TAUBERIAN THEOREMS FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY OF INTEGRALS

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Abstract. Let \( q \) be a positive weight function on \( \mathbb{R}^+ := [0, \infty) \) which is integrable in Lebesgue’s sense over every finite interval \((0, x)\) for \( 0 < x < \infty \), in symbol: \( q \in L^1_{\text{loc}}(\mathbb{R}^+) \) such that \( Q(x) := \int_0^x q(t)dt \neq 0 \) for each \( x > 0 \), \( Q(0) = 0 \) and \( Q(x) \to \infty \) as \( x \to \infty \).

Given a real or complex-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}^+) \), we define \( s(x) := \int_0^x f(t)dt \) and

\[
\tau_q^{(0)}(x) := s(x), \quad \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \quad (x > 0, m = 1, 2, \ldots),
\]

provided that \( Q(x) > 0 \).

We say that \( \int_0^\infty f(x)dx \) is summable to \( L \) by the \( m \)-th iteration of weighted mean method determined by the function \( q(x) \), or for short, \((\mathcal{N}, q, m)\) integrable to a finite number \( L \) if

\[
\lim_{x \to \infty} \tau_q^{(m)}(x) = L.
\]

In this case, we write \( s(x) \to L(\mathcal{N}, q, m) \).

It is known that if the limit \( \lim_{x \to \infty} s(x) = L \) exists, then \( \lim_{x \to \infty} \tau_q^{(m)}(x) = L \) also exists. However, the converse of this implication is not always true. Some suitable conditions together with the existence of the limit \( \lim_{x \to \infty} \tau_q^{(m)}(x) \), which is so called Tauberian conditions, may imply convergence of \( \lim_{x \to \infty} s(x) \).

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for \((\mathcal{N}, q, m)\) summable integrals of real- or complex-valued functions have been obtained. Some classical type Tauberian theorems given for Cesàro summability \((C, 1)\) and weighted mean method of summability \((\mathcal{N}, q)\) have been extended and generalized.

Keywords: Tauberian conditions; weight function; summable integrals; finite interval.

1. Introduction

Let \( q \) be a positive weight function on \( \mathbb{R}^+ := [0, \infty) \) which is integrable in Lebesgue’s sense over every finite interval \((0, x)\) for \( 0 < x < \infty \), in symbol: \( q \in \)
\[ L_{1,\infty}(\mathbb{R}_+) \] such that \( Q(x) := \int_0^x q(t)dt \neq 0 \) for each \( x > 0, \) \( Q(0) = 0 \) and \( Q(x) \to \infty \) as \( x \to \infty. \) Given a real or complex-valued function \( f \in L_{1,\infty}(\mathbb{R}_+), \) we define \( s(x) := \int_0^x f(t)dt \) and

\[
\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \quad (x > 0, m = 1, 2, ...),
\]

provided that \( Q(x) > 0. \)

For each integer \( m \geq 0, \) we define \( v_q^{(m)}(x) \) by

\[
v_q^{(m)}(x) = \begin{cases} 
\frac{Q(x)}{Q(x')} f(x), & m = 0 \\
\frac{1}{Q(x')} \int_0^x f(t)Q(t)dt, & m = 1 \\
\frac{1}{Q(x')} \int_0^x v_q^{(m-1)}(t)q(t)dt, & m \geq 2.
\end{cases}
\]

The identity

\[
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) = v_q^{(m)}(x)
\]

is known as the weighted Kronecker identity for the weighted mean method of summability.

It is clear from (1.1) that

\[
\frac{Q(x)}{q(x)} \frac{d}{dx} \tau_q^{(m)}(x) = v_q^{(m)}(x)
\]

for each integer \( m \geq 0 \) (see [14]). Here, we call \( v_q^{(m)}(x) \) the generator of \( \tau_q^{(m-1)}(x) \) for each integer \( m \geq 1. \)

We say that \( \int_0^\infty f(x)dx \) is summable to \( L \) by the \( m \)-th iteration of weighted mean method determined by the function \( q(x), \) or for short, \((\mathcal{N}, q, m)\) summable to a finite number \( L \) if

\[
\lim_{x \to \infty} \tau_q^{(m)}(x) = L.
\]

It is obvious that \((\mathcal{N}, q, m)\) summability reduces to the ordinary convergence for \( m = 0 \) and \((\mathcal{N}, q, 1)\) is the \((\mathcal{N}, q)\) method. If \( q(x) = 1 \) on \( \mathbb{R}_+, \) then \((\mathcal{N}, q, m)\) method is the Hölder method of order \( m \) and \((\mathcal{N}, q, 1)\) method is the Cesàro summability method \((C, 1)\).

It is well known that condition \( Q(x) \to \infty \) as \( x \to \infty \) is a necessary and sufficient condition that the existence of the integral

\[
\int_0^\infty s(x)dx = L
\]

implies (1.2). That is, the \((\mathcal{N}, q, m)\) summability method is regular, where \( m \) is a nonnegative integer. However, the converse of this implication is not always true. Notice that some suitable condition on \( s(x) \) together with (1.2) may imply (1.3).
Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

Móricz [8] and Fekete and Móricz [6] obtained one-sided and two-sided Tauberian conditions for the weighted mean method \((N, q)\) of integrals. Following these works, Totur and Okur [13] proved one-sided boundedness of \(v_q^{(0)}(x)\) is a Tauberian condition for the weighted mean method of summability \((N, q)\) of integrals. From the fact that condition \(v_q^{(0)}(x) \geq -C\) implies slow decreasing of \(s(x)\), Totur and Okur [13] generalized their first result and proved that slow decrease of \(s(x)\) is also a Tauberian condition for \((N, q)\) method. For a detailed study and some interesting results related to Tauberian theorems for the weighted mean method of summability, we refer the reader to Borwein and Kratz [1], Çanak and Totur [2], Çanak and Totur [3], Çanak and Totur [4], Özsaraç and Çanak [9], Sezer and Çanak [10], Tietz and Zeller [11] and Totur and Çanak [12], etc.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for summable integrals by \(m\)-th iteration of weighted means of real- or complex-valued functions have been obtained, respectively. Some classical type Tauberian theorems given for Cesàro summability \((C, 1)\) and weighted mean method of summability \((N, q)\) have been extended and generalized.

2. Main results

For the main results of the paper, we need the following definitions and notations.

**Definition 2.1.** ([7]) A positive function \(Q\) is called regularly varying of index \(\alpha > 0\) if

\[
\lim_{x \to \infty} \frac{Q(\rho x)}{Q(x)} = \rho^\alpha, \quad \rho > 0. \tag{2.1}
\]

It easily follows from Definition 2.1 that for all \(\rho > 1\) and sufficiently large \(x,\)

\[
\frac{\rho^\alpha}{2 (\rho^\alpha - 1)} \leq \frac{Q(\rho x)}{Q(x)} - \frac{Q(\rho x)}{Q(\rho x)} \leq \frac{3\rho^\alpha}{2 (\rho^\alpha - 1)} \tag{2.2}
\]

and for all \(0 < \rho < 1\) and sufficiently large \(x,\)

\[
\frac{\rho^\alpha}{2 (1 - \rho^\alpha)} \leq \frac{Q(\rho x)}{Q(x)} - \frac{Q(\rho x)}{Q(\rho x)} \leq \frac{3\rho^\alpha}{2 (1 - \rho^\alpha)} \tag{2.3}
\]

We note that if (2.1) holds, then the following equivalent conditions are clearly satisfied (see [5]):

\[
\liminf_{x \to \infty} \frac{Q(x)}{Q(\rho x)} < 1, \quad \text{for every } \rho > 1 \tag{2.4}
\]

and

\[
\liminf_{x \to \infty} \frac{Q(\rho x)}{Q(x)} < 1, \quad \text{for every } 0 < \rho < 1. \tag{2.5}
\]
First, we consider real-valued functions and prove the following Tauberian theorems.

**Theorem 2.1.** Let (2.1) be satisfied. If a real-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, m)\) summable to \( L \) and \( v^{(m-1)}_q(x) \) is one-sided bounded, then \( s(x) \) is \((N, q, m - 1)\) summable to \( L \).

**Corollary 2.1.** ([13]) Let (2.1) be satisfied. If a real-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, 1)\) summable to \( L \) and \( v^{(0)}_q(x) \) is one-sided bounded, then \( s(x) \) converges to \( L \).

**Theorem 2.2.** Let (2.4) be satisfied. If a real-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, m)\) summable to \( L \) and \( v^{(m-1)}_q(x) \) is slowly decreasing, then \( s(x) \) is \((N, q, m - 1)\) summable to \( L \).

**Corollary 2.2.** ([13]) Let (2.4) be satisfied. If a real-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, 1)\) summable to \( L \) and slowly decreasing, then \( s(x) \) converges to \( L \).

A real-valued function \( s(x) \) defined on \( \mathbb{R}_+ \) is said to be slowly decreasing if

\[
(2.6) \quad \lim_{\rho \to 1^+} \lim_{x \to \infty} \min_{x \leq t \leq \rho x} (s(t) - s(x)) \geq 0.
\]

Note that condition (2.6) can be equivalently reformulated as follows:

\[
(2.7) \quad \lim_{\rho \to 1^-} \lim_{x \to \infty} \min_{\rho x \leq t \leq x} (s(x) - s(t)) \geq 0.
\]

Second, we consider complex-valued functions and prove the following Tauberian theorems.

**Theorem 2.3.** Let (2.1) be satisfied. If a complex-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, m)\) summable to \( L \) and \( v^{(m-1)}_q(x) \) is bounded, then \( s(x) \) is \((N, q, m - 1)\) summable to \( L \).

**Corollary 2.3.** Let (2.1) be satisfied. If a complex-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, 1)\) summable to \( L \) and \( v^{(0)}_q(x) \) is bounded, then \( s(x) \) converges to \( L \).

**Theorem 2.4.** Let (2.4) be satisfied. If a complex-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, m)\) summable to \( L \) and \( v^{(m-1)}_q(x) \) is slowly oscillating, then \( s(x) \) is \((N, q, m - 1)\) summable to \( L \).

**Corollary 2.4.** Let (2.4) be satisfied. If a complex-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}_+) \) is such that its integral function \( s(x) \) is \((N, q, 1)\) summable to \( L \) and slowly oscillating, then \( s(x) \) converges to \( L \).
A complex-valued function $s(x)$ defined on $\mathbb{R}_+$ is said to be slowly oscillating if
\begin{equation}
\lim_{\rho \to 1^+} \limsup_{x \to \infty} \max_{x \leq t \leq \rho x} |s(t) - s(x)| = 0.
\end{equation}

Note that condition (2.8) can be equivalently reformulated as follows:
\begin{equation}
\lim_{\rho \to 1^+} \limsup_{x \to \infty} \max_{\rho x \leq t \leq x} |s(x) - s(t)| = 0.
\end{equation}

3. An auxiliary result

The following two representations of $s(x) - \tau_q^{(1)}(x)$ will be needed in the proofs of our main results.

Lemma 3.1. ([13])

(i) For $\rho > 1$ and sufficiently large $x$,
\begin{align*}
s(x) - \tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left( \tau_q^{(1)}(\rho x) - \tau_q^{(1)}(x) \right) \\
&\quad - \frac{1}{Q(\rho x) - Q(x)} \int_{x}^{\rho x} (s(t) - s(x)) q(t) \, dt.
\end{align*}

(ii) For $0 < \rho < 1$ and sufficiently large $x$,
\begin{align*}
s(x) - \tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left( \tau_q^{(1)}(x) - \tau_q^{(1)}(\rho x) \right) \\
&\quad + \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} (s(x) - s(t)) q(t) \, dt.
\end{align*}

4. Proofs of main results

Proof of Theorem 2.1 Suppose that $s(x)$ is $(\omega, q, m)$ summable to $L$ and $v_q^{(m-1)}(x)$ is one-sided bounded. By Lemma 3.1 (i), we have
\begin{align*}
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left( \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
&\quad - \frac{1}{Q(\rho x) - Q(x)} \int_{x}^{\rho x} \left( \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) q(t) \, dt \\
&\quad - \frac{1}{Q(\rho x) - Q(x)} \int_{x}^{\rho x} \left( \int_{x}^{t} \frac{d}{dz} \tau_q^{(m-1)}(z) \, dz \right) q(t) \, dt.
\end{align*}
Since \( v_q^{(m-1)}(x) \) is one-sided bounded, we get
\[
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left( \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left( \int_z^t \frac{q(z)}{Q(z)} dz \right) q(t) dt \\
= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left( \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \log \frac{Q(t)}{Q(x)} dt
\]
(4.1)

By (2.2) and \((\mathcal{N}, q, m)\) summability of \( s(x) \), we have
\[
\lim_{x \to \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left( \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) = 0.
\]
(4.2)

Taking (4.2) into account in (4.1), we obtain
\[
\limsup_{x \to \infty} \left( \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \leq \limsup_{x \to \infty} \left( C \log \frac{Q(\rho x)}{Q(x)} \right) = C \log \rho^\alpha.
\]

Letting \( \rho \to 1^+ \) in the last inequality, we have
\[
\limsup_{x \to \infty} \left( \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \leq 0.
\]
(4.3)

Similarly, from Lemma 3.1 (ii), we have
\[
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) = \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left( \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} \left( \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right) q(t) dt \\
= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left( \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} \left( \int_t^x \frac{d}{dz} \tau_q^{(m-1)}(z) dz \right) q(t) dt.
\]

Since \( v_q^{(m-1)}(x) \) is one-sided bounded, we get
\[
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \geq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left( \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right)
\]
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\[
\begin{align*}
&= \frac{C}{Q(x) - Q(px)} \int_{\rho x}^{x} \left( \frac{\int_{z}^{x} q(z) \, dz}{Q(z)} \right) q(t) \, dt \\
&= \frac{Q(px)}{Q(x) - Q(px)} \left( \tau^{(m)}_{q}(x) - \tau^{(m)}_{q}(px) \right) \\
&= \frac{C}{Q(x) - Q(px)} \int_{\rho x}^{x} q(t) \log \frac{Q(x)}{Q(t)} \, dt \\
\end{align*}
\]

(4.4)

By (2.3) and \((N,q,m)\) summability of \(s(x)\), we obtain

\[
\lim_{x \to \infty} \frac{Q(px)}{Q(x) - Q(px)} \left( \tau^{(m)}_{q}(x) - \tau^{(m)}_{q}(px) \right) = 0.
\]

(4.5)

Taking (4.5) into account in (4.4), we obtain

\[
\lim_{x \to \infty} \sup_{x \to \infty} \left( \tau^{(m-1)}_{q}(x) - \tau^{(m)}_{q}(x) \right) \geq -\lim_{x \to \infty} \inf \left( C \log \frac{Q(x)}{Q(px)} \right) = -C \log \rho^{\alpha}.
\]

Letting \(\rho \to 1^{-}\) in the last inequality, we have

\[
\lim_{x \to \infty} \sup_{x \to \infty} \left( \tau^{(m-1)}_{q}(x) - \tau^{(m)}_{q}(x) \right) \geq 0.
\]

(4.6)

Combining (4.3) and (4.6), we obtain \(s(x)\) is \((N,q,m-1)\) summable to \(L\). \(\square\)

**Proof of Theorem 2.2** Suppose that \(s(x)\) is \((N,q,m)\) summable to \(L\) and \(\tau^{(m-1)}_{q}(x)\) is slowly decreasing. By Lemma 3.1 (i), we have

\[
\begin{align*}
\tau^{(m-1)}_{q}(x) - \tau^{(m)}_{q}(x) &= \frac{Q(px)}{Q(x) - Q(px)} \left( \tau^{(m)}_{q}(px) - \tau^{(m)}_{q}(x) \right) \\
&- \frac{1}{Q(px) - Q(x)} \int_{x}^{\rho x} \left( \tau^{(m-1)}_{q}(t) - \tau^{(m-1)}_{q}(x) \right) q(t) \, dt \\
&\leq \frac{Q(px)}{Q(x) - Q(px)} \left( \tau^{(m)}_{q}(px) - \tau^{(m)}_{q}(x) \right) \\
&- \frac{1}{Q(px) - Q(x)} \int_{x}^{\rho x} q(t) \min_{x \leq t \leq \rho x} \left( \tau^{(m-1)}_{q}(t) - \tau^{(m-1)}_{q}(x) \right) \, dt \\
&= \frac{Q(px)}{Q(px) - Q(x)} \left( \tau^{(m)}_{q}(px) - \tau^{(m)}_{q}(x) \right) \\
&- \min_{x \leq t \leq \rho x} \left( \tau^{(m-1)}_{q}(t) - \tau^{(m-1)}_{q}(x) \right).
\end{align*}
\]

(4.7)
Taking the limit of both sides of (4.7), we get

\[
\limsup_{x \to \infty} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m)} (x) \right) \leq \limsup_{x \to \infty} \frac{Q(px)}{Q(px) - Q(x)} \left( \tau_q^{(m)} (px) - \tau_q^{(m)} (x) \right) \\
- \liminf_{x \to \infty} \min_{z \leq t \leq px} \left( \tau_q^{(m-1)} (t) - \tau_q^{(m-1)} (x) \right).
\]

(4.8)

By (2.4), we have

\[0 < \limsup_{x \to \infty} \frac{Q(px)}{Q(px) - Q(x)} = 1 + \left( \liminf_{x \to \infty} \frac{Q(px)}{Q(x)} - 1 \right)^{-1} < \infty.\]

Since \(s(x)\) is \((N, q, m)\) summable to \(L\), the first term on the right-hand side vanishes in (4.8). From this, we obtain

\[
\limsup_{x \to \infty} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m)} (x) \right) \leq - \liminf_{x \to \infty} \min_{z \leq t \leq px} \left( \tau_q^{(m-1)} (t) - \tau_q^{(m-1)} (x) \right).
\]

Taking the limit of (4.8) as \(p \to 1^+\), we have

\[
\limsup_{x \to \infty} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m)} (x) \right) \leq 0.
\]

(4.9)

Similarly, by Lemma 3.1 (ii), we have

\[
\tau_q^{(m-1)} (x) - \tau_q^{(m)} (x) = \frac{Q(px)}{Q(x) - Q(px)} \left( \tau_q^{(m)} (x) - \tau_q^{(m)} (px) \right) \\
+ \frac{1}{Q(x) - Q(px)} \int_{px}^{x} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m-1)} (t) \right) q(t) \, dt \\
\geq \frac{Q(px)}{Q(x) - Q(px)} \left( \tau_q^{(m)} (x) - \tau_q^{(m)} (px) \right) \\
+ \frac{1}{Q(x) - Q(px)} \int_{px}^{x} q(t) \min_{px \leq t \leq x} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m-1)} (t) \right) \, dt \\
= \frac{Q(px)}{Q(x) - Q(px)} \left( \tau_q^{(m)} (x) - \tau_q^{(m)} (px) \right) \\
+ \min_{px \leq t \leq x} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m-1)} (t) \right).
\]

(4.10)

From (4.10), we get

\[
\liminf_{x \to \infty} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m)} (x) \right) \geq \liminf_{x \to \infty} \frac{Q(px)}{Q(x) - Q(px)} \left( \tau_q^{(m)} (x) - \tau_q^{(m)} (px) \right) \\
+ \liminf_{x \to \infty} \min_{px \leq t \leq x} \left( \tau_q^{(m-1)} (x) - \tau_q^{(m-1)} (t) \right).
\]

(4.11)
By (2.4), we have

$$0 < \liminf_{x \to \infty} \frac{Q(px)}{Q(x)} = \left( \limsup_{x \to \infty} \frac{Q(x)}{Q(px)} - 1 \right)^{-1} < \infty.$$  

Since \(s(x)\) is \((N, q, m)\) summable to \(L\), the first term on the right-hand side vanishes in (4.11). From this, we obtain

$$\liminf_{x \to \infty} \left( \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq \liminf_{x \to \infty} \min_{\rho x \leq t \leq x} \left( \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right).$$

Taking the limit of (4.11) as \(\rho \to 1^-\), we have

$$\liminf_{x \to \infty} \left( \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq 0.$$  

Combining (4.9) and (4.12), we obtain \(s(x)\) is \((N, q, m - 1)\) summable to \(L\).

**Proof of Theorem 2.3** Suppose that \(s(x)\) is \((N, q, m)\) summable to \(L\) and \(v_q^{(m-1)}(x)\) is bounded. By Lemma 3.1 (i), we have

$$\left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \frac{Q(px)}{Q(x)} \left| \tau_q^{(m)}(px) - \tau_q^{(m)}(x) \right|$$

$$+ \frac{1}{Q(px) - Q(x)} \int_x^{px} \left\{ \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right\} q(t) \, dt$$

$$= \frac{Q(px)}{Q(x)} \left| \tau_q^{(m)}(px) - \tau_q^{(m)}(x) \right|$$

$$+ \frac{1}{Q(px) - Q(x)} \int_x^{px} \int_{x}^{t} \frac{d}{dz} \tau_q^{(m-1)}(z) \, dz \, q(t) \, dt.$$  

Since \(v_q^{(m-1)}(x)\) is bounded, we get

$$\left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \frac{Q(px)}{Q(x)} \left| \tau_q^{(m)}(px) - \tau_q^{(m)}(x) \right|$$

$$+ \frac{C}{Q(px) - Q(x)} \int_x^{px} \int_{x}^{t} \frac{q(z)}{Q(z)} \, dz \, q(t) \, dt$$

$$= \frac{Q(px)}{Q(x)} \left| \tau_q^{(m)}(px) - \tau_q^{(m)}(x) \right|$$

$$+ \frac{C}{Q(px) - Q(x)} \int_x^{px} q(t) \log \frac{Q(t)}{Q(x)} \, dt$$

$$\leq \frac{Q(px)}{Q(x)} \left| \tau_q^{(m)}(px) - \tau_q^{(m)}(x) \right| + C \log \frac{Q(px)}{Q(x)}.$$  

(4.13)
By (2.2) and \((\mathcal{N}, q, m)\) summability of \(s(x)\), we have

\[
\lim_{x \to \infty} \frac{Q(px)}{Q(px) - Q(x)} \left| \tau_{q}^{(m)}(px) - \tau_{q}^{(m)}(x) \right| = 0.
\]

Taking the lim sup of both sides of (4.13) gives

\[
\limsup_{x \to \infty} \left| \tau_{q}^{(m-1)}(x) - \tau_{q}^{(m)}(x) \right| \leq \limsup_{x \to \infty} \left( C \log \frac{P(px)}{Q(x)} \right) = C \log \rho^\alpha.
\]

Letting \(\rho \to 1^+\) in last inequality, we have

\[
\text{(4.14)} \quad \limsup_{x \to \infty} \left| \tau_{q}^{(m-1)}(x) - \tau_{q}^{(m)}(x) \right| \leq 0.
\]

Similarly, from Lemma 3.1 (ii), we have

\[
\left| \tau_{q}^{(m-1)}(x) - \tau_{q}^{(m)}(x) \right| \leq \frac{P(px)}{Q(x) - Q(px)} \left| \tau_{q}^{(m)}(x) - \tau_{q}^{(m)}(px) \right| + \frac{1}{Q(x) - Q(px)} \int_{px}^{x} \left| \tau_{q}^{(m-1)}(x) - \tau_{q}^{(m-1)}(t) \right| q(t) \, dt
\]

\[
= \frac{Q(px)}{Q(x) - Q(px)} \left| \tau_{q}^{(m)}(x) - \tau_{q}^{(m)}(px) \right| + \frac{1}{Q(x) - Q(px)} \int_{1}^{x} \frac{d}{dz} \tau_{q}^{(m-1)}(z) \, dz \, q(t) \, dt.
\]

Since \(v_{q}^{(m-1)}(x)\) is bounded, we get

\[
\left| \tau_{q}^{(m-1)}(x) - \tau_{q}^{(m)}(x) \right| \leq \frac{P(px)}{Q(x) - Q(px)} \left| \tau_{q}^{(m)}(x) - \tau_{q}^{(m)}(px) \right| + \frac{C}{Q(x) - Q(px)} \int_{1}^{x} \frac{p(z)}{P(z)} \, dz \, q(t) \, dt
\]

\[
= \frac{Q(px)}{Q(x) - Q(px)} \left| \tau_{q}^{(m)}(x) - \tau_{q}^{(m)}(px) \right| + \frac{C}{Q(x) - Q(px)} \int_{1}^{x} q(t) \log \frac{Q(x)}{Q(t)} \, dt
\]

\[
\text{(4.15)} \quad \leq \frac{Q(px)}{Q(x) - Q(px)} \left| \tau_{q}^{(m)}(x) - \tau_{q}^{(m)}(px) \right| + C \log \frac{Q(x)}{Q(px)}.
\]

By (2.3) and \((\mathcal{N}, p, m)\) summability of \(s(x)\), we have

\[
\lim_{x \to \infty} \frac{Q(px)}{P(x) - Q(px)} \left| \tau_{q}^{(m)}(x) - \tau_{q}^{(m)}(px) \right| = 0.
\]
From (4.15), we get
\[
\limsup_{x \to \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \to \infty} \left( C \log \frac{Q(x)}{Q(\rho x)} \right) = C \log \rho^\alpha.
\]

Letting \( \rho \to 1^- \) in last inequality, we have
\[
(4.16) \quad \limsup_{x \to \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq 0.
\]

From either (4.14) or (4.16), we conclude \( s(x) \) is \((N, q, m - 1)\) summable to \( L \).

**Proof of Theorem 2.4** Suppose that \( s(x) \) is \((N, q, m)\) summable to \( L \) and \( \tau_q^{(m-1)}(x) \) is slowly oscillating. By Lemma 3.1 (i), we have
\[
\left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| = \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right|
\]
\[
+ \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| q(t) \, dt
\]
\[
\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right|
\]
\[
+ \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \max_{x \leq t \leq \rho x} \left( \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| \right) dt
\]
\[
\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right|
\]
\[
+ \max_{x \leq t \leq \rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| .
\]

From (4.17), we get
\[
\limsup_{x \to \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \to \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right|
\]
\[
+ \limsup_{x \to \infty} \max_{x \leq t \leq \rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| .
\]

(4.18)\]

By (2.4), we have
\[
0 < \limsup_{x \to \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} = 1 + \left( \liminf_{x \to \infty} \frac{Q(\rho x)}{Q(x)} - 1 \right)^{-1} < \infty.
\]

Since \( s(x) \) is \((N, q, m)\) summable to \( L \), the first term on the right side vanishes in (4.18). From this, we obtain
\[
\limsup_{x \to \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \to \infty} \max_{x \leq t \leq \rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| .
\]
Taking the limit of (4.18) as \( \rho \to 1^+ \), we have

\[
\limsup_{x \to \infty} \left| r_q^{(m-1)}(x) - r_q^{(m)}(x) \right| \leq 0. 
\]

Similarly, by Lemma 3.1 (ii), we have

\[
\left| r_q^{(m-1)}(x) - r_q^{(m)}(x) \right| = \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| r_q^{(m)}(x) - r_q^{(m)}(\rho x) \right| \\
+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} \left| r_q^{(m-1)}(x) - r_q^{(m-1)}(t) \right| q(t) \, dt \\
\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| r_q^{(m)}(x) - r_q^{(m)}(\rho x) \right| \\
+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} q(t) \max_{\rho x \leq t \leq x} \left( \left| r_q^{(m-1)}(x) - r_q^{(m-1)}(t) \right| \right) \, dt \\
\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| r_q^{(m)}(x) - r_q^{(m)}(\rho x) \right| \\
+ \max_{\rho x \leq t \leq x} \left| r_q^{(m-1)}(x) - r_q^{(m-1)}(t) \right| . 
\]

From (4.20), we get

\[
\limsup_{x \to \infty} \left| r_q^{(m-1)}(x) - r_q^{(m)}(x) \right| \leq \limsup_{x \to \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| r_q^{(m)}(x) - r_q^{(m)}(\rho x) \right| \\
+ \limsup_{x \to \infty} \max_{\rho x \leq t \leq x} \left| r_q^{(m-1)}(x) - r_q^{(m-1)}(t) \right|. 
\]

By (2.4), we have

\[ 0 < \liminf_{x \to \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} = \left( \limsup_{x \to \infty} \frac{Q(x)}{Q(\rho x)} - 1 \right)^{-1} < \infty. \]

Since \( s(x) \) is \((N, q, m)\) summable to \( L \), the first term on the right-hand side vanishes in (4.21). From this, we obtain

\[
\limsup_{x \to \infty} \left| r_q^{(m-1)}(x) - r_q^{(m)}(x) \right| \leq \limsup_{x \to \infty} \max_{\rho x \leq t \leq x} \left| r_q^{(m-1)}(x) - r_q^{(m-1)}(t) \right|. 
\]

Taking the limit of (4.21) as \( \rho \to 1^- \), we have

\[
\limsup_{x \to \infty} \left| r_q^{(m-1)}(x) - r_q^{(m)}(x) \right| \leq 0. 
\]

From either (4.19) or (4.22), we conclude \( s(x) \) is \((N, q, m - 1)\) summable to \( L \). \( \Box \)
5. Conclusion

In this paper, we introduce Tauberian conditions in terms of the generator and its generalizations for summable integrals by \( m \)-th iteration of weighted means of real- or complex-valued functions, respectively. Tauberian conditions for summable double integrals by \( m \)-th iteration of weighted means of real- or complex-valued functions will be illustrated in a forthcoming work.

REFERENCES

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