COUNTING THE NUMBER OF SUBGROUPS AND NORMAL SUBGROUPS OF THE GROUP $U_{2np}$, $p$ IS AN ODD PRIME

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Abstract. The aim of this paper is to compute the number of subgroups and normal subgroups of the group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e, aba^{-1} = b^{-1} \rangle$, where $p$ is an odd prime. Suppose $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ in which $p_i$'s are distinct odd primes, $\alpha_i$'s are positive integers and $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$. It is proved that the number of subgroups is $2\tau(2n) + (p-1) \left( \tau(n) + \tau(t) \right)$, when $p \mid n$ and $2\tau(2n) + (p-1) \left\lfloor \tau(t) \right\rfloor$, otherwise. It will be also proved that this group has $\tau(2n) + \tau(n)$ normal subgroups.

Keywords. group; subgroup; dihedral group; finite group.

1. Introduction

Cavior [1] proved that the number of subgroups of a dihedral group of order $2n$ can be computed by $\tau(n) + \sigma(n)$. After publishing this work Calhoun [2] computed the number of subgroups in certain finite groups. For more information on this problem, we encourage the readers to consult the interesting book of Tărnaţcu [6].

Following Darafsheh and Yaghoobian [3], we define:

$$U_{2nm} = \langle a, b \mid a^{2n} = b^m = e \mid aba^{-1} = b^{-1} \rangle.$$ 

This group has order $2nm$ and can be written as the semi-direct product of two cyclic groups that one of them is of order $m$ and another one has order $2n$. Set $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$, where $p_i$'s are distinct odd prime numbers and $\alpha_i$'s are positive integers. Shelash [4], introduced an algorithm for computing all subgroups and normal subgroups of a finite group. Shelash and Ashrafi [5] applied this algorithm to compute the number of minimal and maximal subgroups of certain finite groups.
Here, we apply this algorithm to obtain the number of subgroups and normal subgroups of the group $U_{2np}$, where $p$ is an odd prime.

The order table of $U_{2np}$ is defined as the matrix $A = [a_{ij}]$ with $a_{ij} = 2^{i-1}c_{j-1}$, $1 \leq i \leq \tau(2r+1)$ and $1 \leq j \leq \tau(\prod_{1 \leq i \leq s} p_i^{\alpha_i})$, where $c_j$ is an odd divisor of $|U_{2np}|$ and the function $\tau(n)$ is defined as the number of positive divisors of $n$. For simplicity of our argument, we assume that $c_0 < c_1 < \cdots < c_{a-1}$, where $a = \tau(\prod_{1 \leq i \leq s} p_i^{\alpha_i})$.

For example if $|G| = 60$, then the order table of $G$ is as follows:

<table>
<thead>
<tr>
<th>$c_j$</th>
<th>$a_{ij}$</th>
<th>$1$</th>
<th>$2$</th>
<th>$2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_{ij}$</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>15</td>
<td>30</td>
<td>60</td>
</tr>
</tbody>
</table>

Throughout this paper our notations are standard and can be taken from the standard books on group theory. The function $\sigma(n)$ is defined as the summation of all divisors of $n$. Furthermore, the number of subgroups and normal subgroups of a group $G$ are denoted by $Sub(G)$ and $NSub(G)$, respectively. Our calculations are done with the aid of GAP [7].

2. Main Results

The group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e \mid aba^{-1} = b^{-1} \rangle$ is a finite group of order $2np$, where $p$ is an odd prime. Suppose $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ in which $p_i$'s are distinct odd primes and $\alpha_i$'s are positive integers. For simplicity of our argument, we assume that $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$. If $p = p_k \mid n$ then the order of $U_{2np}$ is equal to $2^{r+1}p_1^{\alpha_1} \cdots p_k^{\alpha_k+1} \cdots p_s^{\alpha_s}$, otherwise it is $2^{r+1}p \prod_{1 \leq i \leq s} p_i^{\alpha_i}$.

Lemma 2.1. The following hold:

1. If $q$ is even then $a^q b^w = b^w a^q$;

2. If $q$ is odd then $a^q b^w = b^{-w} a^q$.

Proof. By presentation of the group $U_{2np}$, we have $aba^{-1} = b^{-1}$ and so if $q$ is even then $a^q b = b a^q$. Furthermore, if $q$ is odd then $a^q b = b^{-1} a^q$. Choose positive integer $w$. Then $a^q b^w = b a^q b^{w-1}$. If $q$ is even number, thus $a^q b^w = b^w a^q$. If $q$ is odd number then $a^q b^w = b^{-1} a^q b^{w-1}$, then $a^q b^w = b^{-w} a^q$. \(\square\)

Proposition 2.1. Let $n = 2^rt$, $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ and $m = p$ be an odd prime number. Then the structure description of the group $U_{2np}$ is $C_t \times (C_p : C_{2^{r+1}})$.

Proof. Suppose $\Phi = \langle a^{2^{r+1}} \rangle$, $\Psi = \langle b \rangle$ and $\Omega = \langle a^t \rangle$ are subgroups of $U_{2np}$. By Lemma 2.1, one can see that $g\Phi g^{-1} = g\langle a^{2^{r+1}} \rangle g^{-1} = \langle a^{2^{r+1}} \rangle = \Phi$, for all $g \in U_{2np}$. \(\square\)
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Thus $\Phi \leq U_{2np}$. Define $(\Psi : \Omega) = \langle b, a^i \rangle$. If $i$ is odd then,

$$a^i b^j (\Psi : \Omega) b^{-j} a^{-i} = \langle a^i b^j b^{-j} a^{-i}, a^i b^j a^i b^{-j} a^{-i} \rangle = \langle b, a^i b^j \rangle = (\Psi : \Omega),$$

and if $i$ is an even number,

$$a^i b^j (\Psi : \Omega) b^{-j} a^{-i} = \langle a^i b^j b^{-j} a^{-i}, a^i b^j a^i b^{-j} a^{-i} \rangle = \langle b, a^i b^j \rangle = (\Psi : \Omega).$$

Hence $(\Psi : \Omega)$ is a normal subgroup of $U_{2np}$. On the other hand, $\langle a^{2^r+1} \rangle \cap \langle b, a^i \rangle = e$ and $|\langle a^{2^r+1} \rangle \times |\langle b, a^i \rangle| = 2np$, which completes our argument. □

Lemma 2.2. The group $U_{2np}$ has the following types of subgroup:

1. The cyclic subgroups $\langle a^i \rangle$ of order $2^{np}i$, where $i \mid 2n$;
2. The subgroups $\langle a^i, b \rangle$ of order $2^{np}i$, where $i \mid 2n$;
3. The cyclic subgroups $\langle a^i b^j \rangle$, where $i \mid 2n$, $2p^k \not\mid i$ and $j = 1, \ldots, p-1$.

Proof. Set $H = \langle a^i \rangle$ and $K = \langle b \rangle$, $i \mid 2n$. By presentation of $U_{2np}$, $K$ is normal and so $HK = \langle a^i, b \rangle$ has order $2^{np}i$. The result now follows from Lemma 2.1. □

Proposition 2.2. Let $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ be a positive integer and $p$ be an odd prime number. The following hold:

1. There is at most one subgroup of order $k$ such that $2 \mid k, 2^{r+1} \not\mid k$ and $p \mid k$;
2. If $p \mid n$, then there exists one subgroup of order $k$ such that $p^{\alpha_i+1} \mid k$;
3. There exists $p$ subgroups of order $k$ when $p \mid k$ and $2^{r+1} \mid k$;
4. There exists $\sigma(p)$ subgroups of order $k$ when $p \mid k$ and $p^{\alpha_i+1} \not\mid k$.

Proof. Our main proof will consider the following parts:

1. Suppose $p \not\mid 2^h v$, $1 \leq h \leq r$, and $v \mid n$. Then $\langle a^{2^h v} \rangle$ is a cyclic group of order $2^h v$ and the order of subgroups $\langle a^{2^h v + h \cdot b} \rangle$ and $\langle a^{2^h v + h \cdot b} \rangle$ are not $2^h v$. We now apply Lemma 2.2 to get the result.
2. Suppose \( 2^{r+1} \mid k \). Since \( \frac{k}{2} \) is an odd number, by Lemma 2.1 \( \langle a^\frac{k}{2} b^j \rangle \) are cyclic subgroups of order \( 2^{r+1} \), \( 1 \leq j \leq p \).

3. Consider the subgroups \( \langle a^{\frac{2n}{p}} \rangle \) and \( \langle a^{\frac{2n}{p}}, b \rangle \), where \( 1 \leq h \leq r + 1 \). Since there are \( p - 1 \) subgroups of type \( \langle a^{\frac{2n}{p}} b^j \rangle \), \( 1 \leq j \leq p - 1 \), the number of all subgroups of order \( k \) is equal to \( \sigma(p) \).

Hence the result. \( \square \)

**Theorem 2.1.** Let \( p \) be an odd prime and \( n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i} \), where \( p_i \)'s are distinct odd primes, \( \alpha_i \)'s are positive integers and \( t = \prod_{1 \leq i \leq s} p_i^{\alpha_i} \). Then the number of all subgroups of the group \( U_{2np} \) is given by the following:

1. If \( p \mid n \) then \( \text{Sub}(U_{2np}) = 2\tau(2n) + (p - 1) [\tau(\frac{2n}{p}) + \tau(\frac{2n}{p^2})] \).
2. If \( p \nmid n \) then \( \text{Sub}(U_{2np}) = 2\tau(2n) + (p - 1) [\tau(t)] \).

**Proof.** By presentation of the group \( U_{2np} \), it has \( \tau(2n) \) subgroups contained in \( \langle a \rangle \). Since \( \langle b \rangle \) is a normal subgroup, the group \( U_{2np} \) has \( \tau(2n) \) subgroups of the form \( H \langle b \rangle \) such that \( H \) is a subgroup of \( \langle a \rangle \). We now assume that \( p \mid n \). By Lemma 2.2, it is enough to count the number of subgroups in the form \( \langle a^i b^j \rangle \), where \( i \mid 2n \), \( 2p^a \nmid i \) and \( 1 \leq j \leq p - 1 \). Note that \( 2n \) has exactly \( \tau(\frac{2n}{p}) = \tau(\frac{2n}{p^2}) \) odd divisors and the number of all divisors of \( 2n \) such that \( 2p \mid i \) and \( 2p^a \nmid i \) is equal to \( \tau(\frac{2n}{p}) = \tau(\frac{2n}{p^2}) \). So the group \( U_{2np} \) has exactly \( (p - 1)[\tau(\frac{2n}{p}) + \tau(\frac{2n}{p^2})] \) subgroups, when \( p \mid n \). If \( p \nmid n \), then the number of subgroups of type \( \langle a^i b^j \rangle \) is equal to \( (p - 1)\tau(\frac{2n}{p}) = (p - 1)\tau(t) \). \( \square \)

We are now ready to count the number of normal subgroups of the group \( U_{2np} \).

**Lemma 2.3.** The normal subgroup of the group \( U_{2np} \) has one of the following forms:

1. All cyclic subgroups \( \langle a^i \rangle \) such that \( 2 \mid i \mid 2n \);
2. All subgroups \( \langle a^i, b \rangle \), when \( i \mid 2n \).

**Proof.** The first part follows from Lemma 2.1. We apply the presentation of \( U_{2np} \) to prove that \( \langle a^k, b \rangle \) is normal, when \( k \mid 2n \). Choose the element \( a^i b^j \) in \( U_{2np} \). Then we have four cases for the subgroup \( a^i b^j (a^k, b) b^{-j} a^{-i} \) as follows:

1. \( k \) and \( i \) are even numbers. In this case \( a^i b^j a^{-i} b^-j a^{-i} = \langle a^k, b \rangle \), as desired.

2. \( k \) is even and \( i \) is odd. Then, \( a^i b^j a^{-i} b^{-j} a^{-i} = \langle a^k, b \rangle \) which proves our claim.
3. **k and i are odd numbers.** This shows that \( \langle a^ib^ja^{-i}, a^ib^ja^{-i} \rangle = \langle a^k, b \rangle \).

4. **k is even and i is odd.** In this case, \( \langle a^ib^ja^{-i}, a^ib^ja^{-i} \rangle = \langle a^k, b^2 \rangle = \langle a^k, b \rangle \).

Note that \( a^k \) and \( a^k b^j \) has the same order, when \( k \) is odd number. \( \square \)

Choose \( a^i \in U_{2np} \), where \( i \) is an odd number. Then \( a^i \langle a^ib \rangle a^{-i} = \langle a^i a^ib^ja^{-i} \rangle = \langle a^ib^j \rangle \). Since \( \langle a^ib^j \rangle \neq \langle a^ib \rangle \), all subgroups \( \langle a^ib^j \rangle \), \( 1 \leq j \leq p \) and \( i \mid 2n \), are not normal in \( U_{2np} \).

**Theorem 2.2.** The number of normal subgroups in the group \( U_{2np} \) is given by \( NSub(U_{2np}) = \tau(2n) + \tau(n) \).

**Proof.** Let \( p \) be an odd prime and \( n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i} \), where \( p_i \)'s are distinct odd primes, \( \alpha_i \)'s are positive integers and \( t = \prod_{1 \leq i \leq s} p_i^{\alpha_i} \). To prove the theorem, we apply Lemma 2.3. We now that each subgroup of type \( \langle a^i \rangle \), \( i \) is even, is normal. Since

\[
\tau(2^{r+1}t) - \tau(t) = \tau(2^{r+1}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = (r + 2)\tau(p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = (r + 1)\tau(p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = \tau(2^{r+1}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = \tau(n),
\]

\( \tau(2^{r+1}t) \) is the number all divisors of \( 2n \) and \( \tau(t) \) is the number of odd divisors of \( 2n \), \( \tau(2^{r+1}t) - \tau(t) = \tau(2^r t) = \tau(n) \) is the number of even divisors of \( 2n \). On the other hand, the number of all normal subgroups of type \( \langle a^i, b \rangle \), \( i \mid 2n \), is equal to \( \tau(2n) \). Therefore, \( NSub(U_{2np}) = \tau(2n) + \tau(n) \). \( \square \)

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**References**


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