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NEW UPPER BOUND ON THE LARGEST LAPLACIAN EIGENVALUE OF GRAPHS

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Abstract. Let $G = (V, E)$ be a simple, undirected graph with maximum and minimum degree Δ and δ respectively, and let A be the adjacency matrix and Q be the Laplacian matrix of G. In the past decades, the Laplacian spectrum has received much more attention, since it has been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning. In this paper, we will compute lower and upper bounds for the largest Laplacian eigenvalue which is related to a given maximum and minimum degree and a given number of vertices and edges. We will also compare our results in this paper with other published results.
Keywords: Laplacian matrix; Laplacian spectrum; Laplacian eigenvalue; adjacency matrix.

1. Introduction

Let $G = (V, E)$ be a simple graph (i.e. finite, undirected graph without loops or multiple edges) on vertex set $V = \{v_1, ..., v_n\}$ and edge set $E = \{e_1, ..., e_m\}$ (so $n =| V(G) |$ is its order, and $m =| E(G) |$ is its size). For $v_i \in V(G)$, the degree of v_i, written by $d(v_i)$ or d_i , is the number of edges incident with v. Let $\Delta = \max \{d_i : v_i \in V(G)\}$ and $\delta = \min \{d_i : v_i \in V(G)\}$. Spectral graph theory [1, 2, 3] studies properties of graphs using the spectrum of related matrices. The most studied matrix associated with G appears to be the adjacency matrix $A = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. Another much studied matrix is the Laplacian matrix, defined by $Q(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, \ldots, d_n)$ (see [4, 5, 6]). Notice also that $Q = C^T C$, where C is the matrix whose rows are indexed by the edges of G and whose columns are indexed by its vertices, in which each row corresponding to the edge $e = \{u, v\}$,

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$(u < v)$, has a (1) in the column corresponding to u, a (-1) in that corresponding to v and 0 in every other place. Therefore Q is a symmetric, positive semi-definite matrix.

For an $n \times n$ real symmetric matrix M, its eigenvalues are real numbers. The eigenvalues (or spectrum) of $A(G)$ and $Q(G)$ which are real eigenvalues, are called A-eigenvalues (or A-spectrum) Q-eigenvalues (or Q-spectrum) respectively. These eigenvalues will be denoted by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant ... \geqslant \lambda_n(G)$ and $\mu = \mu_1(G) \geqslant \mu_2(G) \geqslant ... \geqslant \mu_n(G) = 0$ respectively.

2. Application

Applications of eigenvalue methods in combinatorics, graph theory and in combinatorial optimization have a long history. For example, eigenvalue bounds on the chromatic number were formulated by Wilf [7] and Homan [8] at the end of the sixties. Historically, the next applications related to combinatorial optimization, according to Fiedler [9] and Donath and Hoffman [10] in 1973, concerned the area of graph partition. A very important use of eigenvalues is the Lovász notion of the theta-function from 1979 [11]. Using it, he solved the long standing Shannon capacity problem for the 5-cycle. The theta-function provides the only known way to compute the chromatic number of perfect graphs in polynomial time.

The next important result was the use of eigenvalues in the construction of superconcentrators and expanders by Alon and Milman [12] in 1985. Their work motivated the study of eigenvalues of random regular graphs. Eigenvalues of random 01-matrices had already been studied by F. Juhász, who analyzed the behavior of the theta-function on random graphs, and introduced eigenvalues in clustering [13]. Isoperimetric properties of graphs and their eigenvalues play a crucial role in the design of several randomized algorithms. These applications are based on the so-called rapidly mixing Markov chains. The most important discoveries in this area include random polynomial time algorithms for approximating the volume of a convex body (cf., e.g., [14, 15, 16]), polynomial time algorithms for approximate counting (e.g., approximating the permanent or counting the number of perfect matchings, see [17] for additional information), etc. Isoperimetric properties and related expansion properties of graphs are the basis for various other applications, ranging from the fast convergence of Markov chains, efficient approximation algorithms, randomized or derandomized algorithms, complexity lower bounds, and building efficient communication networks and networks for parallel computation.

There are several known results that relate μ and to various structural properties of the graph G. In particular, there is a correspondence between μ and the expansion properties of G. Expander graphs have been widely used in Computer Science, in areas ranging from parallel computation to complexity theory and cryptography. See, e.g. [18]. In view of this correspondence, it is interesting to study the maximum possible value of μ for a graph with a given maximum and minimum degree and a given number of vertices and edges.

3. Main Results

There are some known results for upper bounds of μ. Research on the bound involving eigenvalues of A, Q has attracted much attention [19, 20]. In 1985, Anderson and Morley gave an upper bound for largest Laplacian graph eigenvalue in [21]. In 1997, Li and Zhang [22] improved researches of Anderson and Morley. In 1998, Merris [23] showed an upper bound of μ. In 1998, Li and Zhang [24] improved the researches of Merris. In 2000, Rojo et al. [25] obtained an always-nontrivial bound. In 2002, Pan [26] improved researches of Li and Zhang. In 2003, Das [27] improved the bound of Merris. In 2010, Dongmei Zhu gave a new upper bound in [28]. In the following part, we will compute lower and upper bounds for the largest Laplacian eigenvalue of G which is related with given a maximum and minimum degree and a given number of vertices and edges. We have also compared our results with other relevant results.

Theorem 1: Let G be a graph with n vertices and m edges. Then,

$$\mu \geqslant \frac{2m}{n-1} - \sqrt{\frac{n-2}{n-1}(\sum_{i=1}^{n}(d_i)^2 + 2m) - \frac{4m^2}{n-1} + \frac{4m^2}{(n-1)^2}}. \tag{3.1}$$

Proof. Let $\mu = \mu_1(G) \geqslant \mu_2(G) \geqslant ... \geqslant \mu_n(G) = 0$ be eigenvalues of Laplacian matrix of G. We know that

$$2m = \sum_{i=1}^{n} \mu_i, \tag{3.2}$$

and

$$\sum_{i=1}^{n} d_i^2 + 2m = \sum_{i=1}^{n} \mu_i^2. \tag{3.3}$$

Applying (3.2) and using the Cauchy-Schwarz inequality one can obtain

$$2m - \mu = \sum_{i=2}^{n-1} \mu_i^2 \leqslant \sqrt{n-2}\sqrt{\sum_{i=2}^{n-1} \mu_i^2}.$$

Raising both sides to power two and using (3.3), we obtain

$$(2m-\mu)^2 \leqslant (n-2)\left(\sum_{i=1}^{n}\mu_i^2-\mu^2\right) = (n-2)\left(\sum_{i=1}^{n}d_i^2+2m-\mu^2\right).$$

Thus

$$4m^2+\mu^2-4m\mu \leqslant (n-2)\left(\sum_{i=1}^{n}d_i^2+2m-\mu^2\right).$$

Therefore

$$(n-1)\mu^2-4m\mu \leqslant (n-2)\left(\sum_{i=1}^{n}d_i^2+2m\right)-4m^2.$$

Consequently,

$$\mu^2-(\tfrac{4m}{n-1})\mu \leqslant \tfrac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\tfrac{4m^2}{n-1}.$$

As a result, we have

$$(\mu-\tfrac{2m}{n-1})^2 \leqslant \tfrac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\tfrac{4m^2}{n-1}+\tfrac{4m^2}{(n-1)^2}.$$

Hence

$$\mu-\tfrac{2m}{n-1} \geqslant -\sqrt{\tfrac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\tfrac{4m^2}{n-1}+\tfrac{4m^2}{(n-1)^2}}.$$

Finally,

$$\mu \geqslant \frac{2m}{n-1}-\sqrt{\frac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\frac{4m^2}{n-1}+\frac{4m^2}{(n-1)^2}}$$

we complete the proof. □

Theorem 2: Let G be a simple graph with n vertices and m edges, and Δ , δ be the maximum and minimum degree of G respectively. Then we have

$$\mu \leqslant \sqrt{2m-(n-1)-\delta^2+\left(\frac{2\Delta-1}{2}\right)^2}+\left(\frac{2\Delta-1}{2}\right). \tag{3.4}$$

Proof. Let $X=(x_1,x_2,\ldots,x_n)^T$ be the eigenvector of $Q(G)$ and $\|X\|^2=1$ corresponding to $\mu(G)$. Let Q_i denote the ith row of Q. Let $X(i)$ denote the vector obtained from X by replacing x_j with 0 if v_i is not adjacent to v_j and replacing x_j with $(-x_j)$ if v_i is adjacent to v_j. Since

$$Q(G)X = \mu(G)X,$$

and

$$Q_iX(i) = \sum_{a_{ij}=1} x_j,$$

it follows that

$$d_ix_i - \mu x_i = Q_iX(i).$$

Both sides of the above equation are brought to power two, which leads to

$$d_i^2x_i^2 + \mu^2x_i^2 - 2\mu d_ix_i^2 = \|Q_iX(i)\|^2.$$

On the other hand, by the Lagrange identity we have

$$\|Q_iX(i)\|^2 = \|Q_i\|^2\|X(i)\|^2 - d_i^2\left(\sum_{a_{ij}=1} x_j^2\right) - \sum_{\substack{1\leq k<j\leq n\\ a_{ij}=a_{ik}=1}} (x_j - x_k)^2.$$

We also have

$$\|Q_i\|^2\|X(i)\|^2 = (d_i^2 + d_i)\left(\sum_{a_{ij}=1} x_j^2\right).$$

By summing over i and using Raleigh's relation we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\left(\sum_{\substack{1\leqslant k<j\leqslant n\\ a_{ij}=a_{ik}=1}} (x_j - x_k)^2\right) &\geqslant \sum_{\substack{1\leqslant k<j\leqslant n\\ a_{jk}=1}} (x_j - x_k)^2\\
&= \sum_{j=1}^{n} d_jx_j^2 - 2\sum_{a_{jk}=1} x_jx_k \geqslant \mu.
\end{aligned}$$

Note that we have three inequalities, (3.5),(3.6) and (3.7), as below:

$$\begin{aligned}
\sum_{i=1}^{n} d_i\left(\sum_{a_{ij}=1} x_j^2\right) &= \sum_{i=1}^{n} d_i\left(\sum_{i=1}^{n} x_i^2 - \sum_{a_{ij}=0} x_j^2\right)\\
&= \sum_{i=1}^{n} d_i - \left(\sum_{i=1}^{n} d_ix_i^2 + \sum_{i=1}^{n} d_i\Big(\sum_{\substack{a_{ij}=0\\ i\neq j}} x_j^2\Big)\right)\\
&\leqslant \sum_{i=1}^{n} d_i - \left(\sum_{i=1}^{n} d_ix_i^2 + \sum_{i=1}^{n}(n-1-d_i)x_i^2\right)\\
&= \sum_{i=1}^{n} d_i - (n-1),
\end{aligned} \tag{3.5}$$

$$\sum_{i=1}^{n} d_i^2 x_i^2 \geqslant \delta^2 \sum_{i=1}^{n} x_i^2 = \delta^2, \tag{3.6}$$

and

$$\sum_{i=1}^{n} 2\mu d_i x_i^2 \leqslant 2\mu\Delta \sum_{i=1}^{n} x_i^2 = 2\mu\Delta. \tag{3.7}$$

Then, by using the above inequalities it is possible to verify

$$\mu^2 \leqslant \sum_{i=1}^{n} d_i - (n-1) - \delta^2 + 2\Delta\mu - \mu$$

and further

$$\mu^2 + \mu - 2\mu\Delta \leqslant \sum_{i=1}^{n} d_i - (n-1) - \delta^2.$$

Thus

$$\left(\mu - \left(\frac{2\Delta - 1}{2}\right)^2\right) \leqslant \sum_{i=1}^{n} d_i - (n-1) - \delta^2 + \left(\frac{2\Delta - 1}{2}\right)^2.$$

Hence,

$$\mu \leqslant \sqrt{\sum_{i=1}^{n} d_i - (n-1) - \delta^2 + \left(\frac{2\Delta - 1}{2}\right)^2} + \left(\frac{2\Delta - 1}{2}\right)^2.$$

Finally,

$$\mu \leqslant \sqrt{2m - (n-1) - \delta^2 + \left(\tfrac{2\Delta-1}{2}\right)^2} + \left(\tfrac{2\Delta-1}{2}\right). \ \square$$

Remark 1: For circle graph, the upper bound in (3.4) occurs if $n \geqslant 7$. The upper bound in (3.4) is equal when $G = C_7$ be a circle graph with 7 vertices.

Remark 2: The upper bound in (3.4) and [28, 29] are comparable. For instance, let $G = K_n$ be a complete graph with n vertices. Then the upper bound of Laplacian matrix $G = K_n$ in (3.4) is $2\Delta - 1$ and the upper bound of Laplacian matrix $G = K_n$ in [28, 29] is 2Δ.

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