FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 1 (2020), 273-282
https://doi.org/10.22190/FUMI2001273S

SOME RESULTS ON (ϵ)- KENMOTSU MANIFOLDS

Arpan Sardar

Abstract. We have studied curvature symmetries in (ϵ)-Kenmotsu manifolds. Next, we have proved the non-existence of a non-zero parallel 2-form in an (ϵ)-Kenmotsu manifold. Moreover, we have characterised ϕ-Ricci symmetric (ϵ)-Kenmotsu manifolds and finally, we have proved that under certain restriction on the scalar curvature $divR$=0 and $divC$=0 are equivalent, where 'div' denotes divergence.
Keywords: (ϵ)-Kenmotsu manifold, curvature symmetries, ϕ-Ricci symmetric manifold, Weyl curvature tensor.

1. Introduction

The basic difference between Riemannian and semi-Riemannian geometry is the existence of a null vector. In a Riemannian manifold (M, g), the signature of the metric tensor is positive definite, whereas the signature of a semi-Riemannian manifold is indefinite. With the help of indefinite metric Bejancu and Duggal [1] introduced (ϵ)-Sasakian manifolds. Then Xufeng and Xiaoli [16] proved that every (ϵ)-Sasakian manifold must be a real hyperface of some indefinite Kähler manifolds. Since Sasakian manifolds with indefinite metric have applications in Physics [4], we are interested to study various contact manifolds with indefinite metric. Geometry of Kenmotsu manifolds originated from Kenmotsu [10]. In [3] De and Sarkar introduced the notion of (ϵ)-Kenmotsu manifolds with indefinite metric. On the other hand, in [6] Eisenhart proved that if a Riemannian manifold admits a second order parallel syemmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. Later on, several authors investigated the Eisenhart problem on various spaces and obtained some fruitful results. Recently, Haseeb and De [7] have studied η-Ricci solitons in (ϵ)-Kenmotsu manifolds. (ϵ)-Kenmotsu manifolds have also been studied by several authors such as ([2],[8],[9],[13],[15]) and many others. So far, our knowledge about curvature symmetries have not been studied in semi-Riemannian manifolds. In this paper, we are going tol study curvature symmetries in (ϵ)-Kenmotsu manifolds. For curvature symmetries we refer

Received October 24, 2019; accepted January 5, 2020
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

the book of Duggal and Sharma [5].

In [7] Haseeb and De proved the following:

Theorem 1. Let M be an n-dimensional (ϵ)-Kenmotsu manifold. If the manifold has a symmetric parallel second order covariant tensor α, then α is a constant multiple of the metric tensor g.

Using the above theorem, we obtained the following statements.
Proposition 1.1. If a vector field X is an affine Killing in an (ϵ)-Kenmotsu manifold, then the vector field X is homothetic.
Proposition 1.2. An affine conformal vector field in an (ϵ)-Kenmotsu manifold is reduced to a conformal vector field.

Sharma[12] characterised a class of contact manifold admitting a vector field keeping the curvature tensor invariant.
In this paper, wel have considered the same problem in (ϵ)-Kenmotsu manifolds and proved the following:

Theorem 2. In an (ϵ)-Kenmotsu manifold a curvature collineation is Killing.

The nature of a parallel 2-form has been considered by several authors in contact manifolds. In the present paper we consider a parallel 2-form in the context of (ϵ)-Kenmotsu manifolds and prove the following:

Theorem 3. There is no non-zero parallel 2-form in an (ϵ)-Kenmotsu manifold.
As for example $d\eta$ is a 2-form in an (ϵ)-Kenmotsu manifold which is zero.
Next we prove:

Theorem 4. An (ϵ)-Kenmotsu manifold is ϕ-Ricci symmetric if and only if it is an Einstein manifold.

In a Riemannian or semi-Riemannian manifold of dimension n, $divR$ is obtained from the Bianchi identity and given by

$$(divR)(U,V)W = (\nabla_U S)(V,W) - (\nabla_V S)(U,W),$$

where R denotes the curvature tensor, S is the Ricci tensor, ∇ is the Riemannian connection and 'div' denotes the divergence.
Also it is known that

$$(divC)(U,V)W = \frac{n-2}{n-3}[\{(\nabla_U S)(V,W) - (\nabla_V S)(U,W)\} + \frac{1}{2(n-1)}\{dr(U)g(V,W)- dr(V)g(U,W)\}],$$

where C is the Weyl curvature tensor of type (1,3), r is the scalar curvature.

From the above definitions, it follows that $divR = 0$ implies $divC = 0$. However the converse, is not necessarily true. We address

Theorem 5. In an (ϵ)-Kenmotsu manifold $divR = 0$ and $divC = 0$ are equivalent provided the scalar curvature r is invariant under the characteristic vector field ξ.

2. (ϵ)-KENMOTSU MANIFOLDS

Duggal [4] introduced a larger class of contact metric manifolds.

Let M^{2n+1} be a (2n+1)-dimensional differentiable manifold of class C^{∞}. Then a quadruple (ϕ, ξ, η, g) defined on M^{2n+1} satisfying

$$\phi^2(U) = -U + \eta(U)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi, \xi) = \epsilon, \qquad \eta(U) = \epsilon\, g(U, \xi), \tag{2.2}$$

$$g(\phi U, \phi V) = g(U, V) - \epsilon\eta(U)\eta(V), \tag{2.3}$$

where ϕ is a tensor field of type (1,1) , η a tensor field of type (0,1), the Reeb vector field ξ and ϵ is 1 or -1 according as ξ is space like or time like vector field, is called an (ϵ)-almost contact metric manifold. If $d\eta(U, V) = g(U, \phi V)$, for every $U, V \in \chi(M)$, then we say that M is an (ϵ)-contact metric manifold. It can be easily seen that $\phi\xi = 0$, $\eta\phi = 0$.

Moreover, if the manifold satisfies

$$(\nabla_U \phi)V = -g(U, \phi V) - \epsilon\eta(V)\phi U, \tag{2.4}$$

where ∇ denotes the Riemannian connection of g , then we shall call the manifold an (ϵ)-Kenmotsu manifold.

In an (ϵ)-Kenmotsu manifold the following relations hold([3],[7]) :

$$\nabla_U \xi = \epsilon(U - \eta(U)\xi), \tag{2.5}$$

$$(\nabla_U \eta)V = g(U, V) - \epsilon\eta(U)\eta(V), \tag{2.6}$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.7}$$

$$(U, \xi) = -2n\eta(U). \tag{2.8}$$

Example. Let us consider $M^5 = \{(u_1, u_2, u_3, u_4, \mathrm{w}) : u_1, u_2, u_3, u_4, \mathrm{w}$ belongs to $\mathbb{R}$ and $\mathrm{w} \neq 0$ $\}$ and take the basis vector field $\{e_1, e_2, e_3, e_4, e_5\}$, where

$$e_1 = \mathrm{w}\frac{\partial}{\partial u_1},\ e_2 = \mathrm{w}\frac{\partial}{\partial u_2},\ e_3 = \mathrm{w}\frac{\partial}{\partial u_3},\ e_4 = \mathrm{w}\frac{\partial}{\partial u_4},\ e_5 = -\epsilon\ \mathrm{w}\frac{\partial}{\partial w} = \xi.$$

Let us define g as follows :

$$g(e_i, e_j) = 0, i \neq j, i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = 1, g(e_5, e_5) = \epsilon.$$

Then we obtain

$$[e_1,e_2] = [e_1,e_3] = [e_1,e_4] = [e_2,e_3] = [e_2,e_4] = [e_3,e_4] = 0,$$

$$[e_1,e_5] = \epsilon e_1,\ [e_2,e_5] = \epsilon e_2,\ [e_3,e_5] = \epsilon e_3, [e_4,e_5] = \epsilon e_4.$$

By Koszul's formula we have

$$\nabla_{e_1} e_1 = -e_5,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = \epsilon e_1,$$

$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_5,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = \epsilon e_2,$$

$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_5,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = \epsilon e_3,$$

$$\nabla_{e_4} e_1 = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_5,\ \nabla_{e_4} e_5 = \epsilon e_4,$$

$$\nabla_{e_5} e_1 = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

We can easily verify that $(M^5,\ \phi, \xi, \eta, g)$ satisfies all the properties of (ϵ)-Kenmotsu manifolds.

Definition 2.1. A vector field X is said to be an affine Killing vector field if it satisfies

$$\mathcal{L}_X \nabla = 0,$$

where $\mathcal{L}_X$ denotes the Lie differentiation along the vector field X.

Definition 2.2. A vector field X that leaves the Riemann curvature tensor invariant, that is,

$$(\mathcal{L}_X R)(U, V)W = 0$$

is called curvature collineation.

Definition 2.3. A conformal vector field X in a Riemannian or semi-Riemannian manifold (M, g) is defined by

$$\mathcal{L}_X g = 2\rho g, \tag{2.9}$$

for a smooth function ρ on M. If ρ = constant, then the vector field X is called homothetic. If ρ vanishes identically, then X is Killing vector field.

Equation (2.9) yields

$$(\mathcal{L}_X\nabla)(U, V) = (U\rho)V + (V\rho)U - g(U, V)D\rho, \tag{2.10}$$

where $\nabla(U, V) = \nabla_U V$ for any vector field U, V on M and $D\rho$ is the gradient vector field of ρ.

Thus (2.9) implies (2.10), but not conversly.

The vector field X satisfying (2.10) is called conformal collineation and X is then called an affine conformal vector field.

Definition 2.4 An (ϵ)-Kenmotsu manifold is said to be ϕ-Ricci symmetric if

$$\phi^2((\nabla_U Q)W) = 0,$$

where Q is the Ricci operator defined by $g(QU, V) = S(U, V)$.

ϕ-Ricci symmetric manifold is weaker than Ricci symmetric ($\nabla S = 0$) manifold.

If U, W are orthogonal to the characteristic vector field ξ, then ϕ-Ricci symmetric manifold is called locally ϕ-Ricci symmetric. The notion of locally ϕ-symmetric for Sasakian manifolds was introduced by Takahashi[14].

3. PROOFS OF THE RESULTS

Proof of Proposition 1.1. If X is a affine Killing vector field, then

$$\mathcal{L}_X\nabla = 0,$$

which implies that

$$\mathcal{L}_X(\nabla g) = 0.$$

That is,

$$\nabla\mathcal{L}_X g = 0.$$

Thus $\mathcal{L}_X g$ is symmetric second order parallel tensor. Thus, from Theorem 1 we infer that

$$\mathcal{L}_X g = \lambda g,$$

where λ is constant. This implies X is homothetic.

Proof of Proposition 1.2. In [11] Sharma and Duggal prove that a vector field X on a manifold (M, g) is an affine conformal vector field if and only if

$$\mathcal{L}_X g = 2\rho g + K,$$

where K is a second order covariant constant ($\nabla K = 0$) symmetric tensor field. Hence from Theorem 1, we obtain $K = \lambda g$, λ is constant.
Therefore,

$$\mathcal{L}_X g = 2\rho g + \lambda g.$$

This implies

$$\mathcal{L}_X g = 2\sigma g,$$

where $2\sigma = 2\rho + \lambda$, a smooth function. This completes the proof.

Proof of Theorem 2. By definition of curvature collineation, we get

$$\mathcal{L}_X R)(U, V)W = 0, \tag{3.1}$$

which implies

$$(\mathcal{L}_X g)(R(Z, U)V, W) + (\mathcal{L}_X g)(R(Z, U)W, V) = 0. \tag{3.2}$$

Putting $Z = V = W = \xi$ in (3.2), we get

$$(\mathcal{L}_X g)(R(\xi, U)\xi, \xi) + (\mathcal{L}_X g)(R(\xi, U)\xi, \xi) = 0,$$

which implies

$$(\mathcal{L}_X g)(R(\xi, U)\xi, \xi) = 0.$$

Now, using (2.7) in the foregoing equation, we get

$$(\mathcal{L}_X g)(U, \xi) = \eta(U)(\mathcal{L}_X g)(\xi, \xi). \tag{3.3}$$

Again putting $Z = V = \xi$ in (3.2) it follows

$$(\mathcal{L}_X g)(R(\xi, U)\xi, W) + (\mathcal{L}_X g)(R(\xi, U)W, \xi) = 0.$$

Using (2.7) in the above equation we infer that

$$(\mathcal{L}_X g)(U,W) - \eta(U)(\mathcal{L}_X g)(\xi,W) + \eta(W)(\mathcal{L}_X g)(U,\xi) - \epsilon(\mathcal{L}_X g)(\xi,\xi)g(U,W) = 0. \tag{3.4}$$

From (3.3) and (3.4) we get

$$(\mathcal{L}_X g)(U,W) = \epsilon(\mathcal{L}_X g)(\xi,\xi)g(U,W).$$

This implies

$$(\mathcal{L}_X g)(U,W) = \epsilon[\mathcal{L}_X g(\xi,\xi) - 2g(\xi,\mathcal{L}_X\xi)]g(U,W). \tag{3.5}$$

Since $(\mathcal{L}_X R)(U,V)W = 0$ implies $(\mathcal{L}_X S)(V,W) = 0$. Therefore,

$$(\mathcal{L}_X S)(\xi,\xi) = 0,$$

which implies

$$S(\xi,\mathcal{L}_X\xi) = 0.$$

That is,

$$g(Q\xi,\mathcal{L}_X\xi) = 0.$$

Now using (2.8) in the above equation, we obtain

$$g(\xi,\mathcal{L}_X\xi) = 0. \tag{3.6}$$

Using (3.6) in (3.5) we conclude that

$$(\mathcal{L}_X g)(U,W) = 0,$$

that is, X is Killing vector field. Therefore, the Theorem is proved.

Proof of Theorem 3. Let α be a parallel 2-form in an (ϵ)-Kenmotsu manifold. This means α is skew-symmetric and $\nabla\alpha = 0$.
Therefore

$$\alpha(U,V) = -\alpha(V,U). \tag{3.7}$$

Putting $U = V = \xi$ in (3.7) we get

$$\alpha(\xi,\xi) = 0. \tag{3.8}$$

Differentiating (3.8) along U, we get

$$\alpha(\nabla_U \xi, \xi) = 0.$$

Using (2.5) in the above gives

$$\epsilon\alpha(U, \xi) - \epsilon\eta(U)\alpha(\xi, \xi) = 0.$$

Finally, using (3.8), we obtain

$$\alpha(U, \xi) = 0. \tag{3.9}$$

Again, differentiating along V in the foregoing equation we get

$$\alpha(\nabla_V U, \xi) + \alpha(U, \nabla_V \xi) = 0. \tag{3.10}$$

Replacing U by $\nabla_V U$ in (3.9) we get

$$\alpha(\nabla_V U, \xi) = 0. \tag{3.11}$$

Using (3.11), (2.5) in (3.10) and after some calculation we obtain

$$\alpha(U, V) = 0,$$

that is, $\alpha = 0$. This completes the proof.

Proof of Theorem 4. Let M be an (2n+1)-dimensional ϕ-Ricci symmetric (ϵ)-Kenmotsu manifold. Then

$$\phi^2((\nabla_U Q)V) = 0,$$

for arbitary vector fields U, V, which implies

$$-(\nabla_U Q)V + \eta((\nabla_U Q)V)\xi = 0. \tag{3.12}$$

Putting $V = \xi$ in (3.12) and using (2.8), we get

$$2n\nabla_U \xi + Q(\nabla_U \xi) + \eta(-2n\nabla_U \xi - Q(\nabla_U \xi))\xi = 0. \tag{3.13}$$

Now using (2.5) in (3.13) and after some calculations, we obtain

$$S(U, V) = -2ng(U, V),$$

which implies that the manifold is an Einstein manifold.
Conversely, if the manifold is an Einstein manifold, then obviously it becomes ϕ-Ricci symmetric manifold. This completes the proof.

Proof of Theorem 5. Let us assume that $divC = 0$. Hence

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = \frac{1}{2(n-1)}[dr(U)g(V,W) - dr(V)g(U,W)]. \tag{3.14}$$

We know

$$S(U,\xi) = -2n\eta(U).$$

Then

$$(\nabla_U S)(V,\xi) = \nabla_U S(V,\xi) - S(\nabla_U V,\xi) - S(V,\nabla_U \xi).$$

Using (2.5) and (2.8) in the above equation, we get

$$(\nabla_U S)(V,\xi) - (\nabla_V S)(U,\xi) = -4nd\eta(U,V).$$

But in an (ϵ)-Kenmotsu manifold $d\eta = 0$,therefore, the above equation implies that

$$(\nabla_U S)(V,\xi) - (\nabla_V S)(U,\xi) = 0. \tag{3.15}$$

Substituting $W = \xi$ in (3.14) and using (3.15), we have

$$dr(U)\eta(V) - dr(V)\eta(U) = 0.$$

Replacing V by ξ in the above equation, it follows

$$dr(U) = dr(\xi)\eta(U). \tag{3.16}$$

Suppose the scalar curvature is invariant under the characteristic vector field ξ , that is,

$$\mathcal{L}_\xi r = 0,$$

which implies

$$\mathrm{dr}(\xi) = 0.$$

Hence (3.16) gives r = constant.
Therefore from (3.14) we get

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = 0,$$

which implies

$$(divR)(U,V)W = 0.$$

This completes the proof.

REFERENCES

1. A. Bejancu and K. L. Duggal: *Real hypersurfaces of indefinite Kähler manifolds*, Pacific J. Math. **16**(1993), 545–556.
2. A. M. Blaga: *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Applicat. **20**(2015), 1-13.
3. U. C. De and A. Sarkar : *On (ϵ)-Kenmotsu manifold*, Hardonic J. **32**(2009), 231-242.
4. K. L. Duggal: *Space time manifold and contact structures*, Internat. J. Math and Math Sci., **16**(1990), 545-553.
5. K. L. Duggal and R. Sharma: *Symmetries of space time and Riemannian manifolds*, Kluwer Acad. Publishers, 1999.
6. L. P. Eisenhart: *Symmetric tensor of the second order whose first covariant derivatives are zero*, Tran. Amer. Math. Soc. **25**(1923), 297-306.
7. A. Haseeb and U. C. De: *η-Ricci solitions in (ϵ)-Kenmotsu manifolds*, J. Geom. **110**(2019), 1-12.
8. A. Haseeb, M. K. Khan and M. D. Siddiqi: *Some more results on an (ϵ)-Kenmotsu manifold with a semi-symmetric metric connenction*, Acta. Math. Univ. Comenianae, **LXXXV**(2016), 9-20.
9. A. Haseeb: *Some results on projective curvature tensor in an (ϵ)-Kenmotsu manifold*, Palestine J. Math. **6**(2017), 196-203.
10. K. Kenmotsu: *A class of almost contact Riemannian manifolds*, Tohoku Math J. **24**(1972), 93-103.
11. R. Sharma and K. L. Duggal: *A characterization of an affine conformal vector field*, C.R. Math. Rep. Acad. Sci. Canada **7**(1985), 201-205.
12. R. Sharma: *On the curvature of contact metric manifold*, J. Geom. **53**(1995), 179-190.
13. R. N. Sing, S. K. Pandey, G. Pandey and K. Tiwari: *On a semi-symmetric metric connention in an (ϵ)-Kenmotsu manifold*, Commun. Korean Math. Soc. **29**(2014), 331-343.
14. T. Takahasi: *Sasakian ϕ-symmetric spaces*, Tohoku Math. J. **29**(1977), 91-113.
15. V. Venkatesha and S. V. Vishnuvardhana: *(ϵ)-Kenmotsu manifolds admiting a semi-symmetric connection*, Italian J. Pure Appl. Math. **38**(2017), 615-623.
16. X. Xufeng and C. Xiaoli: *Two therems on (ϵ)-Sasakian manifolds*, Internat. J. Math. Math Sci., **21**(1998), 249-254.

Arpan Sardar
Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road, Kol- 700019
arpansardar51@gmail.com