phi hW). \end{aligned} \tag{3.6}$$

Comparing equations (3.5) and (3.6), we get

$$2n\eta(Y)(\kappa A(W) + B(W)) = 2n\kappa g(-\phi W + \phi hW, Y) + S(Y,\phi W - \phi hW). \tag{3.7}$$

Replacing Y by ϕY in the last equation and using (2.1) and (2.11), we obtain

$$\begin{aligned} 0 &= (2n\kappa - 4(1-n))g(W,Y) + (-2n\kappa + 4(1-n))g(W,hY) \\ &\quad +(-2n\kappa + 4(1-n) - 2n\kappa)\eta(Y)\eta(W) + S(Y,W) - S(Y,hW). \end{aligned} \tag{3.8}$$

Employing (2.9) and (2.10) in (3.8),we get

$$\begin{aligned} S(Y,W) &= 2(-n-\kappa+1)g(W,Y) + 2(n\kappa + n - 1)g(hW,Y) \\ &\quad +2(n(\kappa+1) + \kappa - 1)\eta(Y)\eta(W). \end{aligned} \tag{3.9}$$

Putting $W = hW$ in (3.9) and using again (2.9) and (2.10), we have

$$2\kappa g(hW,Y) = 2n\kappa(\kappa+1)g(W,Y) - 2n\kappa(\kappa+1)\eta(Y)\eta(W).$$

By the assumption of $\kappa \neq 0$, the last equations returns to

$$g(hW,Y) = n(\kappa+1)(g(W,Y) - \eta(Y)\eta(W)). \tag{3.10}$$

Using (3.10) in (3.9), we get

$$S(Y,W) = \alpha g(W,Y) + \beta\eta(Y)\eta(W),$$

where $\alpha = 2[(-n-\kappa+1)+n(\kappa+1)(n\kappa+n-1)]$, $\beta = 2[n(\kappa+1)+(\kappa-1)-n(\kappa+1)(n\kappa+n-1)]$. Hence, we can conclude that the manifold is η-Einstein manifold. □

Theorem 3.2. *For a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g), the characteristic vector field ξ and the vector field $\rho_1\kappa+\rho_2$ associated to the 1-form $A\kappa+B$ are co-directional.*

Proof. Two vector fields P and Q are said to be co-directional if $P = fQ$, where f is a non-zero scalar, that is $g(P, X) = fg(Q, X)$ for all X.

Taking inner product of (3.1) with ξ, we have

$$A(W)g(R(X,Y)Z,\xi) + B(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = 0. \tag{3.11}$$

Then by the use of second Bianchi identity, we can write

$$\begin{aligned} & A(W)g(R(X,Y)Z,\xi) + B(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \\ & +A(Y)g(R(W,X)Z,\xi) + B(Y)(g(X,Z)\eta(W) - g(W,Z)\eta(X)) \\ & +A(X)g(R(Y,W)Z,\xi) + B(X)(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)) \\ = \; & 0. \end{aligned} \tag{3.12}$$

From (2.6), it follows that

$$g(R(X,Y)Z,\xi) = \kappa(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z)). \tag{3.13}$$

Using (3.13) in (3.12), we get

$$\begin{aligned} & \kappa\left\{ \begin{array}{c} A(W)[(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z))] \\ +A(Y)[(-\eta(X)g(W,Z) + \eta(W)g(X,Z))] \\ A(X)[(-\eta(W)g(Y,Z) + \eta(Y)g(W,Z))] \end{array} \right\} \\ & +B(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \\ & +B(Y)(g(X,Z)\eta(W) - g(W,Z)\eta(X)) \\ & +B(X)(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)) \\ = \; & 0. \end{aligned} \tag{3.14}$$

Replacing $Y = Z$ by e_i in (3.14) and taking summation over i, $1 \leq i \leq 2n+1$, we obtain

$$(2n-1)\left[\kappa(A(W)\eta(X) - A(X)\eta(W)) + B(W)\eta(X) - B(X)\eta(W)\right] = 0. \tag{3.15}$$

Putting $X = \xi$ in the last equation, we have

$$\begin{aligned} \kappa(A(W) - \eta(W)\eta(\rho_1)) &= -(B(W) - \eta(W)\eta(\rho_2)) \\ \eta(W)(\kappa\eta(\rho_1) + \eta(\rho_2)) &= \kappa A(W) + B(W). \end{aligned} \tag{3.16}$$

where $\eta(\rho_1) = g(\xi, \rho_1) = A(\xi)$ and $\eta(\rho_2) = g(\xi, \rho_2) = B(\xi)$. From (3.16), we complete the proof of the theorem. □

Theorem 3.3. *Let (M^3, g) be a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Then $B(W) = -\kappa A(W)$.*

Proof. We recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \tag{3.17}$$

where Q is the Ricci-operator, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Let (M^3, g) be a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Replacing Z by ξ in (3.17) and using (2.8), we have

$$R(X,Y)\xi = (2\kappa - \frac{r}{2})(\eta(Y)X - \eta(X)Y) + \eta(Y)QX - \eta(X)QY. \tag{3.18}$$

Comparing (2.7) with (3.18), we get

$$(\kappa - \frac{r}{2})(\eta(Y)X - \eta(X)Y) = \eta(X)QY - \eta(Y)QX. \tag{3.19}$$

Putting $Y = \xi$ in (3.19) and using (2.8), we obtain

$$QX = (\frac{r}{2} - \kappa)X + (3\kappa - \frac{r}{2})\eta(X)\xi, \tag{3.20}$$

which gives

$$S(X,Y) = (\frac{r}{2} - \kappa)g(X,Y) + (3\kappa - \frac{r}{2})\eta(X)\eta(Y). \tag{3.21}$$

By taking account of (3.20) and (3.21) in (3.17), one can get

$$\begin{aligned} R(X,Y)Z &= (3\kappa - \frac{r}{2})(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ &\quad +(\frac{r}{2} - 2\kappa)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{3.22}$$

Taking the covariant derivative of the last equation according to W, we deduce that

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= -\frac{dr(W)}{2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ &\quad +\frac{dr(W)}{2}(g(Y,Z)X - g(X,Z)Y) \\ &\quad +(3\kappa - \frac{r}{2})\begin{pmatrix} (g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\nabla_W\xi \\ +(g(Y,Z)\xi - \eta(Z)Y)(\nabla_W\eta)(X) \\ -(g(X,Z)\xi - \eta(Z)X)(\nabla_W\eta)(Y) \\ +(\eta(Y)X - \eta(X)Y)(\nabla_W\eta)(Z). \end{pmatrix}. \end{aligned} \tag{3.23}$$

Now, let Y be a non-zero vector field orthogonal to ξ and $X = Z = \xi$. Using (2.5), (3.23) follows that

$$(\nabla_W R)(\xi,Y)\xi = -2(3\kappa - \frac{r}{2})(\nabla_W\eta)(\xi)Y = 0. \tag{3.24}$$

By virtue of (2.12) and (3.24), we obtain

$$A(W)R(\xi,Y)\xi - B(W)Y = 0. \tag{3.25}$$

From (2.7), we have

$$R(\xi,Y)\xi = -\kappa Y. \tag{3.26}$$

If we use (3.26) in (3.25), it follows that the requested relation holds. This completes the proof of the theorem. □

4. Generalized ϕ-recurrent paracontact metric manifolds of constant curvature

Theorem 4.1. *[10]If a paracontact manifold M^{2n+1} is of constant sectional curvature c and dimension $2n+1 \geqslant 5$, then $c=-1$ and $|h|^2 = 0$.*

Theorem 4.2. *If a generalized ϕ-recurrent paracontact metric manifold (M^{2n+1}, g) is of constant curvature and $(2n+1) \geqslant 5$, then $A(W) = B(W)$.*

Proof. Let (M^{2n+1}, g) be a generalized ϕ-recurrent paracontact metric manifold of constant curvature c and $(2n+1) \geqslant 5$. From Theorem 4.1, we have $c = -1$. So, we can write

$$R(X,Y)Z = -(g(Y,Z)X - g(X,Z)Y). \tag{4.1}$$

Taking the covariant derivative of the last equation according to W, we deduce that

$$(\nabla_W R)(X,Y)Z = 0. \tag{4.2}$$

Now, let Y be a non-zero vector field orthogonal to ξ and $X = Z = \xi$. From (4.1), we have

$$R(\xi,Y)\xi = Y. \tag{4.3}$$

By using (2.12), (4.2) and (4.3), we have

$$0 = A(W) - B(W)$$

which completes the proof. □

Theorem 4.3. *If a generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold (M^{2n+1}, g) is of constant curvature $c \neq 0$, then $B(W) = -\kappa A(W)$.*

Proof. Let us consider a $(2n+1)$-dimensional generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold which has constant curvature c. So, we have

$$R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y). \tag{4.4}$$

Replacing Z by ξ in (4.4), we get

$$R(X,Y)\xi = c(\eta(Y)X - \eta(X)Y). \tag{4.5}$$

From (2.7) and (4.5), we obtain

$$c(\eta(Y)X - \eta(X)Y) = \kappa(\eta(Y)X - \eta(X)Y). \tag{4.6}$$

Now, let Y be a non-zero vector field orthogonal to ξ and $X = \xi$. So, (4.6) returns to $c = \kappa \neq 0$. Because of the manifold is $N(\kappa)$-paracontact metric manifold, we have

$$R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y). \tag{4.7}$$

Taking the covariant derivative of the last equation according to W, we deduce that

$$(\nabla_W R)(X,Y)Z = -\kappa((\nabla_W g)(X,Z)Y - ((\nabla_W g)(Y,Z)X)) = 0. \tag{4.8}$$

Putting $Y = Z = \xi$ in (2.12), and taking account of (4.7) and (4.8), we obtain

$$0 = (X - \eta(X)\xi)(A(W)\kappa + B(W)). \tag{4.9}$$

If X is a non-zero vector field orthogonal to ξ, from (4.9), we get

$$0 = A(W)\kappa + B(W).$$

□

Remark 4.1. If a generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold (M^{2n+1}, g) is of constant curvature $c \neq 0$, and $(2n+1) \geqslant 5$, then $\kappa = -1$.

5. Generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifolds

Theorem 5.1. *For $\kappa \neq -1, 0$, a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold.*

Proof. Let us consider a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold. From (2.15), we have

$$(\nabla_W C)(X,Y)Z - \eta((\nabla_W C)(X,Y)Z)\xi = A(W)C(X,Y)Z + B(W)(g(Y,Z)X - g(X,Z)Y). \tag{5.1}$$

Taking the inner product on both sides of (5.1) with U, we obtain

$$\begin{aligned} g((\nabla_W C)(X,Y)Z,U) - \eta((\nabla_W C)(X,Y)Z)\eta(U) &= A(W)g(C(X,Y)Z,U) \\ &\quad + B(W)(g(Y,Z)g(X,U) \\ &\quad - g(X,Z)g(Y,U)). \end{aligned} \tag{5.2}$$

Let e_i, $1 \leq i \leq 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (5.2) and taking summation over i, we thus get

$$\begin{aligned} (\nabla_W S)(Y,Z) = \frac{dr(W)}{2n+1}g(Y,Z) - \frac{dr(W)}{(2n+1)2n}(g(Y,Z) - \eta(Y)\eta(Z)) \\ = A(W)(S(Y,Z) - \frac{r}{2n+1}g(Y,Z)) + B(W)2ng(Y,Z). \end{aligned} \tag{5.3}$$

If we make use of the property (iii) of (2.5) and (2.8) in (5.3), we obtain

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= \frac{dr(W)}{2n+1}\eta(Y) \\ &+ A(W)\eta(Y)\left(2n\kappa - \frac{r}{2n+1}\right) + B(W)2n\eta(Y). \end{aligned} \tag{5.4}$$

On the other hand, using again the property (iii) of (2.5) and (2.8), we can evaulate $(\nabla_W S)(Y,\xi)$ as

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) \\ &= -2n\kappa g(Y,\phi W - \phi hW) + S(Y,\phi W - \phi hW). \end{aligned} \tag{5.5}$$

Comparing (5.4) to (5.5), we have

$$\begin{aligned} S(Y,\phi W-\phi hW) &= 2n\kappa g(Y,\phi W-\phi hW)+\frac{dr(W)}{2n+1}\eta(Y) \\ &+A(W)\eta(Y)\left(2n\kappa-\frac{r}{2n+1}\right)+B(W)2n\eta(Y). \end{aligned} \tag{5.6}$$

If we use (2.9), (2.10) and (2.11) after putting ϕY instead of Y in (5.6), we get

$$\begin{aligned} S(Y,W) &= 2(-n-\kappa+1)g(Y,W)+2(n-1+n\kappa)g(Y,hW) \\ &+2((n-1)+\kappa(n+1))\eta(Y)\eta(W). \end{aligned} \tag{5.7}$$

If we replace W by hW in the last equation, we can immediately observe that

$$g(Y,hW)=n(1+\kappa)(g(Y,W)-\eta(Y)\eta(W)). \tag{5.8}$$

Using (5.8) in (5.7), we have

$$S(Y,W)=\alpha g(W,Y)+\beta\eta(Y)\eta(W),$$

where $\alpha=2((-n-\kappa+1)+(n-1+n\kappa)n(1+\kappa))$, $\beta=2((n-1)+\kappa(n+1)-(n-1+n\kappa)n(1+\kappa))$. Namely, manifold is η-Einstein manifold. □

Theorem 5.2. *Let (M^{2n+1},g) be a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Then $(\frac{r}{(2n+1)2n}-\kappa)A(W)=B(W)$.*

Proof. Putting $Y=Z=e_i$ in (5.2) and taking summation over i, one can get

$$\begin{aligned} &(\nabla_W S)(X,U)-\frac{dr(W)}{2n+1}g(X,U)-(\nabla_W S)(X,\xi)\eta(U)+\frac{dr(W)}{2n+1}\eta(X)\eta(U) \\ =\ & A(W)(S(X,U)-\frac{r}{2n+1}g(X,U))+B(W)2ng(X,U). \end{aligned} \tag{5.9}$$

Putting $U=\xi$ in (5.9) and using (2.8), we have

$$A(W)(2n\kappa-\frac{r}{2n+1})\eta(X)+2nB(W)\eta(X)=0. \tag{5.10}$$

Setting $X=\xi$ in the last equation, we get the requested relation which completes the proof of the theorem. □

Theorem 5.3. *For a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1},g), the characteristic vector field ξ and the vector field $\rho_1\gamma+\rho_2$ associated to the 1-form $A\gamma+B$ are co-directional.*

Proof. Two vector fields P and Q are said to be co-directional if $P=fQ$, where f is a non-zero scalar, that is $g(P,X)=fg(Q,X)$ for all X.

Taking inner product of (5.1) with ξ, we have

$$A(W)g(C(X,Y)Z,\xi)+B(W)(g(Y,Z)\eta(X)-g(X,Z)\eta(Y))=0. \tag{5.11}$$

In virtue of (2.14) and (5.11), we get

$$A(W)g(R(X,Y)Z,\xi) = (A(W)\frac{r}{(2n+1)2n} - B(W))(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{5.12}$$

Then by the use of second Bianchi identity, we obtain

$$\begin{aligned} & A(W)g(R(X,Y)Z,\xi) + A(Y)g(R(W,X)Z,\xi) + A(X)g(R(Y,W)Z,\xi) \\ = \; & (A(W)\frac{r}{(2n+1)2n} - B(W))(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) + \\ & (A(Y)\frac{r}{(2n+1)2n} - B(Y))(g(X,Z)\eta(W) - g(W,Z)\eta(X)) + \\ & (A(X)\frac{r}{(2n+1)2n} - B(X))(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)). \end{aligned} \tag{5.13}$$

From (2.6), it follows that

$$g(R(X,Y)Z,\xi) = \kappa(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z)).$$

Using the last equation in (5.13), we get

$$\begin{aligned} & A(W)[\kappa(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z))] + \\ & A(Y)[\kappa(-\eta(X)g(W,Z) + \eta(W)g(X,Z))] + \\ & A(X)[\kappa(-\eta(W)g(Y,Z) + \eta(Y)g(W,Z))] \\ = \; & (A(W)\frac{r}{(2n+1)2n} - B(W))(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) + \\ & (A(Y)\frac{r}{(2n+1)2n} - B(Y))(g(X,Z)\eta(W) - g(W,Z)\eta(X)) + \\ & (A(X)\frac{r}{(2n+1)2n} - B(X))(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)). \end{aligned} \tag{5.14}$$

Replacing $Y = Z$ by e_i in (5.14) and taking summation over i, $1 \leq i \leq 2n+1$, we obtain

$$(1-2n)\begin{pmatrix} (\kappa - \frac{r}{(2n+1)2n})(A(X)\eta(W) - A(W)\eta(X)) \\ +B(X)\eta(W) - B(W)\eta(X) \end{pmatrix} = 0. \tag{5.15}$$

Putting $X = \xi$ in the last equation, we have

$$\eta(W)(\eta(\rho_2) + \gamma\eta(\rho_1)) = A(W)\gamma + B(W), \tag{5.16}$$

where $\gamma = (\kappa - \frac{r}{(2n+1)2n})$, $\eta(\rho_1) = g(\xi,\rho_1) = A(\xi)$ and $\eta(\rho_2) = g(\xi,\rho_2) = B(\xi)$. From (5.16), we complete the proof of the theorem. □

Theorem 5.4. *Let (M^3,g) be a generalized concircularly ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Then $B(W) = -\frac{dr(W)}{6} + (\frac{r}{6} - \kappa)A(W)$.*

Proof. Using (3.22) in (2.14), we get

$$\begin{aligned} C(X,Y)Z \;=\; & (3\kappa - \frac{r}{2})(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ & +(\frac{r}{3} - 2\kappa)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{5.17}$$

It is readily taken that the covariant derivative of the above expression

$$(\nabla_W C)(X,Y)Z = -\frac{dr(W)}{2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y)$$
$$+\frac{dr(W)}{3}(g(Y,Z)X - g(X,Z)Y)$$
$$+(3\kappa - \frac{r}{2})\begin{pmatrix} (g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\nabla_W\xi \\ +(g(Y,Z)\xi - \eta(Z)Y)(\nabla_W\eta)(X) \\ -(g(X,Z)\xi - \eta(Z)X)(\nabla_W\eta)(Y) \\ +(\eta(Y)X - \eta(X)Y)(\nabla_W\eta)(Z). \end{pmatrix}. \tag{5.18}$$

Let us assume that Y is a non-zero vector field orthogonal to ξ and $X = Z = \xi$. Using the property *(iii)* of (2.5) and (5.18), we have

$$(\nabla_W C)(\xi,Y)\xi = \frac{dr(W)}{6}Y. \tag{5.19}$$

It follows (2.12) and (5.19) from that

$$A(W)C(\xi,Y)\xi - B(W)Y = \frac{dr(W)}{6}Y. \tag{5.20}$$

From (2.7) and (2.14), we have

$$C(\xi,Y)\xi = (-\kappa + \frac{r}{6})Y. \tag{5.21}$$

If we employ (5.21) in (5.20), we immediately see that one is able to get the requested equation. □

Remark 5.1. In a three dimensional generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold, r is not necessarily be a constant.

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