SOME EQUIVALENT QUASINORMS ON $L_{\phi,E}$

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Abstract. In this paper we define a new operator ideal $L_{\phi,E}$ by using block sequence spaces and symmetric norming function. Also we define different quasi-norms on this class and deal with equivalence of these quasi-norms.

Keywords: Operator ideal; sequence spaces; norming function; quasi-norm.

1. Introduction

The operator ideal theory has a special importance in functional analysis. One of the most important methods of constructing operator ideals is using $s-$numbers. Pietsch defined the approximation numbers of a bounded linear operator in Banach spaces, in 1963 [14]. Later on, the other examples of $s-$numbers, namely Kolmogorov numbers, Weyl numbers, etc. are introduced to the Banach space setting.

In this paper, we denote the set of all natural numbers and nonnegative real numbers by $\mathbb{N}$ and $\mathbb{R}^+$, respectively.

A finite rank operator is defined as a bounded linear operator whose dimension of the range space is finite [10].

Let $\omega$ be the set of all real valued sequences. A sequence space is any vector subspace of $\omega$.

Maddox defined the linear space $l(p)$ as follows in [8]:

$$l(p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} |x_n|^{p_n} < \infty \right\},$$

where $(p_n)$ is a bounded sequence of strictly positive real numbers.
The set of all sequences whose generalized weighted mean transforms are in the space \( l(p) \) is the sequence space \( l(u, v; p) \) which is introduced by Altay and Başar in [1] as follows:

\[
l(u, v; p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} u_k v_k x_k \right|^p < \infty \right\},
\]

where \( u_n, v_k \neq 0 \) for all \( n, k \in \mathbb{N} \).

If \( p_n = p \) for all \( n \in \mathbb{N} \), \( l(u, v; p) = Z(u, v; l_p) \) which is defined by Malkowsky and Savaş [12] as follows:

\[
Z(u, v; l_p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} v_k x_k \right|^p < \infty \right\},
\]

where \( 1 < p < \infty \).

The Cesaro sequence space \( ces_p \) is defined as

\[
ces_p = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\},
\]

where \( 1 < p < \infty \) ([18], [21], [22]). Afterwards, Mursaleen and Khan defined the Cesaro vector-valued sequence space by

\[
Ces(X, p, q) = \left\{ x = (x_k) : \sum_{k=0}^{\infty} \left( \frac{1}{q_k} \sum_{n=1}^{k} |x_n| \right)^{p_k} < \infty \right\},
\]

where \( p = (p_k) \) and \( q = (q_k) \) are bounded sequences of positive real numbers and \( Q_n = \sum_{k=0}^{n} q_k, (n \in \mathbb{N}) \) [13]. Here if \( q_k = 1 \) for each \( k \) then \( Ces(X, p, q) \) is reduced to \( ces_p \).

Let \( E \) and \( F \) be real or complex Banach spaces. \( \mathcal{L}(E, F) \) and \( \mathcal{L} \) denotes the space of all bounded linear operators from \( E \) to \( F \) and the space of all bounded linear operators between any two arbitrary Banach spaces, respectively.

A map \( s = (s_n) : \mathcal{L} \to \mathbb{R}^+ \) assigning to every operator \( T \in \mathcal{L} \) a non-negative scalar sequence \( (s_n(T))_{n \in \mathbb{N}} \) is called an \( s \)-number sequence if the following conditions are satisfied for all Banach spaces \( E, F, E_0 \) and \( F_0 \):

1. \( \|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0 \) for every \( T \in \mathcal{L}(E, F) \),
2. \( s_{m+n-1}(S+T) \leq s_m(S) + s_n(T) \) for every \( S, T \in \mathcal{L}(E, F) \) and \( m, n \in \mathbb{N} \),
3. \( s_n(RST) \leq \|R\| s_n(S) \|T\| \) for some \( R \in \mathcal{L}(F, F_0), S \in \mathcal{L}(E, F) \) and \( T \in \mathcal{L}(E_0, E) \), where \( E_0, F_0 \) are arbitrary Banach spaces,
4. If rank \((T) \leq n \), then \( s_n(T) = 0 \),
5. \( s_n(I : l_2^n \to l_2^n) = 1 \), where \( I \) denotes the identity operator on the \( n \)-dimensional Hilbert space \( l_2^n \), where \( s_n(T) \) denotes the \( n-th \) \( s \)-number of the operator \( T \) [2].
One of the example of s-number sequence is the approximation number, which is defined by Pietsch. The $n$–th approximation number, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf \{ \| T - A \| : A \in \mathcal{L}(E, F), \text{rank}(A) < n \},$$

where $T \in \mathcal{L}(E, F)$ and $n \in \mathbb{N}$ [14].

Pietsch [14] defined an operator $T \in \mathcal{L}(E, F)$ to be $l_p$ type operator if

$$\sum_{n=1}^{\infty} (a_n(T))^p < \infty \text{ for } 0 < p < \infty.$$ Then, in [3] Constantin defined the class of $ces - p$ type operators by using the Cesaro sequence spaces, where an operator $T \in \mathcal{L}(E, F)$ is called $ces - p$ type if

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k(T) \right)^p < \infty, \ 1 < p < \infty.$$ Afterwards Tita in [24] proved that the class of $l_p$ type operators and the class of $ces - p$ type operators are coincides.

As a generalization of $l_p$ type operators, $A - p$ type operators were examined in [5]. Also in [9], [10], [11] Maji and Srivastava studied the class $A^{(s)} - p$ of $s$–type $ces_p$ operators using $s$–number sequence and Cesaro sequence spaces and they introduced a new class $A^{(s)}_{p,q}$ of $s$–type $ces(p,q)$ operators by using weighted Cesaro sequence space for $1 < p < \infty$. Recently, the class of $s$–type $Z(u,v;l_p)$ operators have been defined and studied on some properties of this class in [4].

The idea of quasi-normed operator ideals is developed by the fact that, some important operator ideals which do not possess a natural norm should also be covered. There exists a lot of different quasi-norms on every operator ideal. In addition to this, the nice quasi-norms are determined by the completeness of the corresponding topology [15].

Now give the definitions of operator ideal and quasi-norm:

Let $E'$ be the dual of $E$, which is composed of continuous linear functionals on $E$. Let $x' \in E'$ and $y \in F$, then the map $x' \otimes y : E \to F$ is defined by

$$(x' \otimes y)(x) = x'(x) y, \ x \in E.$$

A subcollection $\mathfrak{I}$ of $\mathcal{L}$ is called an operator ideal if each component $\mathfrak{I}(E, F) = \mathfrak{I} \cap \mathcal{L}(E, F)$ satisfies the following conditions:

(O1) if $x' \in E'$, $y \in F$, then $x' \otimes y \in \mathfrak{I}(E, F)$,

(O2) if $S, T \in \mathfrak{I}(E, F)$, then $S + T \in \mathfrak{I}(E, F)$,

(O3) if $S \in \mathfrak{I}(E, F), T \in \mathcal{L}(E_0, E)$ and $R \in \mathcal{L}(F, F_0)$, then $RST \in \mathfrak{I}(E_0, F_0)$ [15].

A function $\alpha : \mathfrak{I} \to \mathbb{R}^+$ is said to be a quasi-norm on the operator ideal $\mathfrak{I}$ if the following conditions hold:

(QN1) If $x' \in E'$, $y \in F$, then $\alpha(x' \otimes y) = \| x' \| \| y \|$;

(QN2) there exists a constant $C \geq 1$ such that $\alpha(S + T) \leq C [\alpha(S) + \alpha(T)];$

(QN3) if $S \in \mathfrak{I}(E, F), T \in \mathcal{L}(E_0, E)$ and $R \in \mathcal{L}(F, F_0)$, then $\alpha(RST) \leq \| R \| \alpha(S) \| T \| $ [15].
In particular if \( C = 1 \) then \( \alpha \) becomes a norm on the operator ideal \( \mathfrak{F} \).

An ideal \( \mathfrak{F} \) with a quasi-norm \( \alpha \), which is denoted by \([\mathfrak{F}, \alpha]\), is said to be a quasi-Banach operator ideal if each component \( \mathfrak{F}(E, F) \) is complete under the quasi-norm \( \alpha \).

Let \( \ell_\infty \) be the space of all bounded real sequences and \( K \subset \ell_\infty \) be the set of all sequences \( x \) such that \( \text{card} \{ i \in \mathbb{N}, x_i \neq 0 \} < n \) and \( x_1 \geq x_2 \geq \ldots \geq 0 \).

A function \( \phi : K \to \mathbb{R} \) is called symmetric norming function, if the following conditions satisfied

1. \( \phi(x) > 0 \) for every \( x \in K \),
2. \( \phi(\alpha x) = \alpha \phi(x) \) for every \( x \in K \) and \( \alpha \geq 0 \),
3. \( \phi(x + y) \leq \phi(x) + \phi(y) \) for every \( x, y \in K \)
4. \( \phi(1, 0, 0, \ldots) = 1 \)
5. if the inequality \( \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \) holds for \( k = 1, 2, \ldots \), then \( \phi(x) \leq \phi(y) \) holds [28].

It’s given that ([27], [19]) for all symmetric norming functions \( \phi \), the function \( \phi(p) \) defined as

\[
\phi(p) : (x_i) \in K \to (\phi(\{x_i^p\}))^{\frac{1}{p}}, 1 \leq p < \infty
\]

is also a symmetric norming function. For more details on symmetric norming functions we refer to ([7], [20], [23], [25]-[27], [30], [31]).

By using the properties of symmetric norming function and the sequence \( (a_n(T)) \), the class \( L_\phi(E, F) \) is defined in [25] and [29] as follows

\[
L_\phi(E, F) = \{ T \in \mathcal{L}(E, F) : \phi(\{a_n(T)\}) < \infty \}.
\]

Let \( E = (E_n) \) be a partition of finite subsets of the positive integer such that

\[
\max E_n < \min E_{n+1}
\]

for \( n = 1, 2, \ldots \). In [6] Foroutannia defined the sequence space \( l_p(E) \) as

\[
l_p(E) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\}, (1 \leq p < \infty)
\]

with the seminorm \( \|\cdot\|_{p,E} \), which is defined in the following way:

\[
\|\cdot\|_{p,E} = \left( \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{\frac{1}{p}}.
\]

For example, if \( E_n = \{2n - 1, 2n\} \) for all \( n \), then \( x = (x_n) \in l_p(E) \) if and only if \( \sum_{n=1}^{\infty} |x_{2n-1} + x_{2n}|^p < \infty \). It is obvious that \( \|\cdot\|_{p,E} \) cannot be a norm, since we have
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\[ \|x\|_{p,E} = 0 \] while \( x = (1, -1, 0, 0, \ldots) \neq \theta \) and \( E_n = \{2n - 1, 2n\} \) for all \( n \). In the special case \( E_n = \{n\} \) for \( n = 1, 2, \ldots \), we have \( l_p(E) = l_p \) and \( \|x\|_{p,E} = \|x\|_p \).

For more information about block sequence spaces, we refer to [16],[17].

In [32], the class of \( l_p(E) \)-type operators, which is denoted by \( L_{p,E}(E, F) \), is given and it is shown that this class is a quasi-Banach operator ideal by the quasinorm

\[ \|T\|_{p,E} = \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} s_j(T) \right)^p \right)^{\frac{1}{p}}. \]

Also a new class of operators \( L_{\phi(p),E} \) is defined. Further it is proved that by quasinorm

\[ \|T\|_{\phi(p),E} = \phi(p) \left( \left\{ \sum_{j \in E_i} s_j(T) \right\} \right) \]

this class is a quasi-Banach operator ideal.

2. Main Results

Now we define a new class \( L_{\phi,E}(E, F) \) including the class \( L_{\phi}(E, F) \) as

\[ L_{\phi,E}(E, F) = \left\{ T \in \mathcal{L}(E, F) : \phi \left( \left\{ \sum_{j \in E_n} s_j(T) \right\} \right) < \infty \right\}. \]

For example if we take \( E_n = \{n\} \) for \( n = 1, 2, \ldots \), we have \( L_{\phi,E}(E, F) = L_{\phi}(E, F) \).

Also if we take \( E_n = \{2n - 1, 2n\} \) for all \( n \), we get \( \phi \left( \{s_{2n-1}(T) + s_{2n}(T)\} \right) < \infty \).

In this section we show some equivalent quasinorms on operator ideal \( L_{\phi,E}(E, F) \).

**Theorem 2.1.** \( \|T\|_{\phi,E} = \phi \left( \left\{ \sum_{j \in E_n} s_j(T) \right\} \right) \) is a quasinorm on operator ideal \( L_{\phi,E}(E, F) \).

**Proof.** If \( x' \in E \) and \( y \in F \), then the equality

\[ \phi \left( \left\{ \sum_{j \in E_n} s_j(x' \otimes y) \right\} \right) = \phi \left( \left\{ s_1(x' \otimes y) \right\} \right) = \|x' \otimes y\| = \|x'\| \|y\| < \infty \]

holds since \( x' \otimes y \) is a rank one operator, \( s_n(x' \otimes y) = 0 \) for \( n \geq 2 \). Therefore \( \|x' \otimes y\|_{\phi,E} = \|x'\| \|y\| \) and \( x' \otimes y \in L_{\phi,E} \).

Let \( S, T \in L_{\phi,E} \). Then we have that
\[
\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} s_j (S + T) \right) \leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} s_{2j-1} (S + T) + \sum_{j \in E_n} s_{2j} (S + T) \right) \\
\leq \sum_{n=1}^{\infty} \left( 2 \sum_{j \in E_n} s_{2j-1} (S + T) \right) \\
\leq \sum_{n=1}^{\infty} \left( 2 \sum_{j \in E_n} s_j (S) + s_j (T) \right).
\]

By using \((\phi 5)\) we can get
\[
\phi \left( \left\{ \sum_{j \in E_n} s_j (S + T) \right\} \right) \leq \phi \left( \left\{ 2 \left( \sum_{j \in E_n} s_j (S) + \sum_{j \in E_n} s_j (T) \right) \right\} \right) \\
\leq 2 \left[ \phi \left( \left\{ \left( \sum_{j \in E_n} s_j (S) \right) \right\} \right) + \phi \left( \left\{ \left( \sum_{j \in E_n} s_j (T) \right) \right\} \right) \right] \\
< \infty.
\]

It follows that
\[
\|S + T\|_{\phi, E} \leq 2 \left( \|S\|_{\phi, E} + \|T\|_{\phi, E} \right)
\]
and also \(S + T \in L_{\phi, E}\).

We have that
\[
\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} s_j (RST) \right) \leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} \|R\| \|T\| s_j (S) \right) \\
\leq \|R\| \|T\| \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} s_j (S) \right).
\]

By using the properties of \(\phi\) function, we obtain
\[
\phi \left( \left\{ \sum_{j \in E_n} s_j (RST) \right\} \right) \leq \|R\| \|T\| \|S\| \phi \left( \left\{ \sum_{j \in E_n} s_j (S) \right\} \right) < \infty
\]
and also the inequality
\[
\|RST\|_{\phi, E} \leq \|R\| \|T\| \|S\|_{\phi, E}
\]
holds.

Hence, \(\|T\|_{\phi, E} = \phi \left( \left\{ \sum_{j \in E_n} s_j (T) \right\} \right)\) is a quasinorm on operator ideal \(L_{\phi, E} (E, F)\).
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**Proposition 2.1.** The quasinorm $\|T\|_{\phi,E}^+ = \phi\left(\left\{ \sum_{j \in E_n} s_{2j-1}(T) \right\} \right)$ is equivalent with $\|T\|_{\phi,E}$.

**Proof.** The equivalence can be easily seen from the fact that
\[
\sum_{n=1}^{k} \sum_{j \in E_n} s_{2j-1}(T) \leq \sum_{n=1}^{k} \sum_{j \in E_n} s_j(T) \leq 2 \sum_{n=1}^{k} \sum_{j \in E_n} s_{2j-1}(T).
\]

**Remark 2.1.** For the particular case if $E_n = \{n\}$ for $n = 1, 2, \ldots$, we get Proposition 1.1 in [28].

**Proposition 2.2.** The quasinorm $\|T\|_{\phi(p),E}^\vee$ is equivalent with
\[
\|T\|_{\phi(p),E}^\vee = \phi(p)\left(\left\{ \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in E_i} s_j(T) \right\} \right), \quad 1 < p < \infty
\]
where $E_n = \{nN - N + 1, nN - N + 2, \ldots, nN\}$ for all $n$.

**Proof.** This is a consequence of Hardy’s inequality.
\[
\sum_{n=1}^{k} \left( \sum_{j \in E_n} s_j(T) \right)^p \leq \sum_{n=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in E_i} s_j(T) \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{k} \left( \sum_{j \in E_n} s_j(T) \right)^p.
\]

**Remark 2.2.** In particular case if we take $N = 1$ we get Proposition 1.2 in [28].

**Theorem 2.2.** If $(\alpha_n)$ is a nonincreasing positive sequence and $\lim \alpha_{Nn} \neq 0$, then the quasinorm $\|T\|_{\phi(p),E}$ is equivalent with the quasinorm
\[
\|T\|_{\phi(p),E}^o = \phi(p)\left(\left\{ \frac{1}{\alpha_1 + \ldots + \alpha_n} \sum_{i=1}^{n} \sum_{j \in E_i} s_j(T) \right\} \right), \quad 1 < p < \infty
\]
where $E_n = \{nN - N + 1, nN - N + 2, \ldots, nN\}$ for all $n$.

**Proof.** We know that the sequences $(\alpha_n)$ and $(s_n(T))$ are decreasing, so we can write that
\[
\frac{1}{n\alpha_1} n\alpha_{Nn} \sum_{j \in E_i} s_j(T) = \frac{\alpha_{Nn}}{\alpha_1} \sum_{j \in E_i} s_j(T) \leq \frac{1}{\alpha_1 + \ldots + \alpha_n} \sum_{i=1}^{n} \sum_{j \in E_i} \alpha_j s_j(T) \leq \frac{1}{n\alpha_{Nn}} \sum_{i=1}^{n} \sum_{j \in E_i} s_j(T).
\]
If \( \lim_{n \to \infty} \alpha_{Nn} = \alpha \neq 0 \), we get

\[
\frac{\alpha}{\alpha_1} \sum_{j \in E_i} s_j (T) \leq \left( \frac{1}{\alpha_1 + \ldots + \alpha_n} \right) \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in E_i} \alpha_j s_j (T) \leq \frac{\alpha_1}{\alpha} \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in E_i} s_j (T) \right).
\]

By using Hardy’s inequality we obtain

\[
\frac{n}{\alpha_1} \left( \sum_{j \in E_i} s_j (T) \right)^{p} \leq \frac{n}{\alpha_1} \left( \sum_{j \in E_i} \alpha_j s_j (T) \right)^{p} \leq \left( \frac{\alpha_1}{p} \right)^{p} \left( \sum_{j \in E_i} (s_j (T))^{p} \right), \quad 1 < p < \infty.
\]

By the property \((\phi 5)\) it results

\[
\frac{\alpha}{\alpha_1} \|T\|_{\phi(p),E} \leq \|T\|_{\phi(p),E}^{\circ} \leq \frac{\alpha_1}{\alpha} \frac{p}{p-1} \|T\|_{\phi(p),E}.
\]

Hence \(\|T\|_{\phi(p),E}\) is equivalent with \(\|T\|_{\phi(p),E}^{\circ}\). \(\Box\)

**Remark 2.3.** In particular case if we take \(N = 1\) then we get Theorem 1.4 in [28].

**Theorem 2.3.** Let \((u_n)\) and \((w_n)\) are sequences of non-negative real numbers such that \(u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots\) and \(w_1 \leq w_2 \leq \ldots \leq w_n \leq \ldots\) and \(w_n \leq n \leq \frac{w_n}{u_n}\). Let \(\lim_{n \to \infty} u_{Nn} \neq 0\), then the quasinorm \(\|T\|_{\phi(p),E}\) is equivalent to

\[
\|T\|_{\phi(p),E}^{\gamma} = \phi(p) \left( \left\{ \frac{1}{w_n} \sum_{i=1}^{n} \sum_{j \in E_i} u_j s_j (T) \right\} \right) \text{ for } 1 \leq p < \infty.
\]

where

\[E_n = \{nN - N + 1, nN - N + 2, \ldots, nN\} \text{ for all } n.\]

**Proof.** Since the sequences \((u_n)\) and \((a_n (T))\) are decreasing, we can write

\[
\frac{1}{n} \sum_{j \in E_i} s_j (T) \leq \frac{1}{w_n} \sum_{i=1}^{n} \sum_{j \in E_i} u_j s_j (T) \leq \frac{1}{n} \sum_{j \in E_i} s_j (T).
\]
Summing from $n = 1$ to $k$, we get

$$
\sum_{n=1}^{k} \left( u_{Nn} \sum_{j \in E_i} s_j(T) \right)^p \leq \sum_{n=1}^{k} \left( \frac{1}{w_n} \sum_{i=1}^{n} \sum_{j \in E_i} u_j s_j(T) \right)^p \leq \sum_{n=1}^{k} \left( \frac{u_1}{nu_{Nn}} \sum_{i=1}^{n} \sum_{j \in E_i} s_j(T) \right)^p.
$$

If $\lim_{n \to \infty} u_{Nn} = u \neq 0$, then we obtain

$$
u^p \sum_{n=1}^{k} \left( \sum_{j \in E_i} s_j(T) \right)^p \leq \sum_{n=1}^{k} \left( \frac{1}{w_n} \sum_{i=1}^{n} \sum_{j \in E_i} u_j s_j(T) \right)^p \leq \left( \frac{u_1}{u} \right)^p \sum_{n=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in E_i} s_j(T) \right)^p
$$

for every $k \in \mathbb{N}$. By using Hardy's inequality, we get

$$
u^p \sum_{n=1}^{k} \left( \sum_{j \in E_i} s_j(T) \right)^p \leq \sum_{n=1}^{k} \left( \frac{1}{w_n} \sum_{i=1}^{n} \sum_{j \in E_i} u_j s_j(T) \right)^p \leq \left( \frac{u_1}{u} \right)^p \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{k} \left( \sum_{j \in E_i} s_j(T) \right)^p
$$

for every $k \in \mathbb{N}$. From the properties of the function $\phi$, we obtain that

$$
u \|T\|_{\phi(p),E} \leq \|T\|^p_{\phi(p),E} \leq \left( \frac{u_1}{u} \right) \left( \frac{p}{p-1} \right) \|T\|_{\phi(p),E}.$$

\[\square\]

**Remark 2.4.** For the particular case, if we choose $N = 1$, we get Theorem 2.2 in [30]. And also if we take $u_i = \alpha_i$ and $w_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ in Theorem 3, where $N = 1$ then we obtain Theorem 1.4 in [28], where $\alpha_1 \leq 1$. If we take $u_i = 1$ and $w_n = n$ in Theorem 3, then we obtain Proposition 1.2 in [28].
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