\(\eta\)-RICCI SOLITONS IN LORENTZIAN \(\alpha\)-SASAKIAN MANIFOLDS

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Abstract. In the present paper, we have studied \(\eta\)-Ricci solitons in Lorentzian \(\alpha\)-Sasakian manifolds satisfying certain curvature conditions. The existence of \(\eta\)-Ricci soliton in a Lorentzian \(\alpha\)-Sasakian manifold has been proved by a concrete example.

Keywords: \(\eta\)-Ricci solitons; Lorentzian \(\alpha\)-Sasakian manifolds; projective curvature tensor.

1. Introduction

In 1985, J. A. Oubina [14] defined and studied a new class of almost contact metric manifolds known as trans-Sasakian manifolds, which includes \(\alpha\)-Sasakian, \(\beta\)-Kenmotsu and cosymplectic structures. In 2005, A. Yildiz and C. Murathan [5] studied conformally flat and quasi-conformally flat Lorentzian \(\alpha\)-Sasakian manifolds. Lorentzian \(\alpha\)-Sasakian manifolds have been studied by many authors such as [1, 3, 6]. Recently, U. C. De and P. Majhi have studied \(\phi\)-Weyl semisymmetric and \(\phi\)-projectively semisymmetric generalized Sasakian space-forms and obtained some interesting results [21].

In 1982, R. S. Hamilton [20] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. G. Perelman [12, 13] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

\[
\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.
\]

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group
of diffeomorphism and scaling. A Ricci soliton \((g, V, \lambda)\) on a Riemannian manifold \((M, g)\) is a generalization of an Einstein metric such that \([17, 18]\)
\[
(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,
\]
where \(S\) is the Ricci tensor, \(\mathcal{L}_V\) is the Lie derivative operator along the vector field \(V\) on \(M\) and \(\lambda\) is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to \(\lambda\) being negative, zero or positive, respectively. Ricci solitons in the context of general relativity have been studied by M. Ali and Z. Ahsan \([16]\).

As a generalization of Ricci solitons, the notion of \(\eta\)-Ricci solitons was introduced by J. T. Cho and M. Kimura \([15]\). They have studied Ricci solitons of real hypersurfaces in a non-flat complex space form and they defined \(\eta\)-Ricci soliton, which satisfies the equation
\[
(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0,
\]
where \(\lambda\) and \(\mu\) are real number. In particular, if \(\mu = 0\), then the notion \(\eta\)-Ricci soliton \((g, V, \lambda, \mu)\) is reduced to the notion of Ricci soliton \((g, V, \lambda)\). Recently, \(\eta\)-Ricci solitons have been studied by various authors such as A. Singh and S. Kishor \([4]\), A. M. Blaga \([9]\), D. G. Prakasha and B. S. Hadimani \([11]\), S. Ghosh \([19]\) and many others.

The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian \(\alpha\)-Sasakian manifolds. In Section 3, we discuss \(\eta\)-Ricci solitons in Lorentzian \(\alpha\)-Sasakian manifolds. Section 4 is devoted to study \(\eta\)-Ricci solitons in \(\phi\)-projectively semisymmetric Lorentzian \(\alpha\)-Sasakian manifolds. In Section 5, we study \(\eta\)-parallel \(\phi\)-tensor Lorentzian \(\alpha\)-Sasakian manifolds admitting \(\eta\)-Ricci solitons. \(\eta\)-Ricci solitons in Lorentzian \(\alpha\)-Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor have been studied in Section 6. In Section 7, we study \(\eta\)-Ricci solitons in recurrent Lorentzian \(\alpha\)-Sasakian manifolds. Finally, we construct an example of 3-dimensional Lorentzian \(\alpha\)-Sasakian manifold which admits an \(\eta\)-Ricci soliton.

\section{Preliminaries}

A differentiable manifold of dimension \(n\) is called a Lorentzian \(\alpha\)-Sasakian manifold if it admits a \((1, 1)\)-tensor field \(\phi\), a contravariant vector field \(\xi\), a covariant vector field \(\eta\) and a Lorentzian metric \(g\) which satisfy \([5]\)
\[
\begin{align*}
\eta(\xi) &= -1, \\
\phi^2 X &= X + \eta(X)\xi, \\
\phi\xi &= 0, \quad \eta(\phi X) = 0, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
g(X, \xi) &= \eta(X)
\end{align*}
\]
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for all vector fields $X$, $Y$ on $M$.

Also Lorentzian α–Sasakian manifolds satisfy

\begin{eqnarray*}
\nabla_X \xi &=& -\alpha \phi X, \\
(\nabla_X \eta)Y &=& -\alpha g(\phi X, Y),
\end{eqnarray*}

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha \in \mathbb{R}$.

Furthermore, on a Lorentzian α–Sasakian manifold $M$, the following relations hold [5, 6]:

\begin{eqnarray*}
\text{(2.6)} & & g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = \alpha^2 [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \\
\text{(2.7)} & & R(\xi, X)Y = \alpha^2 [g(X, Y)\xi - \eta(Y)X], \\
\text{(2.8)} & & R(X, Y)\xi = \alpha^2 [\eta(Y)X - \eta(X)Y], \\
\text{(2.9)} & & R(\xi, X)\xi = \alpha^2 [X + \eta(X)\xi], \\
\text{(2.10)} & & S(X, \xi) = (n-1)\alpha^2 \eta(X), \\
\text{(2.11)} & & S(\xi, \xi) = -(n-1)\alpha^2, \\
\text{(2.12)} & & Q\xi = (n-1)\alpha^2 \xi, \\
\text{(2.13)} & & (\nabla_X \phi)Y = \alpha g(X, Y)\xi - \alpha \eta(Y)X
\end{eqnarray*}

for any vector fields $X$, $Y$ and $Z$ on $M$.

**Definition 2.1.** A Lorentzian α–Sasakian manifold $M$ is said to be a generalized η-Einstein manifold if its Ricci tensor $S$ is of the form [7]

\[ S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \]

where $a$, $b$ and $c$ are smooth functions on $M$. If $c = 0$, $b = c = 0$ and $b = 0$, then the manifold is said to be an η–Einstein, Einstein and a special type of generalized η-Einstein manifold, respectively.

**Definition 2.2.** The projective curvature tensor $C$ in an $n$–dimensional Lorentzian α–Sasakian manifold $M$ is defined by

\begin{equation}
\text{(2.15)} \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y],
\end{equation}

where $R$ is the Riemannian curvature tensor and $r$ is the scalar curvature of the manifold.
3. \( \eta \)-Ricci solitons in Lorentzian \( \alpha \)-Sasakian manifolds

Suppose that a Lorentzian \( \alpha \)-Sasakian manifold admits an \( \eta \)-Ricci soliton \((g, \xi, \lambda, \mu)\). Then (1.2) holds and thus we have

\[
(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu \eta(X)\eta(Y) = 0.
\]

In a Lorentzian \( \alpha \)-Sasakian manifold, we find

\[
(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = -2\alpha g(X, \phi Y).
\]

Combining (3.1) and (3.2), it follows that

\[
S(X,Y) = -\lambda g(X,Y) + \alpha g(\phi X, Y) - \mu \eta(X)\eta(Y).
\]

It yields

\[
QX = -\lambda X + \alpha \phi X - \mu \eta(X)\xi.
\]

By taking \( Y = \xi \) in (3.3) and using (2.1), (2.3) and (2.5), we get

\[
S(X,\xi) = (\mu - \lambda)\eta(X).
\]

Thus from (2.12) and (3.5), we obtain

\[
\mu - \lambda = (n - 1)\alpha^2.
\]

Hence in view of (3.3) and (3.6), we can state the following theorem:

\textbf{Theorem 3.1.} If \((g, \xi, \lambda, \mu)\) is an \( \eta \)-Ricci soliton in a Lorentzian \( \alpha \)-Sasakian manifold, then the manifold is a generalized \( \eta \)-Einstein manifold of the form (3.3) and \( \mu - \lambda = (n - 1)\alpha^2 \).

In particular, if we take \( \mu = 0 \) in (3.3) and (3.6), then we obtain

\[
S(X,Y) = -\lambda g(X,Y) + \alpha g(\phi X, Y),
\]

\[
\lambda = -(n - 1)\alpha^2,
\]

respectively. Thus we have

\textbf{Corollary 3.1.} If \((g, \xi, \lambda)\) is a Ricci soliton in a Lorentzian \( \alpha \)-Sasakian manifold, then the manifold is a special type of generalized \( \eta \)-Einstein manifold and its Ricci soliton is always shrinking.

Now, let \((g, V, \lambda, \mu)\) be a Ricci soliton in a Lorentzian \( \alpha \)-Sasakian manifold such that \( V \) is pointwise collinear with \( \xi \), i.e., \( V = b\xi \), where \( b \) is a function. Then (1.2) holds and we have

\[
bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(X, \nabla_Y \xi) + (Yb)\eta(X)
\]
\[+2S(X,Y) + 2\lambda g(X,Y) + 2\mu \eta(X)\eta(Y) = 0\]

which in view of (2.6) takes the form

\[\tag{3.9} -2b\alpha g(\phi X,Y) + (Xb)\eta(Y) + (Yb)\eta(X)\]

\[+2S(X,Y) + 2\lambda g(X,Y) + 2\mu \eta(X)\eta(Y) = 0.\]

Putting \(Y = \xi\) in (3.9) and using (2.1), (2.3), (2.5) and (2.12), we find

\[\tag{3.10} -(Xb) + [(\xi b) + 2(n-1)\alpha^2 + 2\lambda - 2\mu]\eta(X) = 0.\]

Again putting \(X = \xi\) in (3.10) and using (2.1), we get

\[\tag{3.11} (\xi b) + (n-1)\alpha^2 + \lambda - \mu = 0.\]

Combining the equations (3.10) and (3.11), it follows that

\[\tag{3.12} db = [(n-1)\alpha^2 + \lambda - \mu]\eta.\]

Now applying \(d\) on (3.12), we get

\[\tag{3.13} [(n-1)\alpha^2 + \lambda - \mu]\eta = 0 \Rightarrow \mu - \lambda = (n-1)\alpha^2, \quad d\eta \neq 0.\]

Thus from (3.12) and (3.13), we obtain \(db = 0\), i.e., \(b\) is a constant. Therefore, (3.9) takes form

\[\tag{3.14} S(X,Y) = -\lambda g(X,Y) + b\alpha g(\phi X,Y) - \mu \eta(X)\eta(Y).\]

Hence in view of (3.13) and (3.14), we can state the following theorem:

**Theorem 3.2.** If \((g, \xi, \lambda, \mu)\) is a \(\eta\)-Ricci soliton in an \(n\)-dimensional Lorentzian \(\alpha\)-Sasakian manifold, such that \(V\) is pointwise collinear with \(\xi\), then \(V\) is a constant multiple of \(\xi\) and the manifold is a generalized \(\eta\)-Einstein manifold of the form (3.14) and \(\mu - \lambda = (n-1)\alpha^2\).

4. \(\eta\)-Ricci solitons in \(\phi\)-projectively semisymmetric Lorentzian \(\alpha\)-Sasakian manifolds

**Definition 4.1.** A Lorentzian \(\alpha\)-Sasakian manifold is said to be \(\phi\)-projectively semisymmetric if [20]

\[P(X,Y) \cdot \phi = 0\]

for all \(X, Y\) on \(M\).
Let $M$ be an $n$-dimensional $\phi$–projectively semisymmetric Lorentzian $\alpha$–Sasakian manifold admits $\eta$-Ricci soliton. Therefore $P(X,Y) \cdot \phi = 0$ turns into

$$P(X,Y)\phi Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0$$

for any vector fields $X,Y,Z \in \chi(M)$. From (2.15), it follows that

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y],$$

$$\phi P(X,Y)Z = \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y].$$

Combining the equations (4.1), (4.2) and (4.3), we have

$$R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y]$$

$$+ \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y] = 0$$

which by taking $Y = \xi$ and using (2.3), (2.9) and (2.12) is reduced to

$$S(X,\phi Z) = (n - 1)\alpha^2 g(X,\phi Z).$$

In view of (3.3), (4.5) takes the form

$$[\lambda + (n - 1)\alpha^2]g(X,\phi Z) - \alpha g(X,\phi Z) = 0.$$}

By replacing $X$ by $\phi X$ in (4.6) and using (2.2), we get

$$[\lambda + (n - 1)\alpha^2]g(\phi X,\phi Z) - \alpha g(X,\phi Z) = 0.$$}

By adding (4.6) and (4.7), we obtain

$$[\lambda + (n - 1)\alpha^2 - \alpha](g(\phi X,\phi Z) + g(X,\phi Z)) = 0$$

from which it follows that $\lambda = -(n - 1)\alpha^2 + \alpha$ and hence from (3.6), we get $\mu = \alpha$. Thus we can state the following theorem:

**Theorem 4.1.** If $(g, \xi, \lambda, \mu)$ is an $\eta$-Ricci soliton in an $n$-dimensional $\phi$–projectively semisymmetric Lorentzian $\alpha$–Sasakian manifold, then $\lambda = -(n - 1)\alpha^2 + \alpha$ and $\mu = \alpha$.

Now from the relations (3.3), (3.6) and (4.7), we obtain

$$S(X,Y) = (n - 1)\alpha^2 g(X,Y).$$

Thus we have

**Corollary 4.1.** An $n$-dimensional $\phi$–projectively semisymmetric Lorentzian $\alpha$–Sasakian manifold admitting an $\eta$-Ricci soliton $(g, \xi, \lambda, \mu)$ is an Einstein manifold.
5. \( \eta \)-parallel \( \phi \)-tensor Lorentzian \( \alpha \)-Sasakian manifolds admitting \( \eta \)-Ricci solitons

In this section, we study the \( \eta \)-parallel \( \phi \)-tensor in Lorentzian \( \alpha \)-Sasakian manifolds. If the \( (1,1) \) tensor \( \phi \) is \( \eta \)-parallel, then we have [10]

\[
g((\nabla_X \phi) Y, Z) = 0 \tag{5.1}
\]

for all \( X, Y, Z \in \chi(M) \). From (2.14) and (5.1), we get

\[
g(X, Y) \eta(Z) - \eta(Y) g(X, Z) = 0, \quad \text{where} \quad \alpha \neq 0. \tag{5.2}
\]

Putting \( Z = \xi \) in (5.2), we find

\[
g(X, Y) = -\eta(X)\eta(Y)
\]

which by replacing \( Y \) by \( QY \) and using (2.12) yields

\[
S(X, Y) = -\alpha^2(n-1)\eta(X)\eta(Y). \tag{5.3}
\]

From (3.3) and (5.3), it follows that

\[
\lambda g(X, Y) - \alpha g(\phi X, Y) + (\mu - (n-1)\alpha^2)\eta(X)\eta(Y) = 0
\]

which by replacing \( Y \) by \( \phi Y \) becomes

\[
\lambda g(X, \phi Y) - \alpha g(\phi X, \phi Y) = 0. \tag{5.4}
\]

Now by replacing \( X \) by \( \phi X \) in (5.4) and using (2.2), we find

\[
\lambda g(\phi X, \phi Y) - \alpha g(X, \phi Y) = 0. \tag{5.5}
\]

By adding (5.4) and (5.5), we obtain \( \lambda = \alpha \) and hence from (3.6) we get \( \mu = \alpha + (n-1)\alpha^2 \). Thus we have the following theorem:

**Theorem 5.1.** If \( (g, \xi, \lambda, \mu) \) is an \( \eta \)-Ricci soliton in an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold and if the tensor \( \phi \) is \( \eta \)-parallel, then \( \lambda = \alpha \) and \( \mu = \alpha + (n-1)\alpha^2 \).

Now from the relations (3.3), (3.6) and (5.5), we obtain

\[
S(X, Y) = -(n-1)\alpha^2\eta(X)\eta(Y). \tag{5.6}
\]

Thus we have

**Corollary 5.1.** If \( (g, \xi, \lambda, \mu) \) is an \( \eta \)-Ricci soliton in an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold and if the tensor \( \phi \) is \( \eta \)-parallel, then the manifold is a special type of \( \eta \)-Einstein manifold.
6. η–Ricci solitons in Lorentzian α–Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor

In this section, we consider η–Ricci solitons in Lorentzian α–Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. A. Gray [2] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor.

**Definition 6.1.** A Lorentzian α–Sasakian manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor $S$ of type $(0, 2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$$

for all $X, Y, Z \in \chi(M)$.

Taking covariant derivative of (3.3) and making use of (2.7) and (2.14), we find

$$\alpha g(\phi Y, Z) = g(\phi X, Z)\eta(Y).$$

If the Ricci tensor $S$ is of Codazzi type, then

$$\alpha g(\phi Y, Z) = g(\phi X, Z)\eta(Y).$$

In view of (6.1), (6.2) takes the form

$$\alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \alpha \mu [g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)] = 0$$

which by putting $X = \xi$ and using (2.1), (2.3)-(2.5) gives

$$\alpha g(\phi Y, Z) - \mu g(\phi Y, Z) = 0, \quad \alpha \neq 0.$$

Now by replacing $Z$ by $\phi Z$ in (6.3) and using (2.2), we find

$$\alpha g(\phi Y, Z) - \mu g(\phi Y, \phi Z) = 0.$$

By adding (6.3) and (6.4), we obtain $\mu = \alpha$ and hence from (3.6) we get $\lambda = \alpha - (n-1)\alpha^2$ and $\mu = \alpha$.

**Theorem 6.1.** Let $(g, \xi, \lambda, \mu)$ be an η–Ricci soliton in an $n$-dimensional Lorentzian α–Sasakian manifold and if the manifold has Ricci tensor of Codazzi type, then $\lambda = \alpha - (n-1)\alpha^2$ and $\mu = \alpha$.

**Definition 6.2.** A Lorentzian α–Sasakian manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor $S$ of type $(0, 2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$$

for all $X, Y, Z \in \chi(M)$. 
Let \((g, \xi, \lambda, \mu)\) be an \(\eta\)--Ricci soliton in an \(n\)-dimensional Lorentzian \(\alpha\)--Sasakian manifold and the manifold has cyclic parallel Ricci tensor, then (6.5) holds. Taking covariant derivative of (3.3) and making use of (2.7) and (2.14), we find

\[
(\nabla_X S)(Y, Z) = \alpha^2 [g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] + \alpha \mu [g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)].
\]

Similarly, we have

\[
(\nabla_Y S)(Z, X) = \alpha^2 [g(Y, Z)\eta(X) - g(Y, X)\eta(Z)] + \alpha \mu [g(\phi Y, Z)\eta(X) + g(\phi Y, X)\eta(Z)],
\]

and

\[
(\nabla_Z S)(X, Y) = \alpha^2 [g(Z, X)\eta(Y) - g(Z, Y)\eta(X)] + \alpha \mu [g(\phi Z, X)\eta(Y) + g(\phi Z, Y)\eta(X)].
\]

By using (6.6)-(6.8) in (6.5), we obtain

\[
\alpha \mu [g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)] = 0
\]

which by taking \(Z = \xi\) reduces to

\[
(6.9) \quad \alpha \mu g(\phi X, Y) = 0.
\]

Since the manifold under consideration is non-cosymplectic and \(g(\phi X, Y) \neq 0\), in general, therefore (6.9) yields \(\mu = 0\). Therefore the \(\eta\)--Ricci soliton becomes Ricci soliton. Thus we have the following:

**Theorem 6.2.** An \(\eta\)--Ricci soliton in a non-cosymplectic Lorentzian \(\alpha\)--Sasakian manifold whose Ricci tensor is of Codazzi-type becomes a Ricci soliton.

### 7. \(\eta\)-Ricci solitons on recurrent Lorentzian \(\alpha\)-Sasakian manifolds

**Definition 7.1.** An \(n\)-dimensional Lorentzian \(\alpha\)-Sasakian manifold is said to be recurrent if there exists a non-zero 1-form \(A\) such that [8]

\[
(7.1) \quad (\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W
\]

for all vector fields \(X, Y, Z\) and \(W\) on \(M\). If the 1-form \(A\) vanishes, then the manifold reduces to a symmetric manifold.
Assume that $M$ is a recurrent Lorentzian $\alpha$-Sasakian manifold. Therefore the curvature tensor of the manifold satisfies (7.1). By a suitable contraction of (7.1), we get
\begin{equation}
\end{equation}
This implies that
\begin{equation}
\nabla_X S(Z, W) - S(\nabla_X Z, W) - S(Z, \nabla_X W) = A(X)S(Z, W)
\end{equation}
which by taking $W = \xi$ and using (2.6) and (2.12) yields
\begin{equation}
S(Z, \phi X) = (n - 1)\alpha^2g(\phi X, Z) + (n - 1)\alpha A(X)\eta(Z), \quad \alpha \neq 0.
\end{equation}
In view of (3.3), (7.4) takes the form
\begin{equation}
\alpha g(X, Z) + \alpha \eta(X)\eta(Z) = [\lambda + (n - 1)\alpha^2]g(\phi X, Z) + (n - 1)\alpha A(X)\eta(Z).
\end{equation}
Suppose the associated 1-form $A$ is equal to the associated 1-form $\eta$, then from (7.5), we have
\begin{equation}
\alpha g(X, Z) = [\lambda + (n - 1)\alpha^2]g(\phi X, Z) + (n - 2)\alpha \eta(X)\eta(Z).
\end{equation}
By replacing $Z$ by $\phi Z$ in (7.6), we get
\begin{equation}
\alpha g(X, \phi Z) = [\lambda + (n - 1)\alpha^2]g(\phi X, \phi Z)
\end{equation}
which by replacing $X$ by $\phi X$ and using (2.2), becomes
\begin{equation}
\alpha g(\phi X, \phi Z) = [\lambda + (n - 1)\alpha^2]g(X, \phi Z).
\end{equation}
By adding (7.7) and (7.8), we obtain $\lambda = -(n - 1)\alpha^2 - \alpha$ and hence from (3.6) we get $\mu = -\alpha$. Thus we can state the following:

**Theorem 7.1.** If $(g, \xi, \lambda, \mu)$ is an $\eta$–Ricci soliton in an $n$-dimensional recurrent Lorentzian $\alpha$-Sasakian manifold, then $\lambda = -(n - 1)\alpha^2 - \alpha$ and $\mu = -\alpha$.

Now from the relations (3.3), (3.6) and (7.7), we obtain
\begin{equation}
S(X, Y) = (n - 1)\alpha^2g(X, Y).
\end{equation}
Thus we have

**Corollary 7.1.** An $n$-dimensional recurrent Lorentzian $\alpha$-Sasakian manifold admitting an $\eta$–Ricci soliton $(g, \xi, \lambda, \mu)$ is an Einstein manifold.

**Example.** We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where $(x, y, z)$ are the standard coordinates of $\mathbb{R}^3$. Let $e_1, e_2$ and $e_3$ be the vector fields on $M$ given by
\begin{align*}
e_1 &= e^{-z}\frac{\partial}{\partial y}, \quad e_2 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \alpha \frac{\partial}{\partial z} = \xi,
\end{align*}
which are linearly independent at each point \( p \) of \( M \). Let \( g \) be the Lorentzian like (semi-Riemannian) metric defined by

\[
g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.
\]

Let \( \eta \) be the 1-form defined by \( \eta(X) = g(X, e_3) = g(X, \xi) \) for all \( X \in \chi(M) \), and let \( \phi \) be the \((1,1)\)-tensor field defined by

\[
\phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0.
\]

By applying linearity of \( \phi \) and \( g \), we have

\[
\eta(\xi) = g(\xi, \xi) = -1, \quad \phi^2 X = X + \eta(X)\xi \quad \text{and} \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)
\]

for all \( X, Y \in \chi(M) \). Thus for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian almost paracontact metric structure on \( M \).

Then we have

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1, \quad [e_2, e_3] = \alpha e_2.
\]

The Levi-Civita connection \( \nabla \) of the Lorentzian metric \( g \) is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X,Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),
\]

which is known as Koszul’s formula. Using Koszul’s formula, we can easily calculate

\[
(7.10) \quad \nabla_{e_1} e_1 = \alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \alpha e_1, \quad \nabla_{e_2} e_1 = 0,
\]

\[
\nabla_{e_2} e_2 = \alpha e_3, \quad \nabla_{e_2} e_3 = \alpha e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\]

Also, one can easily verify that

\[
\nabla_X \xi = -\alpha \phi X \quad \text{and} \quad (\nabla_X \phi)Y = \alpha g(X,Y)\xi - \alpha \eta(Y)X.
\]

Therefore, the manifold is a Lorentzian \( \alpha \)-Sasakian manifold. From the above results, we can easily obtain the components of the curvature tensor as follows:

\[
(7.11) \quad R(e_1, e_2)e_1 = -\alpha^2 e_2, \quad R(e_1, e_3)e_1 = -\alpha^2 e_3, \quad R(e_2, e_3)e_1 = 0,
\]

\[
R(e_1, e_2)e_2 = \alpha^2 e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = -\alpha^2 e_3,
\]

\[
R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = -\alpha^2 e_1, \quad R(e_2, e_3)e_3 = -\alpha^2 e_2
\]

from which it is clear that

\[
(7.12) \quad R(X, Y)Z = \alpha^2 [g(Y, Z)X - g(X, Z)Y].
\]
Hence the manifold $(M, \phi, \xi, g)$ is a Lorentzian $\alpha$-Sasakian manifold of constant curvature $\alpha^2$. With the help of the above results we get the components of Ricci tensor and scalar curvature as follows:

\begin{equation}
S(e_1, e_1) = S(e_2, e_2) = 2\alpha^2, \quad S(e_3, e_3) = -2\alpha^2,
\end{equation}

Therefore, $r = \sum_{i=1}^{3} \epsilon_i S(e_i, e_i) = 6\alpha^2$, where $\epsilon_i = g(e_i, e_i)$. From the equation (3.3) and (7.13), we obtain $\lambda = \alpha(1 - 2\alpha)$ and $\mu = \alpha$. Thus the data $(g, \xi, \lambda, \mu)$ for $\lambda = \alpha(1 - 2\alpha)$ and $\mu = \alpha$ defines an $\eta$–Ricci soliton on $(M, \phi, \xi, g)$.

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**References**

\( \eta \)-Ricci Solitons in Lorentzian \( \alpha \)-SaSakian Manifolds


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