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ON THE WALKS ON CAYLEY GRAPHS

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Abstract. Let G be a group and S be an inverse-closed subset of G which does not contain the identity element of G. The Cayley graph of G with respect to S, $\mathrm{Cay}(G, S)$, is a graph with the vertex set G and the edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. In this paper, we compute the number of walks of any length between two arbitrary vertices of $\mathrm{Cay}(G, S)$ in terms of complex irreducible representations of G. Using our main result, we give exact formulas for the number of walks of any length between two vertices in complete graphs, cycles, complete bipartite graphs, Hamming graphs and complete transposition graphs.
Keywords: Cayley graph; Hamming graphs; complete transposition graphs.

1. Introduction

Let G be a finite group and S be an inverse-closed subset of G not containing the identity element of G. The Cayley graph on G with respect to S, $\mathrm{Cay}(G, S)$, is a graph with the vertex set G and the edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. $\mathrm{Cay}(G, S)$ is an undirected loop-free regular graph of valency $|S|$. Many famous regular graphs can be represented as Cayley graphs. For example, cycles, complete graphs, Hamming graphs and complete transposition graphs are Cayley graphs. Some chemical graphs are Cayley graphs as well. For instance, the Buckyball, a soccer ball like molecule which consists of 60 carbon atoms, is a Cayley graph on the alternating group A_5 on 5 symbols with the connection set $\{(12345), (54321), (12)(23)\}$ [5, p. 209]. Also, the honeycomb toroidal graph is a Cayley graph on a generalized dihedral group [1, Theorem 3.4]. Since Cayley graphs possess many properties such as low degree, low diameter, symmetry, low congestion, high connectivity, high fault tolerance, and efficient routing algorithms, in the past several years there has been a spurt of research on using Cayley graphs in constructions of interconnection networks. For more details see [7].

A *walk* of length r from vertex x to vertex y in a graph Γ is a sequence of vertices $(v_0, v_1, \ldots, v_r)$ such that $v_0 = x$, $v_r = y$ and v_{i-1} is adjacent to v_i for all

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$1 \leq i \leq r$. If $x = y$ then the walk is called a closed walk of length r at vertex x. The number of walks in a graph is often necessary in, for instance, network analysis, epidemiology (requiring slow diffusion of viruses) and network design (aiming for fast data propagation) [3]. Also walks in molecular graphs and their counts for a long time have found applications in theoretical chemistry [6]. Furthermore, using counting closed walks, many non-Cayley vertex-transitive graphs are constructed [10, 11, 12, 13]. So it seems that computing the number of walks in Cayley graphs is important in graph theory. In this paper, we give an exact formula for the number of walks of any length between two vertices of a Cayley graph on a group G in terms of irreducible representations of G. For the representation group's theoretic and graph theoretic terminology not defined here, we refer the reader to [9] and [5], respectively.

2. Main Results

Let G be a finite group and $\mathbb{C}[G]$ be the complex vector space of dimension $|G|$ with basis $\{e_g \mid g \in G\}$. We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Thus a function $\varphi : G \to \mathbb{C}$ corresponds to the vector $\varphi = \sum_{g\in G} \varphi(g)e_g$ and vice versa. In particular, the vector e_g, where $g \in G$, of the standard basis corresponds to the function e_g, where

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

Let $A = [a_{x,y}]_{x,y\in G}$ be the adjacency matrix of $\Gamma = \mathrm{Cay}(G, S)$, $S = S^{-1} \subseteq G \setminus \{1\}$, where

$$a_{x,y} = \begin{cases} 1 & xy^{-1} \in S \\ 0 & xy^{-1} \notin S \end{cases}.$$

Then viewing A as a linear map on $\mathbb{C}[G]$, we have

$$Ae_x = \sum_{y\in G} a_{y,x}e_y = \sum_{y\in G, yx^{-1}\in S} e_y = \sum_{s\in S} e_{sx}. \tag{2.1}$$

Let $\omega_r(\Gamma; x, y)$ be the number of walks of length k from the vertex x to the vertex y in a graph Γ. We denote this by $\omega_r(x, y)$ when there is no ambiguity. Recall that for a graph Γ with adjacency matrix A, $\omega_r(\Gamma; x, y)$ is the xy-entry of A^r [5, Lemma 8.1.2]. In particular, $\omega_r(\Gamma) := \sum_{x\in V(\Gamma)} \omega_r(\Gamma; x, x)$, the total number of closed walks of length r, is the trace of A which is equal to the sum of rth powers of the adjacency eigenvalues of Γ [5, p. 165]. Let us start with an important lemma:

Lemma 2.1. *Let A be the adjacency matrix of $\Gamma = \mathrm{Cay}(G, S)$. Then*

$$A^r e_x = \sum_{y\in G} \omega_r(x, y)e_y.$$

Proof. We use induction on r. Since by (2.1), $Ae_x = \sum_{s\in S} e_{sx}$, and

$$\omega_1(x,y) = \begin{cases} 1 & yx^{-1} \in S \\ 0 & yx^{-1} \notin S, \end{cases}$$

the induction holds for $r = 1$. Now let $r \geq 2$ and the result hold for $r-1$. Since there exists a walk of length r from x to y if and only if there exists a walk of length $r-1$ of x to z where $yz^{-1} \in S$, we have

$$\omega_r(x,y) = \sum_{s\in S} \omega_{r-1}(x, s^{-1}y). \tag{2.2}$$

Now we have

$$\begin{aligned} A^r e_x &= A(A^{r-1}e_x) \\ &= A\Big(\sum_{y\in G} \omega_{r-1}(x,y)e_y\Big) \quad \text{(by induction hypothesis)} \\ &= \sum_{y\in G} \omega_{r-1}(x,y)Ae_y \\ &= \sum_{y\in G} \omega_{r-1}(x,y)\Big(\sum_{s\in S} e_{sy}\Big) \quad \text{(by (2.1))} \\ &= \sum_{z\in G}\sum_{s\in S} \omega_{r-1}(x, s^{-1}z)e_z \\ &= \sum_{z\in G} \omega_r(x,z)e_z, \quad \text{(by (2.2))} \end{aligned}$$

which completes the proof. □

Lemma 2.2. *Let A be the adjacency matrix of $\Gamma = \mathrm{Cay}(G,S)$. Then*

$$A^r e_x = \sum_{s_1,\ldots,s_r\in S} e_{s_r s_{r-1}\ldots s_1 x}.$$

Proof. We prove the result by induction. By 2.1, we have $Ae_x = \sum_{s\in S} e_{sx}$ which proves the result for $r = 1$. Let $r \geq 2$ and the result holds for $r-1$. Then

$$\begin{aligned} A^r e_x &= A(A^{r-1}e_x) \\ &= A\Big(\sum_{s_1,\ldots,s_{r-1}\in S} e_{s_{r-1}s_{r-2}\ldots s_1 x}\Big) \quad \text{(by induction hypothesis)} \\ &= \sum_{s_1,\ldots,s_{r-1}\in S} Ae_{s_{r-1}s_{r-2}\ldots s_1 x} \\ &= \sum_{s_1,\ldots,s_{r-1}\in S}\sum_{s_r\in S} e_{s_r(s_{r-1}\ldots s_1 x)} \quad \text{(by (2.1))} \\ &= \sum_{s_1,\ldots,s_r\in S} e_{s_r s_{r-1}\ldots s_1 x}, \end{aligned}$$

which completes the proof. □

Let $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$ be the set of all irreducible inequivalent $\mathbb{C}$-representations of G. Let d_k and $\varrho^{(k)}$ be the degree and a unitary matrix representation of ρ_k, $k = 1, \ldots, m$, respectively. We keep these notations throughout the paper. In the following lemma, which seems to be well-known, the authors constructed an orthogonal basis for $\mathbb{C}[G]$ using the matrix representations $\varrho^{(k)}$, $1 \le k \le m$.

Lemma 2.3. ([2, Lemma 1]) *Let $\varrho_{ij}^{(k)}(g)$ be the ijth entry of $\varrho^{(k)}(g)$, $1 \le i, j \le d_k$, and $\bar{\varrho}_{ij}^{(k)} = \sum_{g \in G} \overline{\varrho_{ij}^{(k)}(g)} e_g$. Then*

(i) $\{\bar{\varrho}_{ij}^{(k)} \mid 1 \le k \le m, 1 \le i, j \le d_k\}$ *form an orthogonal basis for* $\mathbb{C}[G]$,

(ii) $\rho_{\mathrm{reg}}(g)\bar{\varrho}_{ij}^{(k)} = \sum_{l=1}^{d_k} \varrho_{li}^{(k)}(g)\bar{\varrho}_{lj}^{(k)}$, *for all $g \in G$ and $1 \le i, j \le d_k$, $1 \le k \le m$, where ρ_{reg} is the left regular representation of G,*

(iii) $\mathbb{C}[G] = \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{d_k} W_j^{(k)}$, *where $W_j^{(k)} = \langle \bar{\varrho}_{ij}^{(k)} \mid 1 \le i \le d_k \rangle$ which is a ρ_{reg}-invariant subspace of $\mathbb{C}[G]$ of dimension d_k.*

Now we are ready to prove our main result. Let us denote the ij entry of a matrix X by $[X]_{ij}$. Then we have the following theorem.

Theorem 2.1. *Let $\Gamma = \mathrm{Cay}(G, S)$, $1 \notin S = S^{-1}$ and $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$. Then*

$$\omega_r(x, y) = \frac{1}{|G|} \sum_{k=1}^{m} \sum_{i,j=1}^{d_k} d_k \Big[(\sum_{s \in S} \varrho^{(k)}(s))^r \Big]_{ij} \Big[\varrho^{(k)}(xy^{-1}) \Big]_{ji}.$$

Proof. First, recall that the adjacency matrix A of Γ can be viewed as a linear map on $\mathbb{C}[G]$ and by Lemma $\mathbb{C}[G] = \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{d_k} W_j^{(k)}$, where $W_j^{(k)} = \langle \bar{\varrho}_{ij}^{(k)} \mid 1 \le i \le d_k \rangle$ which is a ρ_{reg}-invariant subspace of $\mathbb{C}[G]$ of dimension d_k. Since $A^r e_x \in \mathbb{C}[G]$, there exist complex numbers $\alpha_{ij}^{(k)}$, $1 \le i, j \le d_k$ such that

$$A^r e_x = \sum_{k=1}^{m} \sum_{i,j=1}^{d_k} \alpha_{ij}^{(k)} \bar{\varrho}_{ij}^{(k)}. \tag{2.3}$$

On the other hand, $\alpha_{ij}^{(k)} = \frac{\langle A^r e_x, \bar{\varrho}_{ij}^{(k)} \rangle}{\langle \bar{\varrho}_{ij}^{(k)}, \bar{\varrho}_{ij}^{(k)} \rangle}$, where $\langle u, v \rangle$ denotes the usual inner product of u and v in complex field vector spaces. Furthermore,

$$\begin{aligned}
\langle A^r e_x, \bar{\varrho}_{ij}^{(k)}\rangle &= \langle \sum_{s_1,\ldots,s_r\in S} e_{s_r s_{r-1}\ldots s_1 x}, \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)} e_g\rangle \quad \text{(by Lemma 2.2)}\\
&= \sum_{s_1,\ldots,s_r\in S} \langle e_{s_r s_{r-1}\ldots s_1 x}, \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)} e_g\rangle\\
&= \sum_{s_1,\ldots,s_r\in S} \sum_{g\in G} \varrho_{ij}^{(k)}(g)\langle e_{s_r s_{r-1}\ldots s_1 x}, e_g\rangle\\
&= \sum_{s_1,\ldots,s_r\in S} \varrho_{ij}^{(k)}(s_r s_{r-1}\ldots s_1 x)\\
&= \sum_{s_1,\ldots,s_r\in S} \Big[\varrho^{(k)}(s_r)\ldots\varrho^{(k)}(s_1)\varrho^{(k)}(x)\Big]_{ij} \quad \text{(since } \varrho^{(k)} \text{ is a homomorphism)}\\
&= \Big[\sum_{s_1,\ldots,s_r\in S} \varrho^{(k)}(s_r)\ldots\varrho^{(k)}(s_1)\varrho^{(k)}(x)\Big]_{ij}\\
&= \Big[(\sum_{s_r\in S}\varrho^{(k)}(s_r))\ldots(\sum_{s_1\in S}\varrho^{(k)}(s_1))\varrho^{(k)}(x)\Big]_{ij}\\
&= \Big[(\sum_{s\in S}\varrho^{(k)}(s))^r\varrho^{(k)}(x)\Big]_{ij}.
\end{aligned}$$

Also

$$\begin{aligned}
\langle \bar{\varrho}_{ij}^{(k)}, \bar{\varrho}_{ij}^{(k)}\rangle &= \langle \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)} e_g, \sum_{h\in G} \overline{\varrho_{ij}^{(k)}(h)} e_h\rangle\\
&= \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)} \sum_{h\in G} \varrho_{ij}^{(k)}(h)\langle e_g, e_h\rangle\\
&= \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)}\varrho_{ij}^{(k)}(g)\\
&= \sum_{g\in G} \varrho_{ji}^{(k)}(g^{-1})\varrho_{ij}^{(k)}(g) \quad \text{(since } \varrho^{(k)} \text{ is unitary)}\\
&= \frac{|G|}{d_k} \quad \text{(by Schur's relations).}
\end{aligned}$$

Hence $\alpha_{ij}^{(k)} = \frac{d_k}{|G|}\Big[(\sum_{s\in S}\varrho^{(k)}(s))^r\varrho^{(k)}(x)\Big]_{ij}$. Now from the equality (2.3), Lemma 2.1 and this fact that $\bar{\varrho}_{ij}^{(k)} = \sum_{g\in G}\varrho_{ji}^{(k)}(g^{-1})e_g$, we have

$$\omega_r(x,y) = \frac{1}{|G|}\sum_{k=1}^{m}\sum_{i,j=1}^{d_k} d_k\Big[(\sum_{s\in S}\varrho^{(k)}(s))^r\Big]_{ij}\Big[\varrho^{(k)}(xy^{-1})\Big]_{ji},$$

which completes the proof. □

Keeping the notations of Theorem 2.1, since $\varrho^{(k)}(1) = I_{d_k}$, we have the following direct consequence.

Corollary 2.1.

$$\omega_r(\Gamma : x, x) = \frac{1}{|G|}\sum_{k=1}^{m} d_k \operatorname{Tr}[(\sum_{s\in S} \varrho^{(k)}(s))^r],$$

where $\operatorname{Tr}[X]$ *denotes the trace of matrix* X*. In particular,*

$$\omega_r(\Gamma) = \sum_{k=1}^{m} d_k \operatorname{Tr}[(\sum_{s\in S} \varrho^{(k)}(s))^r].$$

Corollary 2.2. *([15, Theorem 2]) Let* $\Gamma = \operatorname{Cay}(G, S)$ *and* $1 \notin S = S^{-1}$ *be a union of conjugacy classes of* G*. Then*

$$\omega_r(x, y) = \frac{1}{|G|}\sum_{k=1}^{m} \frac{(\sum_{s\in S} \chi_k(s))^r \chi_k(xy^{-1})}{d_k^{r-1}}.$$

In particular, if G *is abelian then*

$$\omega_r(x, y) = \frac{1}{|G|}\sum_{k=1}^{|G|} (\sum_{s\in S} \chi_k(s))^r \chi_k(xy^{-1}).$$

Proof. First, note that S is a union of conjugacy classes if and only if for all $g \in G$ we have $g^{-1}Sg = S$. Thus for all $g \in G$, we have

$$\begin{aligned}\varrho^{(k)}(g^{-1})(\sum_{s\in S} \varrho^{(k)}(s))\varrho^{(k)}(g) &= \sum_{s\in S} \varrho^{(k)}(g^{-1}sg) \\ &= \sum_{s\in S} \varrho^{(k)}(s) \quad (\text{since } g^{-1}Sg = S).\end{aligned}$$

Hence by Schur's Lemma, $\sum_{s\in S} \varrho^{(k)}(s) = \frac{1}{d_k}\operatorname{Tr}(\sum_{s\in S} \varrho^{(k)}(s))I_{d_k} = \frac{\sum_{s\in S} \chi_k(s)}{d_k} I_{d_k}$. Now the result follows from Theorem 2.1. □

Let $G = \langle a\rangle \cong \mathbb{Z}_n$ be a cyclic group of order n. Then $\operatorname{Irr}(G) = \{\chi_i \mid i = 0, \ldots, n-1\}$, where $\chi_k(a^r) = \exp(2\pi ikr/n)$.

Corollary 2.3. *(See also [14]) Let* K_n *be a complete graph with* n *vertices. Then*

$$\omega_r(K_n; x, y) = \begin{cases} \frac{1}{n}((n-1)^r - (-1)^r) & x \neq y \\ \frac{n-1}{n}((n-1)^{r-1} - (-1)^{r-1}) & x = y. \end{cases}$$

Proof. Let $G = \langle a\rangle$ be a cyclic group of order n and $S = G \setminus \{1\}$. Then for all $g \in G$, $g^{-1}Sg = S$ and $K_n = \operatorname{Cay}(G, S)$. Hence, by Corollary 2.2,

$$\omega_r(K_n; x, y) = \frac{1}{n}\sum_{k=0}^{n-1} (\sum_{s\in S} \chi_k(s))^r \chi_k(xy^{-1}).$$

On the other hand

$$\sum_{s\in S}\chi_k(s)=\left\{\begin{array}{ll}-1 & k\neq 0\\ n-1 & k=0\end{array}\right.$$

Let $x=a^l$ and $y=a^{l'}$. Then $\chi_k(xy^{-1})=\exp(2k(l-l')\pi i/n)$, $k=0,\ldots,n-1$. It is clear that if $x=y$ then $\sum_{k=0}^{n-1}(\sum_{s\in S}\chi_k(s))^r\chi_k(xy^{-1})=(n-1)^r+(n-1)(-1)^r$. Since $z+z^2+\ldots+z^{n-1}=-1$ whenever z is a nth root of unity, we conclude that if $x\neq y$ then $\sum_{k=0}^{n-1}(\sum_{s\in S}\chi_k(s))^r\chi_k(xy^{-1})=(n-1)^r-(-1)^r$, which completes the proof. □

Corollary 2.4. *Let C_n be an n-cycle. Then $C_n=\mathrm{Cay}(G,S)$ where $G=\langle a\rangle$ and $S=\{a,a^{-1}\}$. Furthermore,*

$$\omega_r(C_n;a^l,a^{l'})=\frac{2^r}{n}\sum_{k=0}^{n-1}\cos^r(\frac{2\pi k}{n})\cos(\frac{2\pi k(l-l')}{n}).$$

Proof. Let $\chi_k\in\mathrm{Irr}(G)$. Then $\chi_k(a)+\chi_k(a^{-1})=2\cos(\frac{2\pi k}{n})$. Also $\chi_k(xy^{-1})=\cos(\frac{2\pi k(l-l')}{n})+i\sin(\frac{2\pi k(l-l')}{n})$. Furthermore, $\sum_{k=0}^{n-1}\cos(\frac{2\pi k}{n})^r\sin(\frac{2\pi k(l-l')}{n})=0$. Now the result follows immediately from Corollary 2.2. □

Corollary 2.5. *Let $K_{n,n}$ be the complete bipartite graph with $2n$ vertices, where $n\geq 3$. Then $K_{n,n}=\mathrm{Cay}(G,S)$, where $G=\langle a\rangle\cong\mathbb{Z}_{2n}$ and $S=\{a,a^3,\ldots,a^{2n-1}\}$.*

$$\omega_r(K_{n,n};a^l,a^{l'})=\frac{n^r+(-n)^r(-1)^{l-l'}}{2n}.$$

Proof. Let $w_k=\exp(\pi ik/n)$. Then irreducible characters of G are χ_k, $k=0,\ldots,2n-1$, where $\chi_k(a^l)=w_k^l$. For $k\neq 0,n$ we have $w_k+w_k^3+\ldots+w_k^{2n-1}=0$. Thus

$$\sum_{s\in S}\chi_k(s)=\left\{\begin{array}{ll}0 & k\neq 0,n\\ n & k=0\\ -n & k=n\end{array}\right..$$

Let $x=a^l$ and $y=a^{l'}$. Then $\chi_k(xy^{-1})=w_k^{l-l'}$ which completes the proof. □

Recall that the Hamming graph $H(n,m)$ is the graph whose vertex set is the Cartesian product of n copies of a set with m elements, where two vertices are adjacent if they differ in precisely one coordinate. $H(n,2)=Q_n$ is the familiar n-dimensional hypercube. It is well-known that $\Gamma=\mathrm{Cay}(G_1\times\ldots\times G_n,S)$ where $G_i=\langle a\rangle$, $i=1,\ldots,n$, is of order m and S is the set of all elements of $G_1\times\ldots\times G_n$ with exactly one non-identity coordinate. In the following example, we compute the number of walks between any two vertices in the Hamming graphs.

Corollary 2.6. *Let $\Gamma = H(n,m)$. Then*

$$\omega_r(\Gamma; x, y) = \frac{1}{m^n} \sum_{0 \le j_1, \dots, j_n \le m-1} \Big(n(m-1) - mc(j_1, \dots, j_n)\Big)^r \tau^{(r_1 - s_1)j_1 + \dots + (r_n - s_n)j_n},$$

where $x = (a^{r_1}, \dots, a^{r_n})$, $y = (a^{s_1}, \dots, a^{s_n})$ and $c(j_1, \dots, j_n)$ is the number of non-zero coordinates of $(j_1, \dots, j_n)$.. In particular,

$$\omega_r(Q_n; x, y) = \frac{1}{2^n} \sum_{0 \le j_1, \dots, j_n \le 1} \Big(n - 2c(j_1, \dots, j_n)\Big)^r \tau^{(r_1 - s_1)j_1 + \dots + (r_n - s_n)j_n},$$

where $x = (a^{r_1}, \dots, a^{r_n})$ and $y = (a^{s_1}, \dots, a^{s_n})$.

Proof. Let $\chi \in \mathrm{Irr}(G_1 \times \dots \times G_n)$ and $g = (a^{i_1}, \dots, a^{i_n}) \in G_1 \times \dots \times G_n$. Then there exist $(j_1, \dots, j_n)$, where $0 \le j_i \le m-1$, such that $\chi(g) = \tau^{i_1 j_1 + \dots + i_n j_n}$, where $\tau = \exp(2\pi i/m)$. Hence every irreducible character of $G_1 \times \dots \times G_n$ completely determined by an n-tuple $(j_1, \dots, j_n)$, where $0 \le j_i \le m-1$. Let us denote the corresponding character of this tuple by $\chi_{(j_1, \dots, j_n)}$.

Let $x = a^i \ne 1$ and $x^{(j)}$ be a $1 \times n$ vector that its only non-identity element is x at the jth position. Let $s \in S$. Then $s = (a^i)^{(k)}$ for some $1 \le i \le m-1$ and $1 \le k \le n$. Hence $\chi_{(j_1, \dots, j_n)}(s) = \tau^{ij_k}$ which implies that $\sum_{s \in S} \chi_{(j_1, \dots, j_n)}(s) = \sum_{k=1}^n \sum_{i=1}^{m-1} \tau^{ij_k}$. On the other hand,

$$\sum_{i=1}^{m-1} (\tau^{j_k})^i = \begin{cases} m-1 & j_k = 0 \\ -1 & j_k \ne 0 \end{cases}.$$

Let $c(j_1, \dots, j_n)$ be the number of non-zero coordinates of $(j_1, \dots, j_n)$. Then $\sum_{s \in S} \chi_{(j_1, \dots, j_n)}(s) = n(m-1) - mc(j_1, \dots, j_n)$. Now, by Corollary 2.2,

$$\omega_r(x, y) = \frac{1}{m^n} \sum_{0 \le j_1, \dots, j_n \le m-1} \Big(n(m-1) - mc(j_1, \dots, j_n)\Big)^r \tau^{(r_1 - s_1)j_1 + \dots + (r_n - s_n)j_n},$$

where $x = (a^{r_1}, \dots, a^{r_n})$ and $y = (a^{s_1}, \dots, a^{s_n})$. This completes the proof. $\square$

Recall that a *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ of positive integers such that $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_m$ and $\sum_{i=1}^m \lambda_i = n$. We write $\lambda \vdash n$ to indicate that λ is a partition of n. Since the inequivalent irreducible representations of the symmetric group S_n on n letters are conveniently by partitions of n, we write ρ_λ, χ_λ and d_λ for the irreducible representation, the character and the degree of the representation associated with $\lambda \vdash n$.

For $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$, put $l_i = \lambda_i + m - i$, $1 \le i \le m$. If $m = 1$ then $d_\lambda = 1$ and whenever $m > 1$, by [4, equality (4.11)] we have

$$d_\lambda = \frac{n!}{l_1! l_2! \dots l_m!} \prod_{i<j} (l_i - l_j). \tag{2.4}$$

Furthermore,

(1) if $\tau \in S_n$ is a transposition, then by [8, equality (5.1)],

$$\chi_\lambda(\tau) = \frac{M_2(\lambda)}{n(n-1)} d_\lambda, \tag{2.5}$$

(2) if $\tau \in S_n$ is a 3-cycle, then by [8, equality (5.2)]

$$\chi_\lambda(\tau) = \frac{M_3(\lambda) - 3n(n-1)}{2n(n-1)(n-2)} d_\lambda, \tag{2.6}$$

(3) if τ is a product of two disjoint transpositions, then by [8, equality (5.5)]

$$\chi_\lambda(\tau) = \frac{M_2(\lambda)^2 - 2M_3(\lambda) + 4n(n-1)}{n(n-1)(n-2)(n-3)} d_\lambda, \tag{2.7}$$

where

$$M_2(\lambda) = \sum_{j=1}^{m} \Big((\lambda_j - j)(\lambda_j - j + 1) - j(j-1) \Big)$$

and

$$M_3(\lambda) = \sum_{j=1}^{m} \Big((\lambda_j - j)(\lambda_j - j + 1)(2\lambda_j - 2j + 1) + j(j-1)(2j-1) \Big).$$

Corollary 2.7. *Let $\Gamma = \mathrm{Cay}(S_n, S)$, be the complete transposition graph, where S is the set of all transpositions of $\{1, \ldots, n\}$. Then for all $x \in S_n$, we have*

$$\omega_r(x, x) = \frac{1}{n! 2^r} \sum_{\lambda \vdash n} d_\lambda^2 M_2(\lambda)^r.$$

Furthermore, if $x \neq y$ be two non-disjoint transpositions then

$$\omega_r(x, y) = \frac{1}{n! 2^{r+1} n(n-1)(n-2)} \sum_{\lambda \vdash n} d_\lambda^2 M_2(\lambda)^r (M_3(\lambda) - 3n(n-1)),$$

and if they are disjoint, then

$$\omega_r(x, y) = \frac{1}{n! 2^r n(n-1)(n-2)(n-3)} \sum_{\lambda \vdash n} d_\lambda^2 M_2(\lambda)^r (M_2(\lambda)^2 - 2M_3(\lambda) + 4n(n-1)).$$

Proof. Since S is the set of all transpositions of S_n, it is a conjugacy class of S_n with $\frac{n(n-1)}{2}$ elements. On the other hand, by Equality (2.5), for any $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$ we have

$$\sum_{s \in S} \chi_\lambda(s) = |S| \chi_\lambda((1,2)) = \frac{M_2(\lambda)}{2} d_\lambda.$$

Let $x, y \in S_n$. If $x = y$ then $xy^{-1} = 1$ and $\chi_\lambda(xy^{-1}) = \chi_\lambda(1) = d_\lambda$. If $x \neq y$ and they are not disjoint transpositions then xy^{-1} is a 3-cycle. Now the result follows immediately from Corollary 2.2 and equalities (2.6) and (2.7). $\square$

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