

**SOLVABILITY FOR A CLASS OF NONLINEAR
CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL
EQUATIONS WITH p -LAPLACIAN OPERATOR IN BANACH
SPACES**

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Abstract. This paper is devoted to the existence of solutions for certain classes of nonlinear differential equations involving the Caputo-Hadamard fractional-order with p -Laplacian operator in Banach spaces. The arguments are based on Mönch's fixed point theorem combined with the technique of measures of noncompactness. An example is also presented to illustrate the effectiveness of the main results.

Keywords: Banach spaces; differential equations; Caputo-Hadamard fractional-order; Laplacian operator; Mönch's fixed point theorem.

1. Introduction

Fractional calculus is a branch of mathematical analysis that deals with the derivatives and integrals of arbitrary (non-integer) order. In fact fractional calculus has developed into an important field of research during the last few decades in view of its widespread applications in a variety of disciplines such as physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, financial mathematics, economics, etc. (see [28, 32, 33, 37, 38, 42, 44]).

Fixed point theory is an important tool in Nonlinear Analysis, in particular, in obtaining existence results for a variety of mathematical problems. Although there are many methods (such as Banach contraction principle, Schauder's fixed point theorem, and Krasnoselskii's fixed point theorem, etc.) to analyze, under suitable conditions, the existence and uniqueness of solution of various problems with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions, and periodic boundary conditions for fractional differential equations,

for more details see for instance [5, 13, 18, 19, 20, 21, 26, 29, 43]. We focus here on the so-called measures of noncompactness, see for instance [1, 2, 4, 12, 14, 39]. A key result in the cited works to prove the existence of solutions is the celebrated Mönch fixed point theorem, based on such measures. We also refer the readers to the recent book [11], where several applications of the measure of noncompactness can be found.

A p-Laplacian differential equation was first introduced by Leibenson [31] when he studied the turbulent flow in a porous medium. Since then, fractional differential equations and the differential equation with a p-Laplacian operator are widely applied in different fields of physics and natural phenomena, for example, non-Newtonian mechanics, fluid mechanics, viscoelasticity mechanics, combustion theory, mathematical biology, the theory of partial differential equations. Hence, there have been many published papers that are devoted to the existence of solutions of boundary value problems for the p-Laplacian operator equations, see [9, 22, 23, 34, 35, 41] and their references. On the other hand, it has been noticed that most of the above-mentioned work on the topic is based on Riemann–Liouville or Caputo fractional derivatives. In 1892, Hadamard [27] introduced another fractional derivative, which differs from the above-mentioned ones because its definition involves the logarithmic function of arbitrary exponent and named the Hadamard derivative. For some developments on the f of the Hadamard fractional differential equations, we can refer to [15, 17, 25, 43]. Very recently Jarad et al [30] have modified the Hadamard fractional derivative into a more suitable one having physical interpretable initial conditions similar to the singles in the Caputo setting and called it Caputo–Hadamard fractional derivative. Details and properties of the modified derivative can be found in [30]. To the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo–Hadamard fractional differential equations [2, 7, 8, 16]. There are no contributions, as far as we know, concerning the Caputo–Hadamard fractional differential equations with p-Laplacian operator in Banach spaces. As a result, the goal of this paper is to enrich this academic area. So, in this paper, we mainly study the following problem of Caputo–Hadamard fractional differential equation with p-Laplacian operator of the form

$$(1.1) \quad {}^C_H\mathcal{D}_1^\beta \left(\phi_p \left[{}^C_H\mathcal{D}_1^\alpha u(t) \right] \right) = f(t, u(t)), \quad 1 < \alpha \leq 2 \quad t \in J := [1, T],$$

supplemented with boundary conditions

$$(1.2) \quad \begin{aligned} a_1 u(1) + b_1 {}^C_H\mathcal{D}_1^\gamma u(1) &= \lambda_1 {}^H\mathcal{I}_1^{\sigma_1} u(\eta_1), \quad 0 < \gamma \leq 1 \\ a_2 u(T) + b_2 {}^C_H\mathcal{D}_1^\gamma u(T) &= \lambda_2 {}^H\mathcal{I}_1^{\sigma_2} u(\eta_2), \quad 1 < \eta_1, \eta_2 < T \\ {}^C_H\mathcal{D}_1^\alpha u(1) &= 0, \end{aligned}$$

where ${}^C_H\mathcal{D}_1^\mu$ is the Caputo–Hadamard fractional derivative order $\mu \in \{\alpha, \beta, \gamma\}$ such that $1 < \alpha \leq 2, 0 < \beta, \gamma \leq 1$, ${}^H\mathcal{I}_1^\theta$ is the Hadamard fractional integral of order $\theta > 0, \theta \in \{\sigma_1, \sigma_2\}$ and $f : [1, T] \times E \rightarrow E$ is a given function satisfying some

assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$. $a_i, b_i, \lambda_i, i = 1, 2$ are suitably chosen real constants.

This paper is organized as follows. The second section provides the definitions and preliminary results to be used in this paper. The existence results, which rely on Mönch's fixed point theorem have been presented in Section 3. An example illustrating the obtained results is presented in Section 4, and the paper concludes with some conclusions in Section 5.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Let $C(J, E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm

$$\|u\|_\infty = \sup\{\|u(t)\|, t \in J\}.$$

By $L^1(J)$ we denote the space of Bochner-integrable functions $u : J \rightarrow E$, with the norm

$$\|u\|_1 = \int_1^T \|x(t)\| dt.$$

Next, we define the Kuratowski measure of noncompactness and give some of its important properties.

Definition 2.1. [10] Let E be a Banach space, Ω_E the bounded subsets of E . The Kuratowski measure of noncompactness is the map $\kappa : \Omega_E \rightarrow [0, \infty)$ defined by

$$\kappa(B) = \inf\{\varepsilon > 0 : B \subset \cup_{j=1}^m B_j, B \in \Omega_E \text{ and } \text{diam}(B_j) \leq \varepsilon\}.$$

Properties 2.1. *The Kuratowski measure of noncompactness satisfies some properties. For more details see [10].*

- (1) $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$,
- (2) $\kappa(A) = 0 \Leftrightarrow A$ is relatively compact,
- (3) $\kappa(A) = \kappa(\bar{A}) = \kappa(\text{conv}(A))$, where \bar{A} and $\text{conv}A$ represent the closure and the convex hull of A , respectively,
- (4) $\kappa(A + B) \leq \kappa(A) + \kappa(B)$,
- (5) $\kappa(\lambda A) = |\lambda| \kappa(A), \lambda \in \mathbb{R}$.

Now, we give some results and properties from the theory of fractional calculus. We begin by defining Hadamard fractional integrals and derivatives. In what follows,

Definition 2.2. [33] The Hadamard fractional integral of order $\alpha > 0$, for a function $u \in L^1(J)$, is defined as

$$\left({}^H\mathcal{I}_1^\alpha u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s}, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt, \quad \alpha > 0.$$

Set

$$\delta = t \frac{d}{dt}, \quad \alpha > 0, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of α . Define the space

$$AC_\delta^n[1, T] := \{u : [1, T] \longrightarrow \mathbb{R} : \delta^{n-1}u(t) \in AC([1, T])\}.$$

Definition 2.3. [33] The Hadamard fractional derivative of order $\alpha > 0$ applied to the function $u \in AC_\delta^n[1, T]$ is defined as

$$\left({}^H\mathcal{D}_1^\alpha u\right)(t) = \delta^n \left({}^H\mathcal{I}_1^{n-\alpha} u\right)(t).$$

Definition 2.4. [30, 33] The Caputo–Hadamard fractional derivative of order $\alpha > 0$ applied to the function $u \in AC_\delta^n[1, T]$ is defined as

$$\left({}^C_H\mathcal{D}_1^\alpha u\right)(t) = \left({}^H\mathcal{I}_1^{n-\alpha} \delta^n u\right)(t).$$

Lemmas of the following type are rather standard in the study of fractional differential equations.

Lemma 2.1. [30, 33] *Let $\alpha > 0, r > 0, n = [\alpha] + 1$, and $a > 0$, then the following relations hold*

- $\left({}^H\mathcal{I}_1^\alpha \left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \frac{\Gamma(r)}{\Gamma(\alpha+r)} \left(\log \frac{t}{a}\right)^{\alpha+r-1},$
-

$$\left({}^C_H\mathcal{D}_1^\alpha \left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \frac{\Gamma(r)}{\Gamma(r-\alpha)} \left(\log \frac{t}{a}\right)^{r-\alpha-1}, \quad (r > n),$$

$$\left({}^C_H\mathcal{D}_1^\alpha \left(\log \frac{s}{a}\right)^{r-1}\right)(t) = 0, \quad r \in \{0, \dots, n-1\}.$$

Lemma 2.2. [24, 33] *Let $\alpha > \beta > 0$, and $u \in AC_\delta^n[1, T]$. Then we have:*

- ${}^H\mathcal{I}_1^\alpha {}^H\mathcal{I}_1^\beta u(t) = {}^H\mathcal{I}_1^{\alpha+\beta} u(t),$

- ${}^C_H\mathcal{D}_1^\alpha {}^H\mathcal{I}_1^\alpha u(t) = u(t)$,
- ${}^C_H\mathcal{D}_1^\beta {}^H\mathcal{I}_1^\alpha u(t) = {}^H\mathcal{I}_1^{\alpha-\beta} u(t)$.

Lemma 2.3. [30, 33] *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $u \in AC_\delta^n[1, T]$, then the Caputo-Hadamard fractional differential equation*

$$\left({}^C_H\mathcal{D}_1^\alpha u\right)(t) = 0,$$

has a solution:

$$u(t) = \sum_{j=0}^{n-1} c_j (\log t)^j,$$

and the following formula holds:

$${}^H\mathcal{I}_1^\alpha \left({}^C_H\mathcal{D}_1^\alpha u(t)\right) = u(t) + \sum_{j=0}^{n-1} c_j (\log t)^j,$$

where $c_j \in \mathbb{R}, j = 0, 1, 2, \dots, n - 1$.

Remark 2.1. Note that for an abstract function $u : J \rightarrow E$, the integrals which appear in the previous definitions are taken in Bochner's sense. (see, for instance, [40]).

In the sequel we will make use of the following fixed point theorem.

Theorem 2.2. (Mönch's fixed point theorem [36]). *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let \mathcal{N} be a continuous mapping of D into itself. If the implication*

$$(2.1) \quad V = \overline{\text{conv}}\mathcal{N}(V), \text{ or } V = \mathcal{N}(V) \cup \{0\} \Rightarrow \kappa(V) = 0,$$

holds for every subset $V \subset D$, then \mathcal{N} has a fixed point.

Lemma 2.4. [6] *Let $H \subset C(J, E)$ be a bounded and equicontinuous subset. Then the function $t \rightarrow \kappa(H(t))$ is continuous on J , and*

$$\kappa_C(H) = \max_{t \in J} \kappa(H(t)),$$

and

$$\kappa\left(\int_J u(s) ds\right) \leq \int_J \kappa(H(s)) ds,$$

where $H(s) = \{u(s) : u \in H, s \in J\}$, and κ_C is the Kuratowski measure of noncompactness defined on the bounded sets of $C(J, E)$.

Definition 2.5. [45] A function $f : [1, T] \times E \rightarrow E$ is said to satisfy the Carathéodory conditions, if the following hold

- $f(t, u)$ is measurable with respect to t for $u \in E$,
- $f(t, u)$ is continuous with respect to $u \in E$ for a.e. $t \in J$.

3. Main Results

Before starting and proving our main result, we have introduced the following auxiliary lemma.

Lemma 3.1. *For a given $h \in C(J, \mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$(3.1) \quad {}^C_H\mathcal{D}_1^\alpha u(t) = h(t), \quad 1 < \alpha \leq 2 \quad t \in J := [1, T],$$

supplemented with boundary conditions

$$(3.2) \quad \begin{aligned} a_1 u(1) + b_1 {}^C_H\mathcal{D}_1^\gamma u(1) &= \lambda_1 {}^H\mathcal{I}_1^{\sigma_1} u(\eta_1), \quad 0 < \gamma \leq 1 \\ a_2 u(T) + b_2 {}^C_H\mathcal{D}_1^\gamma u(T) &= \lambda_2 {}^H\mathcal{I}_1^{\sigma_2} u(\eta_2), \quad 1 < \eta_1, \eta_2 < T, \end{aligned}$$

is given by

$$(3.3) \quad \begin{aligned} u(t) &= {}^H\mathcal{I}_1^\alpha h(t) + \mu_1(t) {}^H\mathcal{I}_1^{\sigma_1+\alpha} h(\eta_1) + \mu_2(t) \left(\lambda_2 {}^H\mathcal{I}_1^{\sigma_2+\alpha} h(\eta_2) \right. \\ &\quad \left. - (a_2 {}^H\mathcal{I}_1^\alpha h(T) + b_2 {}^H\mathcal{I}_1^{\alpha-\gamma} h(T)) \right), \end{aligned}$$

where

$$(3.4) \quad \begin{aligned} \mu_1(t) &= \lambda_1 (\Delta_1 - \Delta_2 \log t), \quad \mu_2(t) = \lambda_1 \Delta_3 + \Delta_4 \log t, \\ \Delta_1 &= \frac{1}{\Delta} \left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2+2)} \right); \\ \Delta_2 &= \frac{1}{\Delta} \left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2+1)} \right) \\ \Delta_3 &= \frac{\lambda_1 (\log \eta_1)^{\sigma_1+1}}{\Delta \Gamma(\sigma_1+2)}; \\ \Delta_4 &= \frac{1}{\Delta} \left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1+1)} \right) \\ \Delta &= \left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1+1)} \right) \left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2+2)} \right) \\ &\quad + \frac{\lambda_1 (\log \eta_1)^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2+1)} \right) \neq 0. \end{aligned}$$

Proof. By applying Lemma 2.3, we may reduce (3.1) to an equivalent integral equation

$$(3.5) \quad u(t) = {}^H\mathcal{I}_1^\alpha h(t) + k_0 + k_1 \log(t), \quad k_0, k_1 \in \mathbb{R}.$$

Applying the boundary conditions (3.2) in (3.5) we may obtain

$$\begin{aligned} {}^H\mathcal{I}_1^{\sigma_i} u(\eta_i) &= {}^H\mathcal{I}_1^{\sigma_i+\alpha} h(\eta_i) + k_0 \frac{(\log \eta_i)^{\sigma_i}}{\Gamma(\sigma_i+1)} + k_1 \frac{(\log \eta_i)^{\sigma_i+1}}{\Gamma(\sigma_i+2)}, \quad i = 1, 2. \\ {}^C_H\mathcal{D}_1^\gamma u(T) &= {}^H\mathcal{I}_1^{\alpha-\gamma} h(T) + k_1 \frac{\Gamma(2)}{\Gamma(2-\gamma)} (\log T)^{1-\gamma}. \end{aligned}$$

After collecting the similar terms in one part, we have the following equations:

$$\begin{aligned} \left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) k_0 &- \frac{\lambda_1 (\log \eta_1)^{\sigma_1 + 1}}{\Gamma(\sigma_1 + 2)} k_1 = \lambda_1 {}^H\mathcal{I}_1^{\sigma_1 + \alpha} h(\eta_1), \\ \left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2 + 1)}\right) k_0 &+ \left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2 + 1}}{\Gamma(\sigma_2 + 2)}\right) k_1 \\ &= \lambda_2 {}^H\mathcal{I}_1^{\sigma_2 + \alpha} h(\eta_2) - a_2 {}^H\mathcal{I}_1^\alpha h(T) - b_2 {}^H\mathcal{I}_1^{\alpha-\gamma} h(T). \end{aligned}$$

Therefore, we get

$$\begin{aligned} k_0 &= \frac{\lambda_1}{\Delta} \left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2 + 1}}{\Gamma(\sigma_2 + 2)}\right) {}^H\mathcal{I}_1^{\alpha + \sigma_1} h(\eta_1) \\ &+ \frac{\lambda_1 (\log \eta_1)^{\sigma_1 + 1}}{\Delta \Gamma(\sigma_1 + 2)} (\lambda_2 {}^H\mathcal{I}_1^{\sigma_2 + \alpha} h(\eta_2) - (a_2 {}^H\mathcal{I}_1^\alpha h(T) + b_2 {}^H\mathcal{I}_1^{\alpha-\gamma} h(T))), \\ k_1 &= \frac{\lambda_1}{\Delta} \left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) (\lambda_2 I_{0^+}^{\sigma_2 + \alpha} h(\eta_2) - (a_2 {}^H\mathcal{I}_1^\alpha h(T) + b_2 {}^H\mathcal{I}_1^{\alpha-\gamma} h(1))), \\ &- \frac{\lambda_1}{\Delta} \left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2 + 1)}\right) I_{0^+}^{\sigma_1 + \alpha} h(\eta_1). \end{aligned}$$

Substituting the value of k_0, k_1 in (3.5) we get (3.3), which completes the proof. \square

Lemma 3.2. *Let $1 < \alpha \leq 2, 0 < \beta \leq 1$. Then the boundary value problem of the fractional differential equation*

$$(3.6) \quad {}^C_H\mathcal{D}_1^\beta \left(\phi_p \left[{}^C_H\mathcal{D}_1^\alpha u(t) \right]\right) = f(t, u(t)), \quad 1 < \alpha \leq 2 \quad t \in J := [1, T],$$

supplemented with boundary conditions

$$(3.7) \quad \begin{aligned} a_1 u(1) + b_1 {}^C_H\mathcal{D}_1^\gamma u(1) &= \lambda_1 {}^H\mathcal{I}_1^{\sigma_1} u(\eta_1), \quad 0 < \gamma \leq 1 \\ a_2 u(T) + b_2 {}^C_H\mathcal{D}_1^\gamma u(T) &= \lambda_2 {}^H\mathcal{I}_1^{\sigma_2} u(\eta_2), \quad 1 < \eta_1, \eta_2 < T \\ {}^C_H\mathcal{D}_1^\alpha u(1) &= 0, \end{aligned}$$

has a unique solution

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &+ \frac{\mu_1(t)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &+ \frac{\lambda_2 \mu_2(t)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
& - \frac{a_2 \mu_2(t)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
(3.8) \quad & - \frac{b_2 \mu_2(t)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\end{aligned}$$

Proof. From Lemma 2.3 and the boundary value problem (3.6)–(3.7), we have

$$\begin{aligned}
{}^H\mathcal{I}_1^\beta {}^C\mathcal{D}_1^\beta \left(\phi_p \left[{}^C\mathcal{D}_1^\alpha u(t) \right] \right) &= {}^H\mathcal{I}_1^\beta f(t, u(t)) \\
&= \phi_p \left[{}^C\mathcal{D}_1^\alpha u(t) \right] + d_0, \quad d_0 \in \mathbb{R},
\end{aligned}$$

that is

$$\phi_p \left[{}^C\mathcal{D}_1^\alpha u(t) \right] = {}^H\mathcal{I}_1^\beta f(t, u(t)) - d_0, \quad d_0 \in \mathbb{R},$$

By ${}^C\mathcal{D}_1^\alpha u(1) = 0$, we have $d_0 = 0$. So,

$${}^C\mathcal{D}_1^\alpha u(t) = \phi_q \left[{}^H\mathcal{I}_1^\beta f(t, u(t)) \right].$$

Thus, the boundary value problem (3.6)–(3.7) is equivalent to the following problem:

$${}^C\mathcal{D}_1^\alpha u(t) = \phi_q \left[{}^H\mathcal{I}_1^\beta f(t, u(t)) \right], \quad t \in J := [1, T],$$

$$\begin{aligned}
a_1 u(1) + b_1 {}^C\mathcal{D}_1^\gamma u(1) &= \lambda_1 {}^H\mathcal{I}_1^{\sigma_1} u(\eta_1), \quad 0 < \gamma \leq 1 \\
a_2 u(T) + b_2 {}^C\mathcal{D}_1^\gamma u(T) &= \lambda_2 {}^H\mathcal{I}_1^{\sigma_2} u(\eta_2), \quad 1 < \eta_1, \eta_2 < T.
\end{aligned}$$

Lemma 3.1 implies that boundary value problem (3.6)–(3.7) has a unique solution,

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&+ \frac{\mu_1(t)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&+ \frac{\lambda_2 \mu_2(t)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&- \frac{a_2 \mu_2(t)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&- \frac{b_2 \mu_2(t)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\end{aligned}$$

This completes the proof. \square

In the following, for computational convenience we put

$$\begin{aligned} \mathcal{M}_f &= \frac{\|p_f\|\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}} \left\{ \frac{(\log T)^{\alpha+\beta(q-1)}(1+|a_2|\tilde{\mu}_2)}{\Gamma(\alpha+\beta(q-1)+1)} + \frac{\tilde{\mu}_1(\log \eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{\Gamma(\sigma_1+\alpha+\beta(q-1)+1)} \right. \\ &+ \left. \tilde{\mu}_2 \left[\frac{|\lambda_2|(\log \eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)} + \frac{|b_2|(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)} \right] \right\}, \end{aligned} \tag{3.9}$$

where $\tilde{\mu}_1 = |\lambda_1|(|\Delta_1| + |\Delta_2| \log T)$, $\tilde{\mu}_2 = |\lambda_1 \Delta_3| + |\Delta_4| \log T$,

$$\begin{aligned} \mathcal{L}_f &= \frac{\|p_f\|\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}} \left\{ \frac{(\log T)^{\alpha+\beta(q-1)-1}}{\Gamma(\alpha+\beta(q-1))} + \frac{|\lambda_1 \Delta_2|(\log \eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{T\Gamma(\sigma_1+\alpha+\beta(q-1)+1)} \right. \\ &+ \frac{|\Delta_4|}{T} \left[\frac{|\lambda_2|(\log \eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)} + \frac{|a_2|(\log T)^{\alpha+\beta(q-1)}}{\Gamma(\alpha+\beta(q-1)+1)} \right. \\ &\left. \left. + \frac{|b_2|(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)} \right] \right\} \end{aligned} \tag{3.10}$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1.1)–(1.2)

Theorem 3.1. *Assume that the following hypotheses hold:*

(H1) *The function $f : [1, T] \times E \rightarrow E$ satisfies Carathéodory conditions*

(H2) *There exist $p_f \in L^\infty(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\|f(t, u)\| \leq \phi_p(p_f(t)\psi(\|u\|)) \text{ for a.e. } t \in J \text{ and each } u \in E.$$

(H3) *For each bounded set $D \subset E$, and each $t \in J$, the following inequality holds*

$$\kappa(f(t, D)) \leq \phi_p(p_f(t)\kappa(D)).$$

$$\text{If} \tag{3.11} \quad \mathcal{M}_f < 1,$$

then the problem (1.1)–(1.2) has at least one solution defined on J .

Proof. Consider the operator $\mathcal{N} : C(J, E) \rightarrow C(J, E)$ defined by:

$$\mathcal{N}u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}$$

$$\begin{aligned}
& + \frac{\mu_1(t)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& + \frac{\lambda_2 \mu_2(t)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& - \frac{a_2 \mu_2(t)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& - \frac{b_2 \mu_2(t)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\end{aligned}$$

From Lemma 3.2, the fixed points of the operator \mathcal{N} are solution of the problem (1.1)–(1.2).

Let $R > 0$, such that

$$R \geq \mathcal{M}_f \psi(R),$$

and consider the ball

$$B_R = \{w \in C(J, E) : \|w\|_\infty \leq R\}.$$

We shall show that the operator \mathcal{N} satisfies all the assumptions of Theorem 2.2. Take $u \in B_R, t \in J$ we have

$$\begin{aligned}
& \|\mathcal{N}u(t)\| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \|f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& + \frac{|\mu_1(t)|}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \|f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& + \frac{|\lambda_2 \mu_2(t)|}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \|f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& + \frac{|a_2 \mu_2(t)|}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \|f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& + \frac{|b_2 \mu_2(t)|}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \|f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\end{aligned}$$

Using hypothese (H2) we get

$$\begin{aligned}
& \|\mathcal{N}u(t)\| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
 &+ \frac{|\mu_1(t)|}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\lambda_2\mu_2(t)|}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|a_2\mu_2(t)|}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|b_2\mu_2(t)|}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &\leq \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\mu_1(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\lambda_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|a_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|b_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &= \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \phi_q \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)} \right) \frac{ds}{s} \\
 &+ \frac{|\mu_1(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)} \right) \frac{ds}{s} \\
 &+ \frac{|\lambda_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)} \right) \frac{ds}{s} \\
 &+ \frac{|a_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} \phi_q \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)} \right) \frac{ds}{s} \\
 &+ \frac{|b_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)} \right) \frac{ds}{s} \\
 &= \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)} \right)^{q-1} \frac{ds}{s}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{|\mu_1(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)}\right)^{q-1} \frac{ds}{s} \\
& + \frac{|\lambda_2 \mu_2(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)}\right)^{q-1} \frac{ds}{s} \\
& + \frac{|a_2 \mu_2(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)}\right)^{q-1} \frac{ds}{s} \\
& + \frac{|b_2 \mu_2(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \left(\frac{(\log s)^\beta}{\Gamma(\beta + 1)}\right)^{q-1} \frac{ds}{s} \\
& = \frac{\|p_f\| \psi(\|u\|)}{\Gamma(\beta + 1)^{q-1} \Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} (\log s)^{\beta(q-1)} \frac{ds}{s} \\
& + \frac{|\mu_1(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\beta + 1)^{q-1} \Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} (\log s)^{\beta(q-1)} \frac{ds}{s} \\
& + \frac{|\lambda_2 \mu_2(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\beta + 1)^{q-1} \Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} (\log s)^{\beta(q-1)} \frac{ds}{s} \\
& + \frac{|a_2 \mu_2(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\beta + 1)^{q-1} \Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} (\log s)^{\beta(q-1)} \frac{ds}{s} \\
& + \frac{|b_2 \mu_2(t)| \|p_f\| \psi(\|u\|)}{\Gamma(\beta + 1)^{q-1} \Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} (\log s)^{\beta(q-1)} \frac{ds}{s} \\
& = \frac{\|p_f\| \psi(\|u\|) \Gamma(\beta(q-1) + 1)}{\Gamma(\beta + 1)^{q-1}} \left\{ \frac{(\log T)^{\alpha + \beta(q-1)} (1 + |a_2| \tilde{\mu}_2)}{\Gamma(\alpha + \beta(q-1) + 1)} \right. \\
& + \frac{\tilde{\mu}_1 (\log \eta_1)^{\sigma_1 + \alpha + \beta(q-1)}}{\Gamma(\sigma_1 + \alpha + \beta(q-1) + 1)} + \tilde{\mu}_2 \left[\frac{|\lambda_2| (\log \eta_2)^{\sigma_2 + \alpha + \beta(q-1)}}{\Gamma(\sigma_2 + \alpha + \beta(q-1) + 1)} \right. \\
& + \left. \left. \frac{|b_2| (\log T)^{\alpha - \gamma + \beta(q-1)}}{\Gamma(\alpha - \gamma + \beta(q-1) + 1)} \right] \right\} \\
& \leq \mathcal{M}_f \psi(R) \leq R.
\end{aligned}$$

Thus

$$\|\mathcal{N}u\| \leq R.$$

This proves that \mathcal{N} transforms the ball B_R into itself. Furthermore for any $u \in B_R$ and $t \in J$, we have

$$\begin{aligned}
& \|(\mathcal{N}u)'(t)\| \\
& \leq \frac{1}{\Gamma(\alpha - 1)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 2} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau) \psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
& + \frac{|\lambda_1 \Delta_2|}{T \Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta - 1}}{T \Gamma(\beta)} \phi_p(p_f(\tau) \psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
 &+ \frac{|\lambda_2 \Delta_4|}{T\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|a_2 \Delta_4|}{T\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|b_2 \Delta_4|}{T\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \phi_p(p_f(\tau)\psi(\|u(\tau)\|)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
 \end{aligned}$$

Some computations give

$$\begin{aligned}
 \|(\mathcal{N}u)'(t)\| &\leq \frac{\|p_f\|\psi(\|u\|)\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}} \left\{ \frac{(\log T)^{\alpha+\beta(q-1)-1}}{\Gamma(\alpha+\beta(q-1))} \right. \\
 &+ \frac{|\lambda_1 \Delta_2|(\log \eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{T\Gamma(\sigma_1+\alpha+\beta(q-1)+1)} + \frac{|\Delta_4|}{T} \left[\frac{|\lambda_2|(\log \eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)} \right. \\
 &+ \left. \left. \frac{|a_2|(\log T)^{\alpha+\beta(q-1)}}{\Gamma(\alpha+\beta(q-1)+1)} + \frac{|b_2|(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)} \right] \right\} \\
 &= \mathcal{L}_f \psi(\|u\|),
 \end{aligned}$$

where \mathcal{L}_f is given by (3.10). We shall show that the operator $\mathcal{N} : B_R \rightarrow B_R$ satisfies all the assumptions of Theorem 2.2.

For clarity, we will divide the remain of the proof into several steps.

Step 1: $\mathcal{N} : B_R \rightarrow B_R$ is continuous. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \rightarrow u$ in B_R . Then, for each $t \in J$, we have

$$\begin{aligned}
 &\|(\mathcal{N}u_n)(t) - (\mathcal{N}u)(t)\| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \|f(\tau, u_n(\tau)) - f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\mu_1(t)|}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \|f(\tau, u_n(\tau)) - f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\lambda_2 \mu_2(t)|}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \|f(\tau, u_n(\tau)) - f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|a_2 \mu_2(t)|}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \|f(\tau, u_n(\tau)) - f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|b_2 \mu_2(t)|}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta - 1}}{\Gamma(\beta)} \|f(\tau, u_n(\tau)) - f(\tau, u(\tau))\| \frac{d\tau}{\tau} \right) \frac{ds}{s}.
 \end{aligned} \tag{3.12}$$

Since $u_n \rightarrow u$ as $n \rightarrow \infty$ and $f, \phi_q(\cdot)$ are continuous, then by the Lebesgue dominated convergence theorem, equation (3.12) implies

$$\|\mathcal{N}u_n - \mathcal{N}u\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Step 2: $\mathcal{N}(B_R)$ is bounded and equicontinuous. Since $\mathcal{N}(B_R) \subset B_R$ and B_R is bounded, then $\mathcal{N}(B_R)$ is bounded. Next, let $t_1, t_2 \in J, t_1 < t_2, u \in B_R$. Thus, we have

$$\|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \leq \int_{t_1}^{t_2} \|(\mathcal{N}u)'(t)\| dt \leq \mathcal{L}_f \psi(R) |t_2 - t_1|,$$

where \mathcal{L}_f is given by (3.10). As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero.

Step 3: The implication (2.1) holds. Now let V be a subset of B_R such that $V \subset \mathcal{N}(V) \cup \{0\}$. V is bounded and equicontinuous and therefore the function $t \rightarrow v(t) = \kappa(V(t))$ is continuous on J . By assumption (H3), and the properties of the measure κ we have for each $t \in J$.

$$\begin{aligned} v(t) &\leq \kappa(\overline{\mathcal{N}(V)} \cup \{0\}) \leq \kappa(\mathcal{N}(V)(t)) \\ &\leq \kappa \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \right) \\ &+ \kappa \left(\frac{\mu_1(t)}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s} \right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \right) \\ &+ \kappa \left(\frac{\lambda_2 \mu_2(t)}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s} \right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \right) \\ &+ \kappa \left(\frac{a_2 \mu_2(t)}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \right) \\ &+ \kappa \left(\frac{b_2 \mu_2(t)}{\Gamma(\alpha - \gamma)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha - \gamma - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \right) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \kappa(f(\tau, V(\tau))) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &+ \frac{|\tilde{\mu}_1|}{\Gamma(\sigma_1 + \alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s} \right)^{\sigma_1 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \kappa(f(\tau, V(\tau))) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &+ \frac{|\lambda_2 \tilde{\mu}_2|}{\Gamma(\sigma_2 + \alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s} \right)^{\sigma_2 + \alpha - 1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \kappa(f(\tau, V(\tau))) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &+ \frac{|a_2 \tilde{\mu}_2|}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-1} \phi_q \left(\int_1^s \frac{(\log \frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)} \kappa(f(\tau, V(\tau))) \frac{d\tau}{\tau} \right) \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
 &+ \frac{|b_2\tilde{\mu}_2|}{\Gamma(\alpha-\gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \kappa(f(\tau, V(\tau))) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p(p_f(\tau)v(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\tilde{\mu}_1|}{\Gamma(\sigma_1+\alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p(p_f(\tau)v(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|\lambda_2\tilde{\mu}_2|}{\Gamma(\sigma_2+\alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p(p_f(\tau)v(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|a_2\tilde{\mu}_2|}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p(p_f(\tau)v(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &+ \frac{|b_2\tilde{\mu}_2|}{\Gamma(\alpha-\gamma)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p(p_f(\tau)v(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
 &\leq \mathcal{M}_f \|v\|_\infty,
 \end{aligned}$$

where \mathcal{M}_f is given by (3.9). Which gives

$$\|v\|_\infty \leq \mathcal{M}_f \|v\|_\infty.$$

This means that

$$\|v\|_\infty (1 - \mathcal{M}_f) \leq 0.$$

From (3.11), we get $\|v\|_\infty = 0$, that is, $v(t) = \kappa(V(t)) = 0$, for each $t \in J$, and then $V(t)$ is relatively compact in E . In view of the Ascoli-Arzela theorem, V is relatively compact in B_R . Applying Theorem 2.2 we conclude that \mathcal{N} has a fixed point which is a solution of the problem (1.1)–(1.2). \square

4. An Example

In this section, we give an example to illustrate the usefulness of our main result. Let

$$E = c_0 = \{u = (u_1, u_2, \dots, u_n, \dots) : u_n \rightarrow 0 (n \rightarrow \infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|u\|_\infty = \sup_{n \geq 1} |u_n|.$$

Example 4.1. Consider the following fractional problem posed in c_0 :

$$(4.1) \quad {}^C_H\mathcal{D}_1^{\frac{1}{2}} \left(\phi_p \left[{}^C_H\mathcal{D}_1^{\frac{7}{4}} u(t) \right] \right) = f(t, u(t)), \quad t \in J := [1, e],$$

supplemented with boundary conditions

$$\begin{aligned}
 u(1) + {}^C_H D_1^{\frac{1}{2}} u(1) &= {}^H \mathcal{I}_1^{\frac{5}{2}} u\left(\frac{5}{4}\right), \quad 0 < \gamma \leq 1 \\
 u(e) + {}^C_H D_1^{\frac{1}{2}} u(e) &= {}^H \mathcal{I}_1^{\frac{7}{2}} u\left(\frac{3}{2}\right), \quad 1 < \eta_1, \eta_2 < T \\
 (4.2) \quad {}^C_H D_1^{\frac{7}{4}} u(1) &= (0, 0, \dots, 0, \dots).
 \end{aligned}$$

Note that, this problem is a particular case of BVP (1.1)–(1.2), where

$$\begin{aligned}
 \alpha &= \frac{7}{4}, \beta = \gamma = \frac{1}{2}, T = e \\
 a_1 &= b_1 = a_2 = b_2 = \lambda_1 = \lambda_2 = 1, \\
 \sigma_1 &= \frac{5}{2}, \sigma_2 = \frac{7}{2}, \eta_1 = \frac{5}{4}, \eta_2 = \frac{3}{2}, \\
 p &= 2, q = 2,
 \end{aligned}$$

and $f : J \times c_0 \rightarrow c_0$ given by

$$f(t, u) = \left\{ \phi_p \left(\frac{1}{(t^2 + 2)^2} \left(\frac{1}{n^2} + \sin |u_n| \right) \right) \right\}_{n \geq 1}, \quad \text{for } t \in J, u = \{u_n\}_{n \geq 1} \in c_0.$$

It is clear that condition (H1) holds, and as

$$\begin{aligned}
 \|f(t, u)\| &= \left\| \phi_p \left(\frac{1}{(t^2 + 2)^2} \left(\frac{1}{n^2} + \sin |u_n| \right) \right) \right\| \\
 &\leq \phi_p \left(\frac{1}{(t^2 + 2)^2} (1 + \|u\|) \right) \\
 &= \phi_p (p_f(t) \psi(\|u\|)).
 \end{aligned}$$

Therefore, the assumption (H2) of the Theorem 3.1 is satisfied with $p_f(t) = \frac{1}{(t^2+2)^2}$, $t \in J$, and $\psi(u) = 1 + u$, $u \in [0, \infty)$. On the other hand, for any bounded set $D \subset c_0$, we have

$$\kappa(f(t, D)) \leq \frac{1}{(t^2 + 2)^2} \kappa(D), \quad \text{for each } t \in J.$$

Hence (H3) is satisfied. We shall check that condition (3.11) is satisfied. Using the Matlab program, we can find

$$\mathcal{M}_f = 0.4228 < 1.$$

and

$$(1 + R)\mathcal{M}_f < R,$$

thus

$$R > \frac{\mathcal{M}_f}{1 - \mathcal{M}_f} = 0.7326,$$

Then R can be chosen as $R = 1 \geq 0.7326$. Consequently, Theorem 3.1 implies that the problem (4.1)–(4.2) has at least one solution $u \in C(J, c_0)$.

5. Conclusions

We have proved the existence of solutions for Caputo–Hadamard fractional differential equations with p -Laplacian operator in a given Banach space. The problem is issued by applying Mönch's fixed point theorem combined with the technique of measures noncompactness. We also provide an example to make our results clear.

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