ON STAR COLORING OF DEGREE SPLITTING OF COMB PRODUCT GRAPHS

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Abstract. A star coloring of a graph $G$ is a proper vertex coloring in which every path on four vertices in $G$ is not bi-colored. The star chromatic number $\chi_s(G)$ of $G$ is the least number of colors needed to star color $G$. Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_t \cup T$ where each $S_i$ is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. The degree splitting graph $DS(G)$ is obtained by adding vertices $w_1, w_2, \ldots, w_t$ and joining $w_i$ to each vertex of $S_i$ for $1 \leq i \leq t$. The comb product between two graphs $G$ and $H$, denoted by $G \triangledown H$, is a graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and grafting the $i^{th}$ copy of $H$ at the vertex $o$ to the $i^{th}$ vertex of $G$. In this paper, we give the exact value of star chromatic number of degree splitting of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph.

Keywords: Star coloring; degree splitting graph; comb product

1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph in [4, 5, 10]. The concept of star chromatic number was introduced by Branko Grunbaum in 1973. A star coloring [1, 8, 9] of a graph $G$ is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraph formed by the vertices of any two colors has connected components that are star graph. The star chromatic number $\chi_s(G)$ of $G$ is the least number of colors needed to star color $G$.

Guillaume Fertin et al. [8] determined the star chromatic number of trees, cycles, complete bipartite graphs, outer planar graphs and 2-dimensional grids. They also
investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, graphs with bounded treewidth and cubic graphs and planar graphs with high - girth.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when $G$ is a graph that is both planar and bipartite. Coleman et al. [6] proved that star coloring remains NP-hard problem even on bipartite graphs.

For a given graph $G = (V(G), E(G))$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \ldots S_t \cup T$ where each $S_i$ is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. The degree splitting graph [11, 12] of $G$, denoted by $DS(G)$, is obtained by adding vertices $w_1, w_2, \ldots w_t$ and joining $w_i$ to each vertex of $S_i$ for $1 \leq i \leq t$.

Comb product is also same as the hierarchical product graphs was first introduced by Barrière et al. [3] in 2009. Also, the exact value of metric dimension of hierarchical product graphs was obtained by Tavakoli et al. in [14]. Let $G$ and $H$ be two connected graphs. Let $o$ be a vertex of $H$. The comb product between $G$ and $H$, denoted by $G \triangleright H$, is a graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and grafting the $i^{th}$ copy of $H$ at the vertex $o$ to the $i^{th}$ vertex of $G$. By the definition of comb product, we can say that $V(G \triangleright H) = \{(a, u) | a \in V(G), u \in V(H)\}$ and $(a, u)(b, v) \in E(G \triangleright H)$ whenever $a = b$ and $uv \in E(H)$, or $ab \in E(G)$ and $u = v = o$. Ridho Alfarisi et al. [2] determined the partition dimension of comb product of path and complete graph and in [7] they also determined the star partition dimension of comb product of cycle and complete graph. Saputro et al. showed the metric dimension of comb product of the connected graphs $G$ and $H$ in [13].
In this paper, we have given the exact value of star chromatic number of degree splitting graph of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph denoted by $DS(K_m \bowtie K_n)$, $DS(K_m \bowtie P_n)$, $DS(K_m \bowtie C_n)$, $DS(K_m \bowtie K_{1,n})$, $DS(C_m \bowtie K_n)$, $DS(P_m \bowtie K_n)$ and $DS(C_m \bowtie P_n)$ respectively.

In order to prove our results, we shall make use of the following theorem by Guillaume et al. [8].

**Theorem 1.1.** [8] If $C_n$ is a cycle with $n \geq 3$ vertices, then

$$
\chi_s(C_n) = \begin{cases} 
4, & \text{when } n = 5 \\
3, & \text{otherwise.}
\end{cases}
$$

**Proof.** The proof of the theorem can be found in [8].
2. Main Results

In the following subsections, we will find the star chromatic number of degree splitting graph of comb product of complete with complete graph, complete with path, complete with cycle, complete with star graph, comb product of cycle with complete and cycle with path graph denoted by $DS(K_m \bowtie K_n)$, $DS(K_m \bowtie C_n)$, $DS(K_m \bowtie K_{1,n})$, $DS(K_m \bowtie P_n)$, $DS(C_m \bowtie K_n)$, $DS(P_m \bowtie K_n)$ and $DS(C_m \bowtie P_n)$ respectively. Figure 1 shows an example of degree splitting of comb product $(K_3 \bowtie K_5)$.

2.1. Star Coloring of Degree Splitting of $(K_m \bowtie K_n)$

The comb product between $K_m$ and $K_n$, denoted by $K_m \bowtie K_n$ has vertex set
\[ V(K_m \bowtie K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \]
and edge set
\[ E(K_m \bowtie K_n) = \{v_{i+1,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n-j\}. \]

Thus
\[ |V(K_m \bowtie K_n)| = mn \]
and
\[ |E(K_m \bowtie K_n)| = \frac{mn(n-1) + m(m-1)}{2}. \]

**Theorem 2.1.** Let $K_m$ and $K_n$ be two complete graphs of order $m, n \geq 3$ and $m \leq n$, then
\[ \chi_s(DS(K_m \bowtie K_n)) = m + n. \]

**Proof.** We have,
\[ V(K_m \bowtie K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2 \]
where
\[ S_1 = \{v_{i,1} : 1 \leq i \leq m\} \]
and
\[ S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}. \]
To obtain $DS(K_m \bowtie K_n)$ from $K_m \bowtie K_n$, we add two vertices $w_1$ and $w_2$ corresponding to $S_1$ and $S_2$ respectively. Thus, we get
\[ V(DS(K_m \bowtie K_n)) = V(K_m \bowtie K_n) \cup \{w_1, w_2\}. \]
First we find the upper bound for $\chi_s(DS(K_m \bowtie K_n))$. Clearly, $m + n$ colors are needed at least to star color $DS(K_m \bowtie K_n)$. We now distinguish $n$ as three cases: For every $1 \leq i \leq m$,
Case (i): When $n \equiv 3 \pmod{3}$.

\[
\sigma(v_{i,3k-2}) = i + j - 1, \text{ for } 1 \leq k \leq \frac{n}{3}
\]
\[
\sigma(v_{i,3k-1}) = i + j - 1, \text{ for } 1 \leq k \leq \frac{n}{3}
\]
\[
\sigma(v_{i,3k}) = i + j - 1, \text{ for } 1 \leq k \leq \frac{n}{3}
\]

and

\[
\sigma(w_1) = \sigma(w_2) = m + n
\]

Case (ii): When $n \equiv 1 \pmod{3}$.

\[
\sigma(v_{i,3k-2}) = i + j - 1, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]
\[
\sigma(v_{i,3k-1}) = i + j - 1, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]
\[
\sigma(v_{i,3k}) = i + j - 1, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

and

\[
\sigma(w_1) = \sigma(w_2) = m + n
\]

Case (iii): When $n \equiv 2 \pmod{3}$.

\[
\sigma(v_{i,3k-2}) = i + j - 1, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]
\[
\sigma(v_{i,3k-1}) = i + j - 1, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]
\[
\sigma(v_{i,3k}) = i + j - 1, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

and

\[
\sigma(w_1) = \sigma(w_2) = m + n.
\]

Thus, the upper bound for the star coloring of $(DS(K_m \triangleright K_n)) \leq m + n$.

Now, we prove the lower bound for $\chi_s(DS(K_m \triangleright K_n))$.

Suppose $\chi_s(DS(K_m \triangleright K_n)) < m + n$. Let $\chi_s(DS(K_m \triangleright K_n)) = m + n - 1$, then there exists a bicolored path $P_4$. Since $\{v_{1,i}\}$ induce a clique of order $n$ (say $K_n$). If we assign the same $n$ colors to the second copy of $K_n$, then we get a path on four vertices between these clique which is bicolored, a contradiction for proper star coloring. Thus, $\chi_s(DS(K_m \triangleright K_n)) = m + n - 1$ color is impossible. Therefore, the lower bound of $\chi_s(DS(K_m \triangleright K_n)) \geq m + n$. Thus we get the lower and upper bound of $\chi_s(DS(K_m \triangleright K_n))$. Hence $\chi_s(DS(K_m \triangleright K_n)) = m + n$. This completes the proof of the theorem. \qed
2.2. Star Coloring of Degree Splitting of \((K_m \bowtie C_n)\)

A graph \(K_m \bowtie C_n\) has vertex set 

\[ V(K_m \bowtie C_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \]

and edge set 

\[ |E(K_m \bowtie C_n)| = \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m, 1 \leq k \leq m-i\} \]

\[ \cup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{v_{m,1}v_{1,1}\}. \]

Thus 

\[ |V(K_m \bowtie C_n)| = mn \]

and 

\[ |E(K_m \bowtie C_n)| = \frac{m(m-1) + 2mn}{2}. \]

**Theorem 2.2.** Let \(K_m\) and \(C_n\) be two connected graphs of order \(m \geq 4\) and \(n \geq 5\), then 

\[ \chi_s(DS(K_m \bowtie C_n)) = m + 1. \]

**Proof.** We have 

\[ V(K_m \bowtie C_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2 \]

where 

\[ S_1 = \{v_{i,1} : 1 \leq i \leq m\} \]

and 

\[ S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}. \]

To obtain \(DS(K_m \bowtie C_n)\) from \(K_m \bowtie C_n\), we add two vertices \(w_1\) and \(w_2\) corresponding to \(S_1\) and \(S_2\) respectively. Thus we get 

\[ V(DS(K_m \bowtie C_n)) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{w_1, w_2\}. \]

We first prove the lower bound for the star chromatic number of degree splitting of comb product of complete graph with cycle. For this, we show that any coloring with \(m\) colors will give us at least one bicolored path of length 4. Since each \(\{v_{1,1} : 1 \leq i \leq m\}\) is adjacent to \(w_1\), it gives a complete graph of order \(m + 1\). Thus, no coloring that uses \(m\) colors can be a star coloring. Therefore, the lower bound of star chromatic number is 

\[ \chi_s(DS(K_m \bowtie C_n)) \geq m + 1. \]

Now, we prove the upper bound for the star chromatic number of degree splitting of \((K_m \bowtie C_n)\). Since the complete graph has the chromatic number \(m\). We assign the \(m\) colors to the \(mn\) vertices of the graph \(K_m \bowtie C_n\) alternatively and we assign \(\sigma(w_1) = \sigma(w_2) = m + 1\). Thus the upper bound of the \(\chi_s(DS(K_m \bowtie C_n)) \leq m + 1.\)

Thus we get the lower and upper bound of the \(\chi_s(DS(K_m \bowtie C_n))\). Therefore, 

\[ \chi_s(DS(K_m \bowtie C_n)) = m + 1. \]

This concludes the proof of the theorem. □
2.3. Star Coloring of Degree Splitting of \((K_m \bowtie P_n)\)

A graph \(K_m \bowtie P_n\) has vertex set

\[
V(K_m \bowtie P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}
\]

and edge set

\[
E(K_m \bowtie P_n) = \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m, 1 \leq k \leq m-i\}
\]

\[
\bigcup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}.
\]

Thus

\[
|V(K_m \bowtie P_n)| = mn
\]

and

\[
|E(K_m \bowtie P_n)| = \frac{m(m-1) + 2m(n-1)}{2}.
\]

**Theorem 2.3.** Let \(K_m\) be a complete graph of order \(m \geq 3\) and \(P_n\) be a path graph of order \(n \geq 3\) then,

\[
\chi_s(DS(K_m \bowtie P_n)) = m + 1.
\]

**Proof.** We have

\[
V(K_m \bowtie P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2 \cup S_3
\]

where

\[
S_1 = \{v_{i,1} : 1 \leq i \leq m\},
\]

\[
S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n-1\}
\]

and

\[
S_3 = \{v_{i,n} : 1 \leq i \leq m\}.
\]

To obtain \(DS(K_m \bowtie P_n)\) from \(K_m \bowtie P_n\), we add three vertices \(w_1, w_2\) and \(w_3\) corresponding to \(S_1, S_2\) and \(S_3\) respectively. Thus, \(V(DS(K_m \bowtie P_n)) = V(K_m \bowtie P_n) \cup \{w_1, w_2, w_3\}\). Now, we assign the following coloring pattern:

For every \(1 \leq i \leq m\)

For \(n \equiv 1(\text{mod } 3)\)

\[
\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

\[
\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

\[
\sigma(v_{i,3k}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]
and
\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1. \]

For \( n \equiv 2(\text{mod } 3) \)
\[ \sigma(v_{i,3k-2}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil \]
\[ \sigma(v_{i,3k-1}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil \]
\[ \sigma(v_{i,3k}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil \]
and
\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1. \]

For \( n \equiv 3(\text{mod } 3) \)
\[ \sigma(v_{i,3k-2}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \frac{n}{3} \]
\[ \sigma(v_{i,3k-1}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \frac{n}{3} \]
\[ \sigma(v_{i,3k}) = i + j - 1(\text{mod } m), \text{ for } 1 \leq k \leq \frac{n}{3} \]
and
\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1. \]

Thus the upper bound of star coloring of degree splitting of \((K_m \bowtie P_n) \leq m + 1\).

Now, we prove the lower bound of \( \chi_s(DS(K_m \bowtie P_n)) \). Suppose the lower bound of the
\[ \chi_s(DS(K_m \bowtie P_n)) < m + 1. \]
That is
\[ \chi_s(DS(K_m \bowtie P_n)) = m. \]

We must assign \( m \) colors for \( \{v_{i,1}, 1 \leq i \leq m\} \) for proper star coloring. Since each \( \{v_{i,1}\} \) is adjacent to \( w_1 \), it gives a complete graph of order \( m + 1 \). Therefore \( \chi_s(DS(K_m \bowtie P_n)) \) with \( m \) colors is impossible. Therefore \( \chi_s(DS(K_m \bowtie P_n)) \geq m + 1 \). Hence, \( \chi_s((DS(K_m \bowtie P_n)) = m + 1. \) This concludes the proof of the theorem. \( \square \)
2.4. Star Coloring of Degree Splitting of \((K_m \triangleright K_{1,n})\)

A graph \(K_m \triangleright K_{1,n}\) has a vertex set

\[
V (K_m \triangleright K_{1,n}) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}
\]

and edge set

\[
E (K_m \triangleright K_{1,n}) = \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m-1, 1 \leq k \leq m-i\}
\]

\[
\bigcup \{v_{i,1}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n\}.
\]

Thus

\[
|V (K_m \triangleright K_{1,n})| = mn
\]

and

\[
|E (K_m \triangleright K_{1,n})| = \frac{m(m-1) + 2mn}{2}.
\]

**Theorem 2.4.** Let \(K_m\) be a complete graph of order \(m\), \((m \geq 3)\) and \(K_{1,n}\) be a star graph with \(n + 1\) vertices \((n \geq 2)\) then

\[
\chi_s (DS (K_m \triangleright K_{1,n})) = m + 1.
\]

**Proof.** We have

\[
V (K_m \triangleright K_{1,n}) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2.
\]

To obtain \(DS(K_m \triangleright K_{1,n})\) from \((K_m \triangleright K_{1,n})\), we add two vertices \(w_1\) and \(w_2\) corresponding to \(S_1\) and \(S_2\) respectively, where

\[
S_1 = \{v_{i,1} : 1 \leq i \leq m\}
\]

and

\[
S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.
\]

Thus we get

\[
V (DS (K_m \triangleright K_{1,n})) = V (K_m \triangleright K_{1,n}) \cup \{w_1, w_2\}.
\]

Now we assign the coloring pattern as follows:

For every \(1 \leq i \leq m\), assign \(i\) to \(\sigma(v_{i,1})\), and

For \(2 \leq j \leq n\) assign

\[
\sigma(v_{i,j}) = \begin{cases} 2 & \text{if } i \equiv 1 \text{(mod 3)} \\ 3 & \text{if } i \equiv 2 \text{(mod 3)} \\ 1 & \text{if } i \equiv 3 \text{(mod 3)} \end{cases}
\]

alternatively
Thus \( \chi_s(DS(K_m \triangleright K_1, n)) \geq m + 1 \).

Now, we prove the lower bound of \( \chi_s(DS(K_m \triangleright K_1, n)) \). Suppose the lower bound of the \( \chi_s(DS(K_m \triangleright K_1, n)) < m + 1 \). That is \( \chi_s(DS(K_m \triangleright K_1, n)) = m \). We must assign \( m \) colors for \( \{v_{i,1} : 1 \leq i \leq m\} \) for proper star coloring. Since each \( \{v_{i,1}\} \) is adjacent to \( w_1 \), it gives a complete graph of order \( m + 1 \). Therefore \( \chi_s(DS(K_m \triangleright K_1, n)) \) with \( m \) color is impossible. Therefore \( \chi_s(DS(K_m \triangleright K_1, n)) \geq m + 1 \). Thus we get the lower and upper bound of \( \chi_s(DS(K_m \triangleright K_1, n)) \). Hence, \( \chi_s(DS(K_m \triangleright K_1, n)) = m + 1 \). It concludes the proof of the theorem.

### 2.5. Star Coloring of Degree Splitting of \( (C_m \triangleright K_n) \)

A graph \( C_m \triangleright K_n \) has vertex set

\[
V(C_m \triangleright K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}
\]

and edge set

\[
E(C_m \triangleright K_n) = \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m - 1\} \\
\cup \{v_{m,1}v_{1,1}\} \cup \{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n - j\}.
\]

Thus

\[
|V(C_m \triangleright K_n)| = mn
\]

and

\[
|E(C_m \triangleright K_n)| = \frac{mn^2 - mn + 2m}{2}.
\]

**Theorem 2.5.** Let \( C_m \) and \( K_n \) be two connected graphs of order \( m > n \) and \( m > 3, n \geq 3 \), then

\[
\chi_s(DS(C_m \triangleright K_n)) = m + 1.
\]

**Proof.** We have

\[
V(C_m \triangleright K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2
\]

where

\[
S_1 = \{v_{i,1} : 1 \leq i \leq m\}
\]

and

\[
S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.
\]

To obtain \( DS(C_m \triangleright K_n) \) from \( C_m \triangleright K_n \), we add two vertices \( w_1 \) and \( w_2 \) corresponding to \( S_1 \) and \( S_2 \) respectively. Thus we get \( V(DS(C_m \triangleright K_n)) = V(C_m \triangleright K_n) \cup \{w_1, w_2\} \). First we find the upper bound for \( \chi_s(DS(C_m \triangleright K_n)) \).

We define the coloring pattern as follows:
For every $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\sigma(v_{i,j}) = i + j - 1 \pmod{m}$$

and also

$$\sigma(w_1) = \sigma(w_2) = m + 1.$$ 

Thus the upper bound for star chromatic number of $(DS(C_m \bowtie K_n)) \leq m + 1$.

Now, we prove the lower bound for $\chi_s((DS(C_m \bowtie K_n))) < m + 1$. Let $\chi_s((DS(C_m \bowtie K_n)) = m$, then there exist a bicolored path $P_4$. Since $\{v_{1,1}\}$ induce a clique of order $n$. If we assign the same $n$ colors to the second copy of the clique, then we get a path on four vertices between these cliques which is bicolored, a contradiction for proper star coloring. Thus we obtain $\chi_s((DS(C_m \bowtie K_n)) = m$ color is impossible. It concludes that the lower bound is $\chi_s((DS(C_m \bowtie K_n)) \geq m + 1$. Therefore, $\chi_s((DS(C_m \bowtie K_n)) = m + 1$. Hence the proof of the theorem. 

2.6. Star Coloring of Degree Splitting of $(P_m \bowtie K_n)$

A graph $P_m \bowtie K_n$ has a vertex set

$$V(P_m \bowtie K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$E(P_m \bowtie K_n) = \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m - 1\}$$

$$\cup \{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n - j\}.$$ 

Thus

$$|V(P_m \bowtie K_n)| = mn$$

and

$$|E(P_m \bowtie K_n)| = \frac{mn(n - 1) + 2(m - 1)}{2}.$$ 

**Theorem 2.6.** Let $P_m$ be a path graph of order $m \geq 4$ and $K_n$ be a complete graph with $n \geq 2$, then

$$\chi_s(DS(P_m \bowtie K_n)) = n + 2.$$

**Proof.** We have

$$V(P_m \bowtie K_n) = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{v_{1,1}, v_{m,1}\},$$

$$S_2 = \{v_{i,1} : 2 \leq i \leq m - 1\}.$$
and
\[ S_3 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}. \]

To obtain \( DS(P_m \triangleright K_n) \) from \( P_m \triangleright K_n \), we add three vertices \( w_1, w_2 \) and \( w_3 \) corresponding to \( S_1, S_2 \) and \( S_3 \) respectively. Thus we get \( V(DS(P_m \triangleright K_n)) = \{v_{i,j} : 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{w_1, w_2, w_3\} \). First we find the upper bound for \( \chi_s(DS(P_m \triangleright K_n)) \).

For \( n \equiv 1(\text{mod } 3) \)
\[ \sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil \]
\[ \sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1 \]
\[ \sigma(v_{i,3k}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1 \]
and
\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2. \]

For \( n \equiv 2(\text{mod } 3) \)
\[ \sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil \]
\[ \sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil \]
\[ \sigma(v_{i,3k}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1 \]
and
\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2. \]

For \( n \equiv 3(\text{mod } 3) \)
\[ \sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \frac{n}{3} \]
\[ \sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \frac{n}{3} \]
\[ \sigma(v_{i,3k}) = i + j - 1(\text{mod } n + 1), \text{ for } 1 \leq k \leq \frac{n}{3} \]
and
\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2. \]

Thus \( \chi_s(DS(P_m \triangleright K_n)) \leq n + 2. \)
Now, we prove that $\chi_s(\text{DS}(P_m \triangleright K_n)) \geq n + 2$. Suppose $\chi_s(\text{DS}(P_m \triangleright K_n)) < n + 2$. That is $\chi_s(\text{DS}(P_m \triangleright K_n)) = n + 1$. Since $\{v_{1,i}\}$ induce a clique of order $n$. If we assign the same $n$ colors to the second copy of the clique, then we get a path on four vertices between these cliques which is bicolored, a contradiction for proper star coloring. Thus $\chi_s(\text{DS}(P_m \triangleright K_n)) \geq n + 1$. Therefore, $\chi_s(\text{DS}(P_m \triangleright K_n)) = n + 2$. Hence, there is another proof to the theorem. 

2.7. Star Coloring of Degree Splitting of $C_m \triangleright P_n$

A graph $C_m \triangleright P_n$ has vertex set

$$V(C_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$E(C_m \triangleright P_n) = \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m - 1\} \cup \{v_{m,1}v_{1,1}\} \cup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n - 1\}.$$ 

Thus

$$|V(C_m \triangleright P_n)| = mn$$

and

$$|E(C_m \triangleright P_n)| = m + m(n - 1).$$

Theorem 2.7. Let $C_m$ be a cycle of length $m \geq 3$ and $P_n$ be a path of length $n \geq 3$ then,

$$\chi_s(\text{DS}(C_m \triangleright P_n)) = \begin{cases} 4, & \text{if } m = 3k, k \geq 1 \\ 5, & \text{otherwise} \end{cases}.$$ 

Proof. We have

$$V(C_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\},$$

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n - 1\}$$

and

$$S_3 = \{v_{i,n} : 1 \leq i \leq m\}.$$ 

To obtain $\text{DS}(C_m \triangleright P_n)$ from $C_m \triangleright P_n$, we add three vertices $w_1$, $w_2$ and $w_3$ corresponding to $S_1$, $S_2$ and $S_3$ respectively. Thus we get $V(\text{DS}(C_m \triangleright P_n)) = V(C_m \triangleright P_n) \cup \{w_1, w_2, w_3\}$. First we find the upper bound for $\chi_s(\text{DS}(C_m \triangleright P_n))$.

The star chromatic number is defined as follows:

Case(i):

If $m = 3k, k \geq 1$
For $n \equiv 1 \pmod{3}$

\[
\sigma(v_{i,3k-2}) = i \pmod{3}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

\[
\sigma(v_{i,3k-1}) = i + 1 \pmod{3}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

\[
\sigma(v_{i,3k}) = i + 2 \pmod{3}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

and

\[
\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 4.
\]

For $n \equiv 2 \pmod{3}$

\[
\sigma(v_{i,3k-2}) = i \pmod{3}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

\[
\sigma(v_{i,3k-1}) = i + 1 \pmod{3}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

\[
\sigma(v_{i,3k}) = i + 2 \pmod{3}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor
\]

and

\[
\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 4.
\]

For $n \equiv 3 \pmod{3}$

\[
\sigma(v_{i,3k-2}) = i \pmod{3}, \text{ for } 1 \leq k \leq \frac{n}{3}
\]

\[
\sigma(v_{i,3k-1}) = i + 1 \pmod{3} \text{ for } 1 \leq k \leq \frac{n}{3}
\]

\[
\sigma(v_{i,3k}) = i + 2 \pmod{3}, \text{ for } 1 \leq k \leq \frac{n}{3}
\]

and

\[
\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 4.
\]

Thus, $\chi_s(DS(C_m \triangledown P_n)) = 4$ if $m = 3k$, $k \geq 1$.

Case(ii)(a): When $m = 3k + 1$, $k \in N$.

We color the $3k$ vertices of $C_m$ by $\sigma(v_{i,1}) = i \pmod{3}$ and we assign the remains of one uncolored vertex by 4.

Also, we assign

\[
\sigma(v_{i,j}) = i + j - 1 \pmod{3}.
\]
and

\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 5. \]

Case (ii)(b): When \( m = 3(k - 1) + 2, k \in N \), here \( m = 5 \) is not included. That is \( m = 3(k - 1) + 5, k \geq 2 \).

We color the \( 3(k - 1) \) vertices of \( C_m \) by 1, 2 and 3 and for the remaining five vertices assign the color 4, 1, 2, 3, 4.

Also, we assign

\[ \sigma(v_{i,j}) = i + j - 1 (\text{mod } 3). \]

and

\[ \sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 5. \]

Thus, \( \chi_s(DS(C_m \bowtie P_n)) = 5. \)

When \( m = 5 \), then \( \chi_s(DS(C_m \bowtie P_n)) = 5. \) Hence, the theorem is proved.

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