FAULT-TOLERANT METRIC DIMENSION OF CIRCULANT GRAPHS

Narjes Seyedi and Hamid Reza Maimani

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Abstract. A set \( W \) of vertices in a graph \( G \) is called a resolving set for \( G \) if for every pair of distinct vertices \( u \) and \( v \) of \( G \), there exists a vertex \( w \in W \) such that the distance between \( u \) and \( w \) is different from the distance between \( v \) and \( w \). The cardinality of a minimum resolving set is called the metric dimension of \( G \), denoted by \( \beta(G) \). A resolving set \( W \) for \( G \) is fault-tolerant if \( W \setminus \{w\} \) is also a resolving set, for each \( w \) in \( W \). The fault-tolerant metric dimension of \( G \) is the size of a smallest fault-tolerant resolving set for \( G \), denoted by \( \beta'(G) \). In this paper, we study the fault-tolerant metric dimension of a family of circulant graphs \( X_{n,3} \) with connection set \( C = \{1, \frac{n}{2}, n-1\} \), when \( n \) is even and circulant graphs \( X_{n,4} \) with connection set \( C = \{\pm1, \pm2\} \).

Keywords. Circulant graphs; resolving set; fault-tolerant metric dimension.

1. Introduction

The metric dimension problem was introduced independently by Slater [15] and Harary and Melter [8]. The metric dimension arises in many diverse areas, including telecommunications [3], connected joints in graphs and chemistry [4], the robot navigation [12] and geographical routing protocols [13], etc.

For a connected graph \( G \) with vertex set \( V(G) \) and edge set \( E(G) \), the distance between two vertices \( u \) and \( v \) in \( V(G) \) is the number of edges in a shortest path connecting them, and is denoted by \( d(u, v) \). Consider an ordered set \( W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G) \). For each \( v \in V(G) \) the code of \( v \) with respect to \( W \) is \( (d(v, w_1), d(v, w_2), \cdots, d(v, w_k)) \), denoted by \( c_W(v) \). The set \( W \) is called a resolving set for \( G \), if all vertices of \( G \) have distinct codes. The minimum cardinality of a resolving set of \( G \) is called the metric dimension of \( G \) and is denoted by \( \beta(G) \). A resolving set of order \( \beta(G) \) is called a metric basis of \( G \) [2].

Elements of bases were referred to as sensors in an application given in [5]. If one
In this paper, we consider a family of circulant graphs $X = C$. Salman et al. [14] characterized the metric dimension for family of circulant graphs $k = G$. If $G$ is a circulant graph, then $G$ is a graph with vertex set $\mathbb{Z}_n$, an additive group of integers modulo $n$, and two vertices labeled $i$ and $j$ are adjacent if and only if $i - j \equiv 0 \pmod{n}$, where $C \subset \mathbb{Z}_n$, which is called a connection set, has the property that $C = -C$ and $0 \not\in C$. The circulant graph is denoted by $X_{n,\Delta}$ where $\Delta = |C|$. We compute the codes of all $v \in V(X_{n,3})$ for fixed $i; 0 \leq i \leq k$. The set $C = \{1, 2, n-1\}$, when $n$ is even and prove that the fault-tolerant metric dimension of this family of graphs is independent of choice of $n$ by showing that $\beta'(X_{n,3}) = 4$, for all $n \geq 4$ and $n \equiv 0 \pmod{4}$, in Theorem 2.2, and $\beta'(X_{n,3}) \leq 6$, for all $n \geq 10$ and $n \equiv 2 \pmod{4}$, in Theorem 2.4. We also consider a family of circulant graphs $X_{n,4}$ with connection set $C = \{\pm 1, \pm 2\}$ and prove that the fault-tolerant metric dimension of this family of graphs is independent of choice of $n$ by showing that $\beta'(X_{n,4}) = 4$, for all $n \geq 10$ and $n \equiv 2 \pmod{4}$, in Theorem 3.1.

2. Fault-Tolerant Metric Dimension Of Circulant Graphs $X_{n,3}$

Salman et al. [14] characterized the metric dimension for family of circulant graphs $X_{n,3}$ with connection set $C = \{1, \frac{2}{n} - 1\}$ for even $n$. Now we obtain the fault-tolerant metric dimension of this family of graphs.

Theorem 2.1. [14, Theorem 2.2] Let $n$ be an integer and $n \equiv 0 \pmod{4}$. If $k = \frac{n}{4}$, then for any $1 \leq i \leq k$ the set $W = \{v_i, v_{i+1}, v_{i+2k}\}$ is a resolving set and hence $\beta(X_{n,3}) = 3$.

The following lemma, gave a new family of resolving set of $X_{n,3}$ of size 3, where $n \equiv 0 \pmod{4}$.

Lemma 2.1. Let $n$ be an integer, $n \equiv 0 \pmod{4}$ and $k = \frac{n}{4}$. Then the set $W = \{v_i, v_{i+1}, v_{i+2k+1}\}$ is a resolving set of $X_{n,3}$, for any $1 \leq i \leq k$.

Proof. Let $W = \{v_i, v_{i+1}, v_{i+2k+1}\}$ for fixed $i; 0 \leq i \leq k$ where $k = \frac{n}{4}$. We compute the codes of all $v \in V(X_{n,3}) \setminus W$. We have

$$c_W(v_{i+k}) = (k, k-1, k), c_W(v_{i+k+1}) = (k, k, k), c_W(v_{i+3k}) = (k, k, k-1),$$
The codes of other vertices are listed in Table 1. By a simple computing these codes are distinct and hence $W$ is a resolving set of $X_{n, 3}$.

$\begin{array}{|c|c|c|}
\hline
\text{Shortest paths between} & v_i & v_{i+1} & v_{i+2k+1} \\
\hline
v_{i+j+1} (1 \leq j \leq k - 2) & j + 1 & j & j + 1 \\
v_{i+k+j} (2 \leq j \leq k - 1) & k - j + 1 & k - j + 2 & k - j + 1 \\
v_{i+2k+j} (2 \leq j \leq k - 1) & j & j - 1 & j - 1 \\
v_{i+3k+j} (2 \leq j \leq k - 1) & k - j & k - j + 1 & k - j + 2 \\
\hline
\end{array}$

□

**Theorem 2.2.** For all $n \geq 4$ and $n \equiv 0 \pmod{4}$, $\beta'(X_{n, 3}) = 4$.

**Proof.** From the definition of fault-tolerant metric dimension it can be seen that $\beta'(G) \geq \beta(G) + 1$ [11]. This implies that $\beta'(X_{n, 3}) \geq 4$ since $\beta(X_{n, 3}) = 3$ by Theorem 2.1.

Now for the lower bound, let $W' = \{v_i, v_{i+1}, v_{i+2k}, v_{i+2k+1}\}$ for fixed $i$; $0 \leq i \leq k$ where $k = \frac{n}{4}$. We will show that for each $x \in W'$, the set $W' \setminus \{x\}$ is a resolving set for $X_{n, 3}$. At first note that $\mathbb{Z}_n$ is subgroup of $\text{Aut}(X_{n, 3})$ and if $f = (v_0, v_1, \cdots, v_{n-1})$ is a cycle of order $n$, then $\mathbb{Z}_n \cong < f >$. In addition $f^j(v_i) = v_{i+j}$. Now we consider the following cases:

**Case 1.** Suppose that $x = v_i$. We have

$$f^j(\{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}) = \{v_{j+i+1}, v_{j+i+2k}, v_{j+i+2k+1}\}$$

If $j = 2k$, then

$$f^j(\{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}) = \{v_{i+2k+1}, v_{i}, v_{i+1}\},$$

and

$$f^j(\{v_{i+2k+1}, v_{i}, v_{i+1}\}) = \{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}.$$ 

By Lemma 2.1, $\{v_i, v_{i+1}, v_{i+2k+1}\}$ is a resolving set for $X_{n, 3}$ and since automorphisms of graphs preserves the properties of the graph, we conclude that $W' \setminus \{x\} = \{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}$ is a resolving set for $X_{n, 3}$.

**Case 2.** Let $x = v_{i+1}$. We have

$$f^j(\{v_i, v_{i+1}, v_{i+2k}\}) = \{v_{j+i}, v_{j+i+1}, v_{j+i+2k}\}.$$ 

If $j = 2k$, then

$$f^j(\{v_i, v_{i+1}, v_{i+2k}\}) = \{v_{i+2k}, v_{i+2k+1}, v_i\}.$$
Proof. By the same argument of Case 1, the set $W' \setminus \{x\} = \{v_{i+2k}, v_{i+2k+1}, v_i\}$ is a resolving set for $X_{n,3}$.

Case 3. If $x = v_{i+2k}$, then according to the Lemma 2.1, $W' \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Case 4. If $x = v_{i+2k+1}$, then according to the Theorem 2.1, $W' \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Therefore, $W'$ is the fault-tolerant resolving set for this family of graphs. Thus $\beta'(X_{n,3}) \leq 4$, for all $n \geq 10$ and $n \equiv 2 \pmod{4}$. This completes the proof.

Now we study the fault-tolerant metric dimension of $X_{n,3}$ in the case $n \equiv 2 \pmod{4}$.

**Theorem 2.3.** [14, Theorem 2.5] Let $n \geq 6$ be an integer and $n \equiv 2 \pmod{4}$. If $k = \frac{n-2}{4}$, then $W = \{v_i, v_{i+1}, v_{i+2k}, v_{i+2k+1}\}$ is a resolving set for $X_{n,3}$ for any $1 \leq i \leq k$. In addition $\beta(X_{n,3}) = 4$.

In the following lemma we gave some resolving sets of size 3 for $X_{n,3}$.

**Lemma 2.2.** Let $n \geq 10$ be an integer and $n \equiv 2 \pmod{4}$. For $k = \frac{n-2}{4}$ and any $1 \leq i \leq k$ the following sets are resolving sets of size 4 of $X_{n,3}$.

i) $W_1 = \{v_i, v_{i+1}, v_{i+2k+1}, v_{i+2k+2}\}$;

ii) $W_2 = \{v_{i+1}, v_{i+2k}, v_{i+2k+2}, v_{i+4k+1}\}$;

iii) $W_3 = \{v_i, v_{i+2k+1}, v_{i+2k+2}, v_{i+4k+1}\}$.

**Proof.** Suppose that $k = \frac{n-2}{4}$ and $W = \{v_i, v_{i+1}, v_{i+2k+1}, v_{i+2k+2}\}$ where $0 \leq i \leq k$. We compute $c_{W_1}(v)$ for $v \in V(X_{n,3}) \setminus W_1$. We have

$$c_{W_1}(v_{i+k}) = (k, k-1, k+1, k), c_{W_1}(v_{i+k+1}) = (k+1, k, k, k+1),$$

$$c_{W_1}(v_{i+2k}) = (2, 3, 1, 2), c_{W_1}(v_{i+2k+2}) = (k, k+1, k+1, k).$$

The codes of other vertices respect to $W_1$, are shown in Table 2. It is not difficult to see that all codes are distinct and hence $W_1$ is a resolving set of $X_{n,3}$.

<table>
<thead>
<tr>
<th>Shortest paths between</th>
<th>$v_i$</th>
<th>$v_{i+1}$</th>
<th>$v_{i+2k+1}$</th>
<th>$v_{i+2k+2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{i+j+1} (1 \leq j \leq k-2)$</td>
<td>$j+1$</td>
<td>$j$</td>
<td>$j$</td>
<td>$j+1$</td>
</tr>
<tr>
<td>$v_{i+k+1} (2 \leq j \leq k-1)$</td>
<td>$k-j+2$</td>
<td>$k-j+3$</td>
<td>$k-j+1$</td>
<td>$k-j+2$</td>
</tr>
<tr>
<td>$v_{i+2k+j+1} (2 \leq j \leq k)$</td>
<td>$j+1$</td>
<td>$j$</td>
<td>$j$</td>
<td>$j-1$</td>
</tr>
<tr>
<td>$v_{i+3k+j+1} (2 \leq j \leq k)$</td>
<td>$k-j+1$</td>
<td>$k-j+2$</td>
<td>$k-j+2$</td>
<td>$k-j+3$</td>
</tr>
</tbody>
</table>

For $W_2$ we have,

$$c_{W_2}(v_{i+k}) = (k-1, k, k+1), c_{W_2}(v_{i+k+1}) = (k, k-1, k+1, k),$$

$$c_{W_2}(v_{i+2k}) = (2, 3, 1, 2), c_{W_2}(v_{i+2k+2}) = (k, k+1, k+1, k).$$
and the codes of other vertices are listed Table 3. These codes are distinct and we conclude that \( W_2 \) is a resolving set for \( X_{n,3} \).

Similarly for \( W_3 \), we have
\[
cw_3(v_1) = (1, 2, 1, 2), cw_3(v_{i+k}) = (k+1, k, k+1, k), cw_3(v_{i+k+1}) = (k+1, k, k+1, k),
\]
and for other vertices, the codes are listed in Table 4. By these codes, we conclude that \( W_3 \) is a resolving set for \( X_{n,3} \).

**Theorem 2.4.** For all \( n \geq 6 \) and \( n \equiv 2 \ (mod \ 4) \), \( \beta'(X_{n,3}) \leq 6 \).

**Proof.** For \( n = 6 \), \( X_{6,3} \simeq K_{3,3} \). This implies that \( \beta'(X_{6,3}) = 6 \) since \( \beta'(K_{m,n}) = m + n \ [6, \text{Proposition 1}] \).

Now suppose that \( n \geq 10 \). Let \( k = \frac{n-2}{4} \). For a fixed \( i \), where \( 0 \leq i \leq k \), consider the set
\[
W = \{ v_i, v_{i+1}, v_{i+2k}, v_{i+2k+1}, v_{i+2k+2}, v_{i+4k+1} \}.
\]
Since \( W \) contains the set \( W_1 \) of Theorem 2.3 (i), so \( W \) is a resolving set for \( X_{n,3} \). Now we will show that for each \( x \in W' \), the set \( W' \setminus \{ x \} \) is a resolving set for \( X_{n,3} \). We have the following cases:

**Case 1.** If \( x \in \{ v_i, v_{i+2k+1} \} \), then \( W \setminus \{ x \} \) contains a set \( W_2 \) listed in Lemma 2.2 (ii). Thus \( W \setminus \{ x \} \) is a resolving set for \( X_{n,3} \).

**Case 2.** If \( x = v_{i+1} \), then \( W \setminus \{ x \} \) contains a set \( W_3 \) listed in Lemma 2.2 (iii). So \( W \setminus \{ x \} \) is a resolving set for \( X_{n,3} \).

### Table 3

<table>
<thead>
<tr>
<th>Shortest paths between</th>
<th>( v_{i+1} )</th>
<th>( v_{i+2k} )</th>
<th>( v_{i+2k+2} )</th>
<th>( v_{i+4k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_{i+j+1}(1 \leq j \leq k-2) )</td>
<td>( j )</td>
<td>( j+3 )</td>
<td>( j+1 )</td>
<td>( j+2 )</td>
</tr>
<tr>
<td>( v_{i+k+j}(2 \leq j \leq k-1) )</td>
<td>( k-j+3 )</td>
<td>( k-j )</td>
<td>( k-j+2 )</td>
<td>( k-j+1 )</td>
</tr>
<tr>
<td>( v_{i+2k+j+1}(2 \leq j \leq k) )</td>
<td>( j )</td>
<td>( j+1 )</td>
<td>( j-1 )</td>
<td>( j )</td>
</tr>
<tr>
<td>( v_{i+3k+j+1}(2 \leq j \leq k-1) )</td>
<td>( k-j+2 )</td>
<td>( k-j+1 )</td>
<td>( k-j+3 )</td>
<td>( k-j )</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>Shortest paths between</th>
<th>( v_i )</th>
<th>( v_{i+2k+1} )</th>
<th>( v_{i+2k+2} )</th>
<th>( v_{i+4k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_{i+j+1}(1 \leq j \leq k-2) )</td>
<td>( j+1 )</td>
<td>( j+2 )</td>
<td>( j+1 )</td>
<td>( j+2 )</td>
</tr>
<tr>
<td>( v_{i+k+j}(2 \leq j \leq k-1) )</td>
<td>( k-j+2 )</td>
<td>( k-j+1 )</td>
<td>( k-j+2 )</td>
<td>( k-j+1 )</td>
</tr>
<tr>
<td>( v_{i+2k+j+1}(2 \leq j \leq k) )</td>
<td>( j+1 )</td>
<td>( j )</td>
<td>( j-1 )</td>
<td>( j )</td>
</tr>
<tr>
<td>( v_{i+3k+j+1}(2 \leq j \leq k-1) )</td>
<td>( k-j+1 )</td>
<td>( k-j+2 )</td>
<td>( k-j+3 )</td>
<td>( k-j )</td>
</tr>
</tbody>
</table>
Case 3. If \( x = v_{i+2k} \), then \( W \setminus \{x\} \) contains a set \( W \) listed in Lemma 2.2 (i). Hence \( W \setminus \{x\} \) is a resolving set for \( X_{n,3} \).

Case 4. If \( x \in \{v_{i+2k+2}, v_{i+4k+1}\} \), then \( W \setminus \{x\} \) contains a set \( W' \) listed in Theorem 2.3. Hence \( W \setminus \{x\} \) is a resolving set for \( X_{n,3} \).

Therefore, \( W \) is the fault-tolerant resolving set for this family of graphs. Thus \( \beta'(X_{n,3}) \leq 6 \), for all \( n \geq 10 \) and \( n \equiv 2 \pmod{4} \).

3. Fault-Tolerant Metric Dimension Of Circulant Graphs \( X_{n,4} \)

In this section consider \( X_{n,4} \) with connection set \( C = \{\pm 1, \pm 2\} \). In [1], Borchert and Gosselin showed that \( \dim(X_{n,4}) = 4 \) if \( n = 1 \pmod{4} \) and \( \dim(X_{n,4}) = 3 \) otherwise. Now we study the fault-tolerant metric dimension of this family of graphs in the case \( n \equiv 2 \pmod{4} \).

In the following lemma we obtain some resolving sets for \( X_{n,4} \).

**Lemma 3.1.** Let \( n \geq 10 \) and \( n \equiv 2 \pmod{4} \). For \( k = \frac{n-2}{4} \) and any \( 1 \leq i \leq k \), the following sets are resolving sets for \( X_{n,4} \),

i) \( W_1 = \{v_i, v_{i+1}, v_{i+2}\} \);
ii) \( W_2 = \{v_i, v_{i+1}, v_{i+3}\} \);
iii) \( W_3 = \{v_i, v_{i+2}, v_{i+3}\} \).

**Proof.** The set \( W_1 \) is a resolving set by [10, Theorem 5]. For the parts (ii) and (iii), we prove that the sets \( W_2 \) and \( W_3 \) are resolving sets of \( X_{n,4} \) for \( i = 0 \). The remaining cases, obtained by this fact that \( \mathbb{Z}_n \) is a subgraph of \( Aut(X_{n,4}) \). By a simple computing we can obtain the codes of vertices respect to \( W_2 \) and \( W_3 \). These codes listed in Table 5 and Table 6. Clearly these codes are distinct and hence the sets \( W_2 \) and \( W_3 \) are resolving sets.

**Table 5**

<table>
<thead>
<tr>
<th>Shortest paths between</th>
<th>( v_0 )</th>
<th>( v_1 )</th>
<th>( v_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( v_3 ) (4 \leq j \leq \frac{n}{2})</td>
<td></td>
<td>( \frac{n-j}{2} )</td>
<td></td>
</tr>
<tr>
<td>( v_{j+1} )</td>
<td>( \frac{n-j}{2} )</td>
<td>( \frac{n-j}{2} )</td>
<td>( \frac{n-j+1}{2} )</td>
</tr>
<tr>
<td>( v_{j+2} )</td>
<td>( \frac{n-j}{2} )</td>
<td>( \frac{n-j}{2} )</td>
<td>( \frac{n-j-2}{2} )</td>
</tr>
<tr>
<td>( v_{j+3} )</td>
<td>( \frac{n-j}{2} )</td>
<td>( \frac{n-j}{2} )</td>
<td>( \frac{n-j-3}{2} )</td>
</tr>
<tr>
<td>( v_3 ) (( \frac{n}{4} + 4 \leq j \leq n - 1 ))</td>
<td></td>
<td>( \frac{n-j}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

\[\square\]

**Theorem 3.1.** For \( n \geq 10 \) and \( n \equiv 2 \pmod{4} \), \( \beta'(X_{n,4}) = 4 \).
Table 6

<table>
<thead>
<tr>
<th>Shortest paths between</th>
<th>v₀</th>
<th>v₂</th>
<th>v₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>v₁</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>v₂ (4 ≤ j ≤ ( \frac{n}{2} ))</td>
<td>( \left\lfloor \frac{j}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{j-1}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{j-2}{2} \right\rfloor )</td>
</tr>
<tr>
<td>v₃₊₁</td>
<td>( \left\lfloor \frac{n-j}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-1}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-2}{2} \right\rfloor )</td>
</tr>
<tr>
<td>v₃₊₂</td>
<td>( \left\lfloor \frac{n-j}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-1}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-2}{2} \right\rfloor )</td>
</tr>
<tr>
<td>v₃₊₃</td>
<td>( \left\lfloor \frac{n-j}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-1}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-2}{2} \right\rfloor )</td>
</tr>
<tr>
<td>v₄ (( \frac{n}{2} + 4 \leq j \leq n - 1 ))</td>
<td>( \left\lfloor \frac{n-j}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-1}{2} \right\rfloor )</td>
<td>( \left\lfloor \frac{n-2}{2} \right\rfloor )</td>
</tr>
</tbody>
</table>

Proof. From the definition of fault-tolerant metric dimension it can be seen that \( \beta'(G) \geq \beta(G) + 1 \) [11]. This implies that \( \beta'(X_{n,4}) \geq 4 \) since \( \beta(X_{n,4}) = 3 \) [1].

Now for the lower bound, consider the set \( W' = \{v₁, v₂, v₃, v₄\} \). Since \( W \) contains the set \( W₁ \) listed in Theorem 3.1, so \( W \) is a resolving set for \( X_{n,4} \). Now we will show that for each \( x \in W \), the set \( W \setminus \{x\} \) is a resolving set for \( X_{n,4} \). If \( x \in \{v₁, v₄\} \), then \( W \setminus \{x\} \) is a resolving set by setting \( i = 1 \) and \( i = 2 \) in part (i) of Lemma 3.1. If \( x = v₂ \), then by setting \( i = 1 \) in part (iii) of Lemma 3.1, we conclude that \( W \setminus \{x\} \) is a resolving set. Finally \( \{v₁, v₂, v₄\} \) is a resolving set of \( X_{n,4} \) by Lemma 3.1 (ii). Therefore \( \beta'(X_{n,4}) = 4 \)

\( \square \)

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