STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS AND SOME PROPERTIES IN 2-NORMED SPACES

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Abstract. In this study, we introduced the concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Also, we studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces. Keywords: Uniform convergence, Statistical Convergence, Double sequences of Functions, Statistical Cauchy sequence, 2-normed Spaces.

1. Introduction and Background

Throughout the paper, \( \mathbb{N} \) and \( \mathbb{R} \) denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [16] and Schoenberg [35]. Gökhan et al. [21] introduced the concepts of pointwise statistical convergence and statistical Cauchy sequence of real-valued functions. Balcerzak et al. [5] studied statistical convergence and ideal convergence for sequence of functions. Duman and Orhan [7] studied \( \mu \)-statistically convergent function sequences. Gökhan et al. [22] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. Dündar and Altay [8,9] studied the concepts of pointwise and uniformly \( \mathcal{I} \)-convergence and \( \mathcal{I}' \)-convergence of double sequences of functions and investigated some properties about them. Also, a lot of development have been made about double sequences of functions (see [4, 14, 20]).

The concept of 2-normed spaces was initially introduced by Gähler [18, 19] in the 1960’s. Gürdal and Pehlivan [25] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Sharma and Kumar [32] introduced statistical convergence, statistical Cauchy sequence, statistical limit points and statistical cluster points in probabilistic 2-normed space. Statistical convergence and statistical Cauchy sequence

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of functions in 2-normed space were studied by Yegül and Dündar [37]. Sarabadan and Talebi [31] presented various kinds of statistical convergence and $I$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $I$-equistatistically convergence and study $I$-equistatistically convergence of sequences of functions. Furthermore, a lot of development have been made in this area (see [1–3, 6, 15, 23, 24, 26–29, 33, 34]).

2. Definitions and Notations

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A 2-norm on $X$ is a function $\| \cdot, \cdot \| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\| x, y \| = 0$ if and only if $x$ and $y$ are linearly dependent.

(ii) $\| x, y \| = \| y, x \|$. 

(iii) $\| \alpha x, y \| = |\alpha| \| x, y \|$, $\alpha \in \mathbb{R}$. 

(iv) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$. 

The pair $(X, \| \cdot, \cdot \|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\| \cdot, \cdot \| :=$ the area of the parallelogram based on the vectors $x$ and $y$ which may be given explicitly by the formula 

$$\| x, y \| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), \quad y = (y_1, y_2) \in \mathbb{R}^2.$$ 

In this study, we suppose $X$ to be a 2-normed space having dimension $d$; where $2 \leq d < \infty$.

Let $(X, \| \cdot, \cdot \|)$ be a finite dimensional 2-normed space and $u = \{u_1, \cdots, u_d\}$ be a basis of $X$. We can define the norm $\| \cdot \|_\infty$ on $X$ by $\| x \|_\infty = \max \{ \| x, u_i \| : i = 1, \cdots, d \}$. 

Associated to the derived norm $\| \cdot \|_\infty$, we can define the (closed) balls $B_u(x, \varepsilon)$ centered at $x$ having radius $\varepsilon$ by $B_u(x, \varepsilon) = \{ y : \| x - y \|_\infty \leq \varepsilon \}$, where $\| x - y \|_\infty = \max \{ \| x - y, u_j \|, j = 1, \cdots, d \}$. 

Throughout the paper, we let $X$ and $Y$ be two 2-normed spaces, $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be two sequences of functions and $f, g$ be two functions from $X$ to $Y$. 

The sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is said to be convergent to $f$ if $f_n(x) \to f(x)(\| \cdot, \cdot \|_Y)$ for each $x \in X$. We write $f_n \to f(\| \cdot, \cdot \|_Y)$. This can be expressed by the formula $(\forall y \in Y) (\forall x \in X)(\exists \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \| f_n(x) - f(x), y \| < \varepsilon$. 
If $K \subseteq \mathbb{N}$, then $K_n$ denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of $K_n$. The natural density of $K$ is given by $\delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n|$, if it exists.

The sequence $\{f_n\}_{n \in \mathbb{N}}$ is said to be (pointwise) statistical convergent to $f$, if for every $\varepsilon > 0$, $\lim_{n \to \infty} \frac{1}{|n|} |\{n \in \mathbb{N} : \|f_n(x) - f(x)\| > \varepsilon\}| = 0$, for each $x \in X$ and each nonzero $z \in Y$. It means that for each $x \in X$ and each nonzero $z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$, a.a. (almost all) $n$. In this case, we write

$$st - \lim_{n \to \infty} \|f_n(x), z\| = \|f(x), z\| \text{ or } f_n \to st f.$$

The sequence of functions $\{f_n\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exists a number $k = k(\varepsilon, z)$ such that $\delta(\{n \in \mathbb{N} : \|f_n(x) - f_k(x), z\| \geq \varepsilon\}) = 0$, for each $x \in X$, i.e., $\|f_n(x) - f_k(x), z\| < \varepsilon$, a.a. $n$.

Let $X$ be a 2-normed space. A double sequence $(x_{mn})$ in $X$ is said to be convergent to $L \in X$, if for every $z \in X$, $\lim_{m,n \to \infty} \|x_{mn} - L, z\| = 0$. In this case, we write $\lim_{n,m \to \infty} x_{mn} = L$ and call $L$ the limit of $(x_{mn})$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$. Let $K_{mn}$ be the number of $(j, k) \in K$ such that $j \leq m, k \leq n$. That is, $K_{mn} = |\{(j, k) : j \leq m, k \leq n\}|$, where $|A|$ denotes the number of elements in $A$. If the double sequence $\{K_{mn}\}$ has a limit then we say that $K$ has double natural density and is denoted by $d_2(K) = \lim_{m,n \to \infty} \frac{K_{mn}}{mn}$.

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

Let $\{x_{mn}\}$ be a double sequence in 2-normed space $(X, \|\|, \|\|)$. The double sequence $(x_{mn})$ is said to be statistically convergent to $L$, if for every $\varepsilon > 0$, the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L, z\| \geq \varepsilon\}$ has natural density zero for each nonzero $z \in X$. In other words, $(x_{mn})$ statistically converges to $L$ in 2-normed space $(X, \|\|, \|\|)$ if $\lim_{m,n \to \infty} \frac{1}{mn} |\{(m, n) : \|x_{mn} - L, z\| \geq \varepsilon\}| = 0$, for each nonzero $z \in X$. It means that for each $z \in X$, $\|x_{mn} - L, z\| < \varepsilon$, a.a. $(m, n)$. In this case, we write $st - \lim_{m,n \to \infty} \|x_{mn}, z\| = \|L, z\|$.

A double sequence $(x_{mn})$ in 2-normed space $(X, \|\|, \|\|)$ is said to be statistically Cauchy sequence in $X$, if for every $\varepsilon > 0$ and each nonzero $z \in X$ there exist two number $M = M(\varepsilon, z)$ and $N = N(\varepsilon, z)$ such that $d_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{MN}, z\| \geq \varepsilon\}) = 0$, i.e., for each nonzero $z \in X$, $\|x_{mn} - x_{MN}, z\| < \varepsilon$, a.a. $(m, n)$. A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to $f$ on a set $S \subseteq \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that $|f_{mn}(x) - f(x)| < \varepsilon$, for all $m, n > N$. In this case we write $\lim_{m,n \to \infty} f_{mn}(x) = f(x)$ or $f_{mn} \to f$, on $S$.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly convergent to $f$ on a set $S \subseteq \mathbb{R}$, if for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that
for all $m, n > N$ implies $|f_{mn}(x) - f(x)| < \varepsilon$, for all $x \in S$. In this case we write $f_{mn} \Rightarrow f$, on $S$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise statistically convergent to $f$ on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{i,j \to \infty} \frac{1}{ij} \left| \{(m, n), m \leq i \text{ and } n \leq j : |f_{mn}(x) - f(x)| \geq \varepsilon \} \right| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon$, a.a. $(m, n)$. In this case, we write $st-lim_{m,n \to \infty} f_{mn}(x) = f(x)$ or $f_{mn} \to_{st} f$, on $S$.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly statistically convergent to $f$ on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{i,j \to \infty} \frac{1}{ij} \left| \{(m, n), m \leq i \text{ and } n \leq j : |f_{mn}(x) - f(x)| \geq \varepsilon \} \right| = 0,$$

for all $x \in S$, i.e., for all $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon$, a.a. $(m, n)$. In this case we write $f_{mn} \Rightarrow f$, on $S$.

Let $\{f_{mn}\}$ be a double sequence of functions defined on a set $S$. A double sequence $\{f_{mn}\}$ is said to be statistically Cauchy if for every $\varepsilon > 0$, there exist $N(= N(\varepsilon))$ and $M(= M(\varepsilon))$ such that $|f_{mn}(x) - f_{MN}(x)| < \varepsilon$ a.a. $(m, n)$ and for each (fixed) $x \in S$, i.e.,

$$\lim_{i,j \to \infty} \frac{1}{ij} \left| \{(m, n), m \leq i \text{ and } n \leq j : |f_{mn}(x) - f_{MN}(x)| \geq \varepsilon \} \right| = 0$$

for each (fixed) $x \in S$

**Lemma 2.1.** [9] Let $f$ and $f_{mn}$, $m, n = 1, 2, \ldots$, be continuous functions on $D = [a, b] \subset \mathbb{R}$. Then $f_{mn} \Rightarrow f$ on $D$ if and only if $\lim_{m,n \to \infty} c_{mn} = 0$, where $c_{mn} = \max_{x \in D} |f_{mn}(x) - f(x)|$.

### 3. Main Results

In this paper, we study concepts of convergence, statistical convergence and statistical Cauchy sequence of double sequences of functions and investigate some properties and relationships between them in 2-normed spaces.

Throughout the paper, we let $X$ and $Y$ be two 2-normed spaces, $\{f_{mn}\}_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ and $\{g_{mn}\}_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ be two double sequences of functions, $f$ and $g$ be two functions from $X$ to $Y$.

**Definition 3.1.** A double sequence $\{f_{mn}\}$ is said to be pointwise convergent to $f$ if, for each point $x \in X$ and for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x, \varepsilon)$ such that for all $m, n \geq k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \to f(\|\cdot\|_Y)$. 
**Definition 3.2.** A double sequence \( \{f_{mn}\} \) is said to be uniformly convergent to \( f \), if for each \( \varepsilon > 0 \), there exists a positive integer \( k_0 = k_0(\varepsilon) \) such that for all \( m, n > k_0 \) implies \( \|f_{mn}(x) - f(x), z\| < \varepsilon \), for all \( x \in X \) and for every \( z \in Y \). In this case, we write \( f_{mn} \Rightarrow f(\|\cdot\|_Y) \).

**Theorem 3.1.** Let \( D \) be a compact subset of \( X \) and \( f \) and \( f_{mn} \), \( (m, n = 1, 2, \ldots) \), be continuous functions on \( D \). Then,

\[
f_{mn} \Rightarrow f(\|\cdot\|_Y)
\]

on \( D \) if and only if

\[
\lim_{m, n \to \infty} c_{mn} = 0,
\]

where \( c_{mn} = \max_{x \in D} \|f_{mn}(x) - f(x), z\| \).

**Proof.** Suppose that \( f_{mn} \Rightarrow f(\|\cdot\|_Y) \) on \( D \). Since \( f \) and \( f_{mn} \) are continuous functions on \( D \), so \( (f_{mn}(x) - f(x)) \) is continuous on \( D \), for each \((m, n) \in \mathbb{N} \times \mathbb{N}\). Since \( f_{mn} \Rightarrow f(\|\cdot\|_Y) \) on \( D \) then, for each \( \varepsilon > 0 \), there is a positive integer \( k_0 = k_0(\varepsilon) \in \mathbb{N} \) such that \( m, n > k_0 \) implies

\[
\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}
\]

for all \( x \in D \) and every \( z \in Y \). Thus, when \( m, n > k_0 \) we have

\[
c_{mn} = \max_{x \in D} \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2} < \varepsilon.
\]

This implies

\[
\lim_{m, n \to \infty} c_{mn} = 0.
\]

Now, suppose that

\[
\lim_{m, n \to \infty} c_{mn} = 0.
\]

Then, for each \( \varepsilon > 0 \), there is a positive integer \( k_0 = k_0(\varepsilon) \in \mathbb{N} \) such that

\[
0 \leq c_{mn} = \max_{x \in D} \|f_{mn}(x) - f(x), z\| < \varepsilon,
\]

for \( m, n > k_0 \) and every \( z \in Y \). This implies that \( \|f_{mn}(x) - f(x), z\| < \varepsilon \), for all \( x \in D \), every \( z \in Y \) and \( m, n > k_0 \). Hence, we have

\[
f_{mn} \Rightarrow f(\|\cdot\|_Y),
\]

for all \( x \in D \) and every \( z \in Y \). \( \square \)

**Definition 3.3.** A double sequence \( \{f_{mn}\} \) is said to be (pointwise) statistical convergent to \( f \), if for every \( \varepsilon > 0 \),

\[
\lim_{i, j \to \infty} \frac{1}{ij} \sum_{m \leq i, n \leq j} \mathbb{1}\{|(m, n), m \leq i, n \leq j : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} = 0.
\]
for each (fixed) \( x \in X \) and each nonzero \( z \in Y \). It means that for each (fixed) \( x \in X \) and each nonzero \( z \in Y \),
\[
\|f_{mn}(x) - f(x), z\| < \varepsilon, \quad \text{a.a. (} m, n \text{).}
\]

In this case, we write
\[
st - \lim_{m,n \to \infty} \|f_{mn}(x) - z\| = \|f(x), z\| \quad \text{or} \quad f_{mn} \longrightarrow st f(\|., .\|_Y).
\]

**Remark 3.1.** \( \{f_{mn}\} \) is any double sequence of functions and \( f \) is any function from \( X \) to \( Y \), then set
\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon, \quad \text{for each } x \in X \text{ and each } z \in Y\} = \emptyset,
\]
since if \( z = \overrightarrow{0} \) (0 vector), \( \|f_{mn}(x) - f(x), z\| = 0 \nexists \varepsilon \) so the above set is empty.

**Theorem 3.2.** If for each \( x \in X \) and each nonzero \( z \in Y \),
\[
st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \text{and} \quad st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|g(x), z\|
\]
then, for each \( x \in X \) and each nonzero \( z \in Y \)
\[
\|f_{mn}(x), z\| = \|g_{mn}(x), z\|
\]
(i.e., \( f = g \)).

**Proof.** Assume \( f \neq g \). Then, \( f - g \neq \overrightarrow{0} \), so there exists a \( z \in Y \) such that \( f, g \) and \( z \) are linearly independent (such a \( z \) exists since \( d \geq 2 \)). Therefore, for each \( x \in X \) and each nonzero \( z \in Y \),
\[
\|f(x) - g(x), z\| = 2\varepsilon, \quad \text{with} \quad \varepsilon > 0.
\]

Now, for each \( x \in X \) and each nonzero \( z \in Y \), we get
\[
2\varepsilon = \|f(x) - g(x), z\| = \|(f(x) - f_{mn}(x)) + (f_{mn}(x) - g(x)), z\|
\]
\[
\leq \|f_{mn}(x) - g(x), z\| + \|f_{mn}(x) - f(x), z\|
\]
and so
\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - g(x), z\| < \varepsilon\} \subseteq \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}.
\]
But, for each \( x \in X \) and each nonzero \( z \in Y \),
\[
d_2\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - g(x), z\| < \varepsilon\} = 0,
\]
then contradicting the fact that \( f_{mn} \longrightarrow st g(\|., .\|_Y) \).

**Theorem 3.3.** If \( \{g_{mn}\} \) is a convergent sequence of double sequences of functions such that \( f_{mn} = g_{mn}, \) a.a. \( (m,n) \) then, \( \{f_{mn}\} \) is statistically convergent.
Proof. Suppose that for each \( x \in X \) and each nonzero \( z \in Y \),
\[
d_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\}) = 0 \quad \text{and} \quad \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|,
\]
then for every \( \varepsilon > 0 \),
\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}
\subseteq \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}
\cup \{(m, n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\}.
\]
Therefore,
\[
ed_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\})
\leq d_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\})
+ d_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \neq g_{mn}\}).
\]
Since \( \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\| \), for each \( x \in X \) and each nonzero \( z \in Y \), the set \( \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\} \) contains finite number of integers and so
\[
d_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0.
\]
Using inequality (3.1) we get for every \( \varepsilon > 0 \)
\[
ed_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,
\]
for each \( x \in X \) and each nonzero \( z \in Y \) and so consequently
\[
st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|.
\]
\]
\]
\]
\]

\[\text{Theorem 3.4.} \quad \text{If} \quad st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \text{for each} \quad x \in X \quad \text{and each nonzero} \quad z \in Y, \quad \text{then} \quad \{f_{mn}\} \quad \text{has a subsequence of function} \quad \{f_{mn,i}\} \quad \text{such that}
\]
\[
\lim_{i \to \infty} \|f_{mn,i}(x), z\| = \|f(x), z\|,
\]
\[
\text{for each} \quad x \in X \quad \text{and each nonzero} \quad z \in Y.
\]
\]

\[\text{Proof.} \quad \text{Proof of this Theorem is as an immediate consequence of Theorem 3.3.} \]

\[\text{Theorem 3.5.} \quad \text{Let} \quad \alpha \in \mathbb{R}. \quad \text{If for each} \quad x \in X \quad \text{and each nonzero} \quad z \in Y,
\]
\[
st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \text{and} \quad st - \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|g(x), z\|,
\]
\[
\begin{align*}
(i) \quad & st - \lim_{m,n \to \infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\| \quad \text{and} \\
(ii) \quad & st - \lim_{m,n \to \infty} \|\alpha f_{mn}(x), z\| = \|\alpha f(x), z\|.
\end{align*}
\]
Proof. (i) Suppose that
\[ st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } st - \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|g(x), z\| \]
for each \( x \in X \) and each nonzero \( z \in Y \). Then, \( \delta(K_1) = 0 \) and \( \delta(K_2) = 0 \) where
\[ K_1 = K_1(\varepsilon, z) : \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\} \]
and
\[ K_2 = K_2(\varepsilon, z) : \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|g_{mn}(x) - g(x), z\| \geq \frac{\varepsilon}{2}\} \]
for every \( \varepsilon > 0 \), each \( x \in X \) and each nonzero \( z \in Y \). Let
\[ K = K(\varepsilon, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x)), z\| \geq \varepsilon\}. \]
To prove that \( \delta(K) = 0 \), it suffices to show that \( K \subseteq K_1 \cup K_2 \). Let \( (m_0, n_0) \in K \) then, for each \( x \in X \) and each nonzero \( z \in Y \),
\[ \|(f_{mn_0}(x) + g_{mn_0}(x)) - (f(x) + g(x)), z\| \geq \varepsilon. \]
Suppose to the contrary that \( (m_0, n_0) \not\in K_1 \cup K_2 \). Then, \( (m_0, n_0) \not\in K_1 \) and \( (m_0, n_0) \not\in K_2 \). If \( (m_0, n_0) \not\in K_1 \) and \( (m_0, n_0) \not\in K_2 \) then, for each \( x \in X \) and each nonzero \( z \in Y \),
\[ \|f_{mn_0}(x) - f(x), z\| < \frac{\varepsilon}{2} \text{ and } \|g_{mn_0}(x) - g(x), z\| < \frac{\varepsilon}{2}. \]
Then, we get
\[
\|(f_{mn_0}(x) + g_{mn_0}(x)) - (f(x) + g(x)), z\| \\
\leq \|f_{mn_0}(x) - f(x), z\| + \|g_{mn_0}(x) - g(x), z\| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon,
\]
for each \( x \in X \) and each nonzero \( z \in Y \), which contradicts (3.2). Hence, \( (m_0, n_0) \in K_1 \cup K_2 \) and so \( K \subseteq K_1 \cup K_2 \).

(ii) Let \( \alpha \in \mathbb{R} (\alpha \neq 0) \) and for each \( x \in X \) and each nonzero \( z \in Y \),
\[ st - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|. \]
Then, we get
\[
d_2 \left( \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|\alpha f_{mn}(x) - \alpha f(x), z\| \geq \frac{\varepsilon}{|\alpha|}\right\} \right) = 0.
\]
Therefore, for each \( x \in X \) and each nonzero \( z \in Y \), we have
\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|\alpha f_{mn}(x) - \alpha f(x), z\| \geq \varepsilon\} \\
= \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \\
= \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{|\alpha|}\right\}.
\]
Hence, density of the right hand side of above equality equals 0. Therefore, for each \( x \in X \) and each nonzero \( z \in Y \), we have

\[
\mathcal{S}t - \lim_{m,n \to \infty} \| \alpha f_{mn}(x), z\| = \| \alpha f(x), z\|
\]

\( \square \)

**Theorem 3.6.** A double sequence of functions \( \{ f_{mn} \} \) is pointwise statistically convergent to a function \( f \) if and only if there exists a subset \( K_x = \{(m,n)\} \subseteq \mathbb{N} \times \mathbb{N} \), \( m, n = 1, 2, \ldots \) for each (fixed) \( x \in X \) \( d_2(K_x) = 1 \) and \( \lim_{m,n \to \infty} \| f_{mn}(x), z\| = \| f(x), z\| \) for each (fixed) \( x \in X \) and each nonzero \( z \in Y \).

**Proof.** Let \( \mathcal{S}t_2 = \lim_{m,n \to \infty} \| f_{mn}(x), z\| = \| f(x), z\| \). For \( r = 1, 2, \ldots \) put

\[
K_{r,x} = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \| f_{mn}(x), z\| \geq \frac{1}{r}\}
\]

and

\[
M_{r,x} = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \| f_{mn}(x), z\| < \frac{1}{r}\}
\]

for each (fixed) \( x \in X \) and each nonzero \( z \in Y \). Then, \( d_2(K_{r,x}) = 0 \) and

\[
M_{1,x} \supset M_{2,x} \supset \ldots \supset M_{i,x} \supset M_{i+1,x} \supset \ldots
\]

and

\[
d_2(M_{r,x}) = 1, \quad r = 1, 2, \ldots
\]

for each (fixed) \( x \in X \) and each nonzero \( z \in Y \).

Now, we have to show that for \( (m,n) \in M_{r,x} \), \( \{ f_{mn} \} \) is convergent to \( f \). Suppose that \( \{ f_{mn} \} \) is not convergent to \( f \). Therefore, there is \( \varepsilon > 0 \) such that

\[
\| f_{mn}(x), z\| = \| f(x), z\| \geq \varepsilon
\]

for infinitely many terms and some \( x \in X \) and each nonzero \( z \in Y \). Let

\[
M_{r,x} = \{(m,n) : \| f_{mn}(x) - f(x), z\| < \varepsilon\}
\]

and \( \varepsilon > \frac{1}{r} \) \( (r = 1, 2, \ldots) \). Then, \( d_2(M_{r,x}) = 0 \) and by (3.3) \( M_{r,x} \subseteq (M_{r,x}) \). Hence, \( d_2(M_{r,x}) = 0 \) which contradicts (3.4). Therefore, \( \{ f_{mn} \} \) is convergent to \( f \).

Conversely, suppose that there exists a subset \( K_x = \{(m,n)\} \subseteq \mathbb{N} \times \mathbb{N} \) for each (fixed) \( x \in X \) and each nonzero \( z \in Y \) such that \( d_2(K_x) = 1 \) and \( \lim_{m,n \to \infty} \| f_{mn}(x), z\| = \| f(x), z\| \), i.e., there exist an \( N(x, \varepsilon) \) such that for each (fixed) \( x \in X \), each nonzero \( z \in Y \) and each \( \varepsilon > 0 \), \( m, n \geq N \) implies \( \| f_{mn}(x), z\| = \| f(x), z\| < \varepsilon \). Now,

\[
K_{r,x} = \{(m,n) : \| f_{mn}(x), z\| \geq \varepsilon\} \subseteq \mathbb{N} \times \mathbb{N} - \{(m_{N+1}, n_{N+1}), (m_{N+2}, n_{N+2}), \ldots\}
\]

for each (fixed) \( x \in X \) and each nonzero \( z \in Y \). Therefore, \( d_2(K_{r,x}) \leq 1 - 1 = 0 \) for each (fixed) \( x \in X \) and each nonzero \( z \in Y \). Hence, \( \{ f_{mn} \} \) is pointwise statistically convergent to \( f \). \( \square \)
Definition 3.4. A double sequence of functions \( \{f_{mn}\} \) is said to uniformly statistically convergent to \( f \), if for every \( \varepsilon > 0 \) and for each nonzero \( z \in Y \),
\[
\lim_{i,j \to \infty} \frac{1}{ij} \| \{(m,n), m \leq i, n \leq j : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \| = 0,
\]
for all \( x \in X \). That is, for all \( x \in X \) and for each nonzero \( z \in Y \)
\[
(3.5) \quad \|f_{mn}(x) - f(x), z\| < \varepsilon, \quad a.a. \quad (m,n).
\]
In this case, we write \( f_{mn} \mathbin{\Rightarrow}_{st} f(\|\cdot\|,\|\cdot\|_Y) \).

Theorem 3.7. Let \( D \) be a compact subset of \( X \) and \( f \) and \( \{f_{mn}\}, m,n = 1,2, \ldots \)
be continuous functions on \( D \). Then,
\[
f_{mn} \mathbin{\Rightarrow}_{st} f(\|\cdot\|,\|\cdot\|_Y) \quad \text{on} \quad D \quad \text{if and only if}
\]
\[
st_2 - \lim_{m,n \to \infty} \|c_{mn}(x), z\| = 0,
\]
where \( c_{mn} = \max_{x \in S} \|f_{mn}(x) - f(x), z\| \).

Proof. Suppose that \( \{f_{mn}\} \) uniformly statistically convergent to \( f \) on \( D \). Since \( f \)
and \( \{f_{mn}\} \) are continuous functions on \( D \), so \( (f_{mn}(x) - f(x)) \) is continuous on \( D \),
for each \( m,n \in \mathbb{N} \). By statistically convergence for \( \varepsilon > 0 \)
\[
d_2((m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon) = 0,
\]
for each \( x \in D \) and for each nonzero \( z \in Y \). Hence, for \( \varepsilon > 0 \) it is clear that
\[
c_{mn} = \max_{x \in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}
\]
for each \( x \in D \) and for each nonzero \( z \in Y \). Thus we have
\[
st_2 - \lim_{m,n \to \infty} c_{mn} = 0.
\]
Now, suppose that \( st_2 - \lim_{m,n \to \infty} c_{mn} = 0 \). We let following set
\[
A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \max_{x \in D} \|f_{mn}(x) - f(x), z\| \geq \varepsilon \},
\]
for \( \varepsilon > 0 \) and for each nonzero \( z \in Y \). Then, by hypothesis we have \( d_2(A(\varepsilon)) = 0 \).
Since for \( \varepsilon > 0 \)
\[
\max_{x \in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \varepsilon
\]
we have
\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon \} \subset A(\varepsilon)
\]
and so
\[
d_2((m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon) = 0,
\]
for each \( x \in D \) and for each nonzero \( z \in Y \). This proves the theorem. \( \blacksquare \)
Now, we can give the relations between well-known convergence models and our studied models as the following result.

**Corollary 3.1.** (i) $f_{mn} \Rightarrow f(\|\cdot\|Y) \Rightarrow f_{mn} \rightarrow f(\|\cdot\|Y) \Rightarrow f_{mn} \rightarrow_{st} f(\|\cdot\|Y)$.

(ii) $f_{mn} \Rightarrow f(\|\cdot\|Y) \Rightarrow f_{mn} \Rightarrow_{st} f(\|\cdot\|Y) \Rightarrow f_{mn} \rightarrow_{st} f(\|\cdot\|Y)$.

Now, we give the concept of statistical Cauchy sequence and investigate relationships between statistical Cauchy sequence and statistical convergence of double sequences of functions in 2-normed space.

**Definition 3.5.** The double sequences of functions $\{f_{mn}\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exist two numbers $k = k(\varepsilon, z)$, $t = t(\varepsilon, z)$ such that

$$d_2((m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{kt}(x), z\| \geq \varepsilon) = 0$$

i.e., for each nonzero $z \in Y$,

$$\|f_{mn}(x) - f_{kt}(x), z\| < \varepsilon, \ a.a. \ (m, n).$$

**Theorem 3.8.** Let $\{f_{mn}\}$ be a statistically Cauchy sequence of double sequence of functions in a finite dimensional 2-normed space $(X, \|\cdot\|)$. Then, there exists a convergent sequence of double sequences of functions $\{g_{mn}\}$ in $(X, \|\cdot\|)$ such that $f_{mn} = g_{mn}$, for a.a. $(m, n)$.

**Proof.** First note that $\{f_{mn}\}$ is a statistically Cauchy sequence of functions in $(X, \|\cdot\|)$. Choose a natural number $k(1)$ and $j(1)$ such that the closed ball $B^u_1 = B_u(f_{k(1)}j(1)(x), 1)$ contains $f_{mn}(x)$ for a.a. $(m, n)$ and for each $x \in X$. Then, choose a natural number $k(2)$ and $j(2)$ such that the closed ball $B_2 = B_u(f_{k(2)}j(2)(x), 1)$ contains $f_{mn}(x)$ for a.a. $(m, n)$ and for each $x \in X$. Note that $B^u_1 = B^u_1 \cap B_2$ also contains $f_{mn}(x)$ for a.a. $(m, n)$ and for each $x \in X$. Thus, by continuing this process, we can obtain a sequence $\{B^u_r\}_{r \geq 1}$ of nested closed balls such that diam $(B^u_r) \leq \frac{1}{r}$.

Therefore,

$$\bigcap_{r=1}^{\infty} B^u_r = \{h(x)\},$$

where $h$ is a function from $X$ to $Y$. Since each $B^u_r$ contains $f_{mn}(x)$ for a.a. $(m, n)$ and for each $x \in X$, we can choose a sequence of strictly increasing natural numbers $\{S_r\}_{r \geq 1}$ such that for each $x \in X$,

$$\frac{1}{mn} |\{(m, n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \notin B^u_r\}| < \frac{1}{r}, \quad \text{if} \quad m, n > S_r.$$

Put $T_r = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m, n > S_r, \ f_{mn}(x) \notin B^u_r\}$ for each $x \in X$, for all $r \geq 1$ and $R = \bigcup_{r=1}^{\infty} T_r$. Now, for each $x \in X$, define the sequence of functions $\{g_{mn}\}$ as following

$$g_{mn}(x) = \begin{cases} h(x) & \text{if } (m, n) \in R \times R \\ f_{mn}(x) & \text{otherwise} \end{cases}$$
Note that \( \lim_{m,n \to \infty} g_{mn}(x) = h(x) \), for each \( x \in X \). In fact, for each \( \varepsilon > 0 \) and for each \( x \in X \), choose a natural number \( m \) such that \( \varepsilon > \frac{1}{k} > 0 \). Then, for each \( m,n > S_r \), and for each \( x \in X \), \( g_{mn}(x) = h(x) \) or \( g_{mn}(x) = f_{mn}(x) \in B_{u}^{r} \), and so in each case
\[
||g_{mn}(x) - h(x)||_{\infty} \leq \text{diam}(B_{u}^{r}) \leq \frac{1}{2r-1}.
\]
Since, for each \( x \in X \),
\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\} \subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \notin B_{u}^{r}\},
\]
we have
\[
\frac{1}{mn}||\{(m,n) \in \mathbb{N} \times \mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\}| \\
\leq \frac{1}{mn}||\{(n,m) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \notin B_{u}^{r}\}| \\
< \frac{1}{r},
\]
and so
\[
d_{d}(\{(m,n) \in \mathbb{N} \times \mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\}) = 0.
\]
Thus, \( g_{mn}(x) = f_{mn}(x) \) for a.a. \( m,n \) and for each \( x \in X \) in \( \langle X, \| \cdot \|_{\infty} \rangle \). Suppose that \( \{u_1, \ldots, u_d\} \) is a basis for \( \langle X, \| \cdot \| \rangle \). Since, for each \( x \in X \),
\[
\lim_{m,n \to \infty} ||g_{mn}(x) - h(x)||_{\infty} = 0 \quad \text{and} \quad ||g_{mn}(x) - h(x), u_i|| \leq ||g_{mn}(x) - h(x)||_{\infty}
\]
for all \( 1 \leq i \leq d \), then we have
\[
\lim_{m,n \to \infty} ||g_{mn}(x) - h(x), z||_{\infty} = 0,
\]
for each \( x \in X \) and each nonzero \( z \in X \). It completes the proof. \( \Box \)

**Theorem 3.9.** The sequence \( \{f_{mn}\} \) is statistically convergent if and only if \( \{f_{mn}\} \) is a statistically Cauchy sequence of double sequence of functions.

**Proof.** Assume that \( f \) be function from \( X \) to \( Y \) and \( s - \lim_{m,n \to \infty} ||f_{mn}(x), z|| = ||f(x), z|| \) for each \( x \in X \) and each nonzero \( z \in Y \) and \( \varepsilon > 0 \). Then, for each \( x \in X \) and each nonzero \( z \in Y \), we have
\[
||f_{mn}(x) - f(x), z|| < \frac{\varepsilon}{2}, \quad \text{a.a.} \ (m,n).
\]
If \( k = k(\varepsilon, z) \) and \( t = t(\varepsilon, z) \) are chosen so that for each \( x \in X \) and each nonzero \( z \in Y \),
\[
||f_{kt}(x) - f(x), z|| < \frac{\varepsilon}{2},
\]
and so we have
\[ \|f_{mn}(x) - f_{kt}(x), z\| \leq \|f_{mn}(x) - f(x), z\| + \|f(x) - f_{kt}(x), z\| \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{a.a.} \quad (m, n). \]

Hence, \( \{f_{mn}\} \) is statistically Cauchy sequence of double sequence of functions.

Now, assume that \( \{f_{mn}\} \) is statistically Cauchy sequence of double sequence of function. By Theorem 3.8, there exists a convergent sequence \( \{g_{mn}\} \) from \( X \) to \( Y \) such that \( f_{mn} = g_{mn} \) for a.a. \((m, n)\). By Theorem 3.3, we have
\[ \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|, \]
for each \( x \in X \) and each nonzero \( z \in Y \).

**Theorem 3.10.** Let \( \{f_{mn}\} \) be a double sequence of functions. The following statements are equivalent

(i) \( \{f_{mn}\} \) is (pointwise) statistically convergent to \( f(x) \),

(ii) \( \{f_{mn}\} \) is statistically Cauchy,

(iii) There exists a subsequence \( \{g_{mn}\} \) of \( \{f_{mn}\} \) such that \( \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\| \).

**Proof.** Proof of this Theorem is as an immediate consequence of Theorem 3.6 and Theorem 3.9. \( \square \)

**Definition 3.6.** Let \( D \) be a compact subset of \( X \) and \( \{f_{mn}\} \) be a double sequence of functions on \( D \). \( \{f_{mn}\} \) is said to be statistically uniform Cauchy if for every \( \varepsilon > 0 \) and each nonzero \( z \in Y \), there exists \( k = k(\varepsilon, z) \), \( t = t(\varepsilon, z) \) such that
\[ d_2(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{kt}(x), z\| \geq \varepsilon\}) = 0 \]
for all \( x \in X \).

**Theorem 3.11.** Let \( D \) be a compact subset of \( X \) and \( \{f_{mn}\}, \) be a sequence of bounded functions on \( D \). Then, \( \{f_{mn}\} \) is uniformly statistically convergent if and only if it is uniformly statistically Cauchy on \( D \).

**Proof.** Proof of this theorem is similar the Theorem 3.9. So, we omit it. \( \square \)

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