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COMMON FIXED POINT THEOREMS INVOLVING C-CLASS FUNCTIONS IN G-METRIC SPACES

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Abstract. The purpose of this paper is to prove some common fixed point theorems using the concept of C class function in G-metric spaces. Moreover, some examples are presented to illustrate the validity of our results.
Key words: fixed-point theorems, C-class functions, G-metric space.

1. Introduction and Preliminaries

In [13], Mustafa and Sims introduced a new class of generalized metric space, called G-metric, as generalization of a metric space (X, d). In fact, various researchers studied several and many fixed point theorems for self mappings in this structure (G-metric), for example we refer readers to References ([2, 3, 4, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22]).

In this paper, we will obtain common fixed point results for three mappings satisfying certain contractive conditions on G-metric space. The obtained results extend many recent results in the literature.

The following definitions and results will be needed:

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Definition 1.1. [13] Let X be a nonempty set. Suppose that the mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies:

(a) $G(x, y, z) = 0$ if $x = y = z$;

(b) $0 < G(x, x, y)$ for all $x, y \in X$. with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables); and

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X, G) is called a G-metric space.

Note that if $G(x, y, z) = 0$ then $x = y = z$.

Definition 1.2. [13] A sequence $\{x_n\}$ in a G-metric space X is:

(i) a G-Cauchy sequence if, for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$

(ii) a G-Convergent sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$ such that for all $n, m \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that $\{x_n\}$ G-converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to +\infty$.

Proposition 1.1. [13] *Let X be a G-metric space. Then the following are equivalent:*

(1) The sequence $\{x_n\}$ is G-convergent to x.

(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$.

(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$.

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Proposition 1.2. [13] *Let X be a G-metric space. Then the following are equivalent:*

(1) The sequence $\{x_n\}$ is G-Cauchy.

(2) For every for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $G(x_n, x_m, x_m) < \varepsilon$; that is, if $G(x_n, x_m, x_m) \to 0$ as $n, m \to +\infty$.

Definition 1.3. [13] A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$, and called nonsymmetric if it is not symmetric.

Definition 1.4. [13] A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Proposition 1.3. [13] *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three variables.*

Recently, Arslan Hojat Ansari in [5] introduced the concept of a C-class functions which covers a large class of contractive conditions.

Definition 1.5. [5] A continuous function $F : [0, +\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s, t \in [0, +\infty)$; the following conditions hold

$c1$ $F(s, t) \leq s$;

$c2$ $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0, 0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Example 1.1. The following examples shows that the class C is nonempty:

1. $F(s, t) = s - t$:
2. $F(s, t) = ms$; for some $m \in (0, 1)$.
3. $F(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, 1)$.
4. $F(s, t) = \frac{\log(t+a^s)}{(1+t)}$, for some $a > 1$.

Let Φ_u denote the class of the functions $\varphi : [0, +\infty) \to [0, +\infty), \varphi(0) \geq 0$ Therefore, the condition $\varphi(0) \geq 0$ is meaningless. It may be $\varphi(0) = 0$.

In 1984, Khan et al. [11] introduced altering distance function as follows:

Definition 1.6. [11] A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

i) ψ is non-decreasing and continuous,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 1.7. A tripled (ψ, φ, F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x, y \in [0, 1)$,

$$x \leq y \text{ implies } F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

Example 1.2. Let $F(s, t) = s - t$, $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

2. Main results

Now, we are ready to state our main theorem

Theorem 2.1. *Let (X, G) be a complete G-metric space and suppose mappings f, g and $h : X \to X$ satisfy*

$$\psi\left(G(fx, gy, hz)\right) \leq F\left(\psi\left(M(x, y, z)\right), \varphi\left(M(x, y, z)\right)\right), \tag{2.1}$$

for all $x, y, z \in X$, where $F : [0, +\infty)^2 \to \mathbb{R}$ is C-class function, $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function, $\varphi : [0, +\infty) \to [0, +\infty)$ is an ultra altering distance function and

$$\begin{aligned} M(x, y, z) \quad = \quad & \max\{G(x, y, z), G(x, x, fx), G(y, y, gy), G(z, z, hz), \\ & G(x, fx, gy), G(y, gy, hz), G(z, hz, fx)\}. \end{aligned}$$

Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. Suppose that x_0 is an arbitrary point in X. Define a sequence $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$.

Firstly, taking $G(x_{3n}, x_{3n+1}, x_{3n+2}) = 0$, for some n. Using (2.1), we obtain

$$\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \leq F\left(\psi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right), \varphi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right)\right),$$

where

$$\begin{aligned} M(x_{3n}, x_{3n+1}, x_{3n+2}) \quad = \quad & \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, fx_{3n}), \\ & G(x_{3n+1}, x_{3n+1}, gx_{3n+1}), G(x_{3n+2}, x_{3n+2}, hx_{3n+2}), \\ & G(x_{3n}, fx_{3n}, gx_{3n+1}), G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), \\ & G(x_{3n+2}, hx_{3n+2}, fx_{3n})\} \\ = \quad & \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\ & G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\ & G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ & G(x_{3n+2}, x_{3n+3}, x_{3n+1})\}. \end{aligned}$$

So

$$\begin{aligned} \psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \quad \leq \quad & F\left(\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3}\right), \varphi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right)\right) \\ \leq \quad & \psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3}\right) \end{aligned}$$

implies that $\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3}\right) = 0$ and

$$x_{3n+1} = x_{3n+2} = x_{3n+3}. \tag{2.2}$$

The same arguments, we obtain $x_{3n+2} = x_{3n+3} = x_{3n+4}$ and hence x_{3n} becomes a common fixed point of f, g and h.

Now, by taking $G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for every n and using (2.1), we obtain

$$\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \leq F\left(\psi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right), \varphi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right)\right),$$

where

$$\begin{aligned} M(x_{3n}, x_{3n+1}, x_{3n+2}) &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, fx_{3n}), \\ &\quad G(x_{3n+1}, x_{3n+1}, gx_{3n+1}), G(x_{3n+2}, x_{3n+2}, hx_{3n+2}), \\ &\quad G(x_{3n}, fx_{3n}, gx_{3n+1}), G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), \\ &\quad G(x_{3n+2}, hx_{3n+2}, fx_{3n})\} \\ &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\ &\quad G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\ &\quad G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ &\quad G(x_{3n+2}, x_{3n+3}, x_{3n+1})\}. \end{aligned}$$

Hence

$$\begin{aligned} &\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \\ &\leq F\begin{pmatrix} \psi\left(\max\left\{G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n}, x_{3n+1}, x_{3n+2})\right\}\right), \\ \varphi(\max\left\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right\}). \end{pmatrix} \end{aligned}$$

Suppose $\max\left\{G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n}, x_{3n+1}, x_{3n+2})\right\} = G(x_{3n+1}, x_{3n+2}, x_{3n+3})$, so, we find the same result of (2.2) , we obtain $G(x_{3n}, x_{3n+1}, x_{3n+2}) = 0$, This contradicts the assumption. Thus,

$$\begin{aligned} &\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \\ \leq\ & F\left(\psi\left(G(x_{3n}, x_{3n+1}, x_{3n+2})\right), \varphi(G(x_{3n}, x_{3n+1}, x_{3n+2}))\right) \\ \leq\ & \psi\left(G(x_{3n}, x_{3n+1}, x_{3n+2})\right). \end{aligned}$$

Then

$$\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \leq \psi\left(G(x_{3n}, x_{3n+1}, x_{3n+2})\right).$$

By the nondecreasing of ψ , it follows that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Similarly, we find

$$\begin{aligned} G(x_{3n+3}, x_{3n+4}, x_{3n+5}) &\leq G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \\ &\leq G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq G(x_{3n}, x_{3n+1}, x_{3n+2}), \end{aligned}$$

Consequently, it can be shown that for all n,

$$G(x_{n+1}, x_{n+2}, x_{n+3}) \leq G(x_n, x_{n+1}, x_{n+2}).$$

Therefore, $\{G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}$ is a non increasing sequence, then there exists $L \geq 0$, such that

$$\psi\left(\lim_{n\to+\infty} G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \leq F\left(\begin{array}{c} \psi\left(\lim_{n\to+\infty} G(x_{3n}, x_{3n+1}, x_{3n+2})\right), \\ \varphi(\lim_{n\to+\infty}\inf G(x_{3n}, x_{3n+1}, x_{3n+2})). \end{array}\right)$$

Then, we have

$$\psi(L) \leq F(\psi(L), \varphi(L)) \leq \psi(L)$$

Thus $\psi(L) = 0$ and we conclude that

$$\lim_{n\to+\infty} G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = 0. \tag{2.3}$$

Now, we shall show that $\{x_n\}$ is a G-Cauchy sequence. It is sufficient to show that $\{x_{3n}\}$ is G-Cauchy in X. If it is not, there is $\varepsilon > 0$ and integers $3n_k$, $3m_k$ with $3m_k > 3n_k > k$ such that

$$G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \geq \varepsilon \quad \textit{and} \quad G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) < \varepsilon \tag{2.4}$$

Now, (2.3) and (2.4) give

$$\begin{aligned} \varepsilon &\leq G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k}, x_{3m_k}) \\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k-1}, x_{3m_k-1}) \\ &\quad + G(x_{3m_k-1}, x_{3m_k}, x_{3m_k}) \\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-1}, x_{3m_k-2}, x_{3m_k-3}) \\ &\quad + G(x_{3m_k-1}, x_{3m_k}, x_{3m_k+1}), \end{aligned}$$

which implies that

$$\lim_{k\to+\infty} G(x_{3n_k}, x_{3m_k}, x_{3m_k}) = \varepsilon. \tag{2.5}$$

Also, in the same manner, we obtain

$$\lim_{k\to+\infty} G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}) = \varepsilon. \tag{2.6}$$

However, by using (2.3) and (2.6), we obtain

$$\lim_{k\to+\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) = \varepsilon. \tag{2.7}$$

Also, using (2.3) and (2.7) we have

$$\lim_{k\to+\infty} G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}) = \varepsilon. \tag{2.8}$$

Now, from the definition of $M(x,y,z)$ and from (2.3), (2.6), (2.7), (2.8) we get

$$\begin{aligned} & M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) \\ = \ & \max\{G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}), \\ & G(x_{3m_k+1}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3m_k+2}, x_{3m_k+2}, x_{3m_k+3}), \\ & G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}), G(x_{3m_k+1}, x_{3m_k+2}, x_{3m_k+3}), \\ & G(x_{3m_k+2}, x_{3m_k+3}, x_{3n_k+1})\} \end{aligned}$$

Hence

$$\lim_{k\to+\infty} M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) = \max\{\varepsilon, 0, 0, 0, \varepsilon, \varepsilon, \varepsilon\} = \varepsilon.$$

From (2.1), we obtain

$$\begin{aligned} \psi\left(G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3})\right) &= \psi\left(G(fx_{3n_k}, gx_{3m_k+1}, hx_{3m_k+2})\right) \\ &\leq F\begin{pmatrix} \psi\left(M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})\right), \\ \varphi\left(M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})\right) \end{pmatrix}, \end{aligned}$$

So, as $k \to +\infty$, we have

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon)$$

which leads to a contradiction because $\varepsilon > 0$.

It follows that $\{x_{3n}\}$ is a G-Cauchy sequence and by the G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to +\infty$. We claim that $fu = u$. For this, consider

$$\psi\left(G(fu, x_{3n+2}, x_{3n+3})\right) \leq F\left(\psi\left(M(u, x_{3n+1}, x_{3n+2})\right), \varphi\left(M(u, x_{3n+1}, x_{3n+2})\right)\right),$$

where

$$\begin{aligned} & M(u, x_{3n+1}, x_{3n+2}) \\ = \ & \max\{G(u, x_{3n+1}, x_{3n+2}), G(u,u,fu), G(x_{3n+1}, x_{3n+1}, gx_{3n+1}), \\ & G(x_{3n+2}, x_{3n+2}, hx_{3n+2}), G(u, fu, gx_{3n+1}), \\ & G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), G(x_{3n+2}, hx_{3n+2}, fu)\} \\ = \ & \max\{G(u, x_{3n+1}, x_{3n+2}), G(u,u,fu), G(x_{3n+1}, x_{3n+1}, x_{3n+2}), \\ & G(x_{3n+2}, x_{3n+2}, x_{3n+3}), G(u, fu, x_{3n+2}), \\ & G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, x_{3n+3}, fu)\}. \end{aligned}$$

Letting $n \to +\infty$, we obtain that

$$\psi\left(G(fu,u,u)\right) \leq F\left(\psi\left(G(fu,u,u)\right), \varphi G(fu,u,u)\right)\right) \leq \psi\left(G(fu,u,u)\right)$$

Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$.

Finally, to show the uniqueness of common fixed point. Suppose that v is another common fixed point of f, g and h. Then

$$\psi\left(G(u,v,v)\right)=\psi\left(G(fu,gv,hv)\right)\leq F\left(\psi M(u,v,v),\varphi\left(M(u,v,v)\right)\right),$$

where

$$\begin{aligned}M(u,v,v) &= \max\{G(u,v,v),G(u,u,fu),G(v,v,gv),\\ &\quad G(v,v,hv),G(u,fu,v),G(v,gv,hv),G(v,hv,fu)\}\\ &= \max\{G(u,v,v),G(u,u,u),G(v,v,v),\\ &\quad G(v,v,v),G(u,u,v),G(v,v,v),G(v,v,u)\}\\ &= \max\{G(u,v,v),G(u,u,v)\}\end{aligned}$$

If $M(u,v,v)=G(u,v,v)$, then

$$\psi\left(G(u,v,v)\right)\leq F\left(\psi\left(G(u,v,v)\right),\varphi\left(G(u,v,v)\right)\right)\leq\psi\left(G(u,v,v)\right)$$

which implies that $G(u,v,v)=0$, a contradiction.

If

$$M(u,v,v)=G(u,u,v),$$

we can find

$$\psi\left(G(u,v,v)\right)\leq F\left(\psi\left(G(u,u,v)\right),\varphi\left(G(u,u,v)\right)\right)\leq\psi\left(G(u,u,v)\right)$$

so, by nondecreasing of ψ, it follows that

$$G(u,v,v)\leq G(u,u,v) \tag{2.9}$$

Again applying (2.1), we have

$$\psi\left(G(u,u,v)\right)\leq F\left(\psi\left(G(u,v,v)\right),\varphi\left(G(u,v,v)\right)\right)\leq\psi\left(G(u,v,v)\right).$$

This implies that

$$G(u,u,v)\leq G(u,v,v) \tag{2.10}$$

by (2.9) and (2.10), we get $G(u,u,v)=G(u,v,v)$, a contradiction. Hence u is a unique common fixed point of f,g and h.

Now, we prove that every fixed point of f is a fixed point of g and h. suppose that for some p in X, we have $f(p)=p$. We claim that $p=g(p)=h(p)$.

If not then in the case when $p\neq g(p)$ or $p\neq h(p)$ we obtain

$$\psi\left(G(p,gp,hp)\right)=\psi\left(G(fp,gp,hp)\right)\leq F\left(\psi M((p,p,p),\varphi\left(M((p,p,p)\right)\right),$$

where

$$\begin{aligned}M(p,p,p) &= \max\{G(p,p,p),G(p,p,fp),G(p,p,gp),G(p,p,hp),\\ &\quad G(p,fp,gp),G(p,gp,hp),G(p,hp,fp)\}\\ &= \max\left\{0,G(p,p,gp),G(p,p,hp),G(p,gp,hp)\right\}\\ &= G(p,gp,hp)\end{aligned}$$

Thus

$$\psi\left(G(p,gp,hp)\right) \leq F\left(\psi\left(G(p,gp,hp)\right), \varphi(G(p,gp,hp))\right) \leq \psi\left(G(p,gp,hp)\right)$$

a contradiction.Therefore in all cases, we conclude that, $f(p) = g(p) = h(p) = p$. Hence, every fixed point of f is a fixed point of g and h, and conversely. □

Now, we give an example to support Theorem 2.1.

Example 2.1. Let $X = [0,1]$ and $G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$ be a G- metric on X. Define $f, g, h : X \to X$ by

$$f(x) = \begin{cases} \frac{x}{15}, x \in \left[0, \frac{1}{2}\right) \\ \frac{x}{11}, x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$g(x) = \begin{cases} \frac{x}{9}, x \in \left[0, \frac{1}{2}\right) \\ \frac{x}{7}, x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$h(x) = \begin{cases} \frac{x}{7}, x \in \left[0, \frac{1}{2}\right) \\ \frac{x}{4}, x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

We take $\psi(t) = t$ and $F(t,s) = \frac{9}{10}t$ for $t \in [0, +\infty)$, so that

$$F\left(\psi\left(M(x,y,z)\right), \varphi\left(M(x,y,z)\right)\right) = \frac{9}{10}\psi\left(M(x,y,z)\right) = \frac{9}{10}M(x,y,z)$$

where

$$M(x,y,z) = \max\left\{ \begin{array}{c} G(x,y,z), G(x,x,fx), G(y,y,gy), G(z,z,hz), \\ G(x,fx,gy), G(y,gy,hz), G(z,hz,fx) \end{array} \right\}$$

a) If $x, y, z \in \left[0, \frac{1}{2}\right)$

$G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$
$G(x,x,fx) = \frac{14}{15}x$
$G(y,y,gy) = \frac{8}{9}y$
$G(z,z,hz) = \frac{6}{7}y$

Then, $M(x,y,z) = \max\left\{\max\{|x-y|, |y-z|, |z-x|\}, \frac{14}{15}x, \frac{8}{9}x, \frac{6}{7}x\right\}$.

So,

$$\begin{aligned} \psi\left(G(fx,gy,hz)\right) &= G(fx,gy,hz) = \max\{|fx-gy|, |gy-hz|, |hz-fx|\} \\ &= \max\{|\frac{x}{15} - \frac{y}{9}|, |\frac{y}{9} - \frac{z}{7}|, |\frac{z}{7} - \frac{x}{15}|\} \\ &\leq \frac{9}{10}\max\left\{\max\{|x-y|, |y-z|, |z-x|\}, \frac{14}{15}x, \frac{8}{9}y, \frac{6}{7}z\right\} \\ &= \frac{9}{10}M(x,y,z) \end{aligned}$$

b) If $x, y, z \in \left[\frac{1}{2}, 1\right]$

$G(x,y,z)=\max\{\lvert x-y\rvert,\lvert y-z\rvert,\lvert z-x\rvert\}$
$G(x,x,fx)=\frac{10}{11}x$
$G(y,y,gy)=\frac{6}{7}y$
$G(z,z,hz)=\frac{3}{4}z$

Then, $M(x,y,z)=\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{10}{11}x,\frac{6}{7}x,\frac{3}{4}x\right\}$.

We have,

$$\begin{aligned}\psi\left(G(fx,gy,hz)\right) &= G(fx,gy,hz)=\max\{|fx-gy|,|gy-hz|,|hz-fx|\}\\ &= \max\{|\frac{x}{11}-\frac{y}{7}|,|\frac{y}{7}-\frac{z}{4}|,|\frac{z}{4}-\frac{x}{11}|\}\\ &\leq \frac{9}{10}\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{10}{11}x,\frac{6}{7}y,\frac{3}{4}z\right\}\\ &= \frac{9}{10}M(x,y,z)\end{aligned}$$

c) If $x\in\left[0,\frac{1}{2}\right)$ and $y,z\in\left[\frac{1}{2},1\right)$

$G(x,y,z)=\max\{\lvert x-y\rvert,\lvert y-z\rvert,\lvert z-x\rvert\}$
$G(x,x,fx)=\frac{14}{15}x$
$G(y,y,gy)=\frac{6}{7}y$
$G(z,z,hz)=\frac{3}{4}z$

Then, $M(x,y,z)=\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{14}{15}x,\frac{6}{7}y,\frac{3}{4}z\right\}$

We get,

$$\begin{aligned}\psi\left(G(fx,gy,hz)\right) &= G(fx,gy,hz)=\max\{|fx-gy|,|gy-hz|,|hz-fx|\}\\ &= \max\{|\frac{x}{11}-\frac{y}{7}|,|\frac{y}{7}-\frac{z}{4}|,|\frac{z}{4}-\frac{x}{11}|\}\\ &\leq \frac{9}{10}\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{14}{15}x,\frac{6}{7}y,\frac{3}{4}z\right\}\\ &= \frac{9}{10}M(x,y,z)\end{aligned}$$

d) As above results, we can find that the other cases are the same.

Therefore, all the conditions of Theorem 2.1 are satisfied. Then 0 is the unique common fixed point of f, g and h. Moreover, each fixed point of f is a fixed point of g and h, and conversely.

Corollary 2.1. *Let f, g and h be self maps on a complete G-metric space X satisfying the inequality*

$$\psi\left(G(fx,gy,hz)\right)\leq F\left(\psi\left(G(x,y,z)\right),\varphi\left(G(x,y,z)\right)\right), \tag{2.11}$$

for all $x,y,z\in X$, where $F:[0,+\infty)^2\to\mathbb{R}$ is C-class function, $\psi:[0,+\infty)\to[0,+\infty)$ is an altering distance function, $\varphi:[0,+\infty)\to[0,+\infty)$ is an ultra altering distance function. Then f,g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Corollary 2.2. [1] *Let f, g and h be self maps on a complete G-metric space X satisfying the inequality*

$$\psi\left(G(fx,gy,hz)\right) \leq \psi\left(M(x,y,z)\right) - \varphi\left(M(x,y,z)\right)$$

where $\varphi \in \Psi$, $\psi \in \Psi$ and

$$M(x,y,z) = \max\{G(x,y,z), G(x,x,fx), G(y,y,gy), G(z,z,hz), G(x,fx,gy), G(y,gy,hz), G(z,hz,fx)\}$$

for all $x,y,z \in X$. Then f,g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. Set $F(s,t) = s - t$ in Theorem 2.1. □

Remark 2.1. Put $\psi(t) = t$, $F(s,t) = ks$ with $k \in (0,1)$,we can find corollary 2.3 of [14]

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