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SOME FRACTIONAL HERMITE-HADAMARD INEQUALITIES FOR CONVEX AND GODUNOVA-LEVIN FUNCTIONS*

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Abstract. In this paper, two new integral equalities involving left-sided and right-sided Riemann-Liouville fractional integrals are established. Thereafter, some new fractional Hermite-Hadamard inequalities are presented by using the above fractional integral equalities and applying the concepts of s- and (s,m)-convex functions and s- and (s,m)-Godunova-Levin functions. Some applications to special means of real numbers are given as well.
Keywords: Hermite-Hadamard inequalities; Riemann-Liouville fractional integrals; fractional integral equalities; (s,m)-convex functions. equalities

1. Introduction and preliminaries

There are many interesting results which generalize, improve, and extend the classical Hermite-Hadamard inequality (see the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein). Throughout this paper, we use $I = [a,b] \subseteq R^+ \cup \{0\}$ to denote an interval and R^+ to denote a set of positive real numbers.

At the beginning of the paper, we recall the concepts of s-convex and (s,m)-convex functions.

Definition 1.1. (see [11]) The function $f : I \to R$ is said to be s-convex, where $s \in (0,1]$, if for every $x, y \in I$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y).$$

Definition 1.2. (see [12]) The function $f : I \to R$ is said to be (s,m)-convex, where $(s,m) \in [0,1] \times [0,1]$, if for every $x, y \in I$ and $t \in [0,1]$, we have

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t^s) f(y).$$

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Next, we recall the concept of s-Godunova-Levin functions.

Definition 1.3. (see [13, Definition 3]) The function $f : I \to R$ is said to be s-Godunova-Levin function, where $s \in (0,1]$, if for every $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}.$$

Motivated by definitions 1.2 and 1.3, we introduce another generalization of Godunova-Levin type of functions called (s, m)-Godunova-Levin functions.

Definition 1.4. The function $f : I \to R$ is said to be (s, m)-Godunova-Levin functions, where $(s, m) \in (0,1] \times (0,1]$, if for every $x, y \in I$ and $t \in (0,1)$, we have

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{m(1-t^s)}.$$

In the sequel, we recall the concepts of the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$.

Definition 1.5. (see [14]) Let $f \in L[a,b]$. The symbol ${}_{RL}J^{\alpha}_{a^+} f$ and ${}_{RL}J^{\alpha}_{b^-} f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ and are defined by

$$({}_{RL}J^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ (0 \leq a < x \leq b),$$

$$({}_{RL}J^{\alpha}_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ (0 \leq a \leq x < b),$$

respectively. Here $\Gamma(\cdot)$ is the Gamma function.

Next, we give two new type fractional integral equalities which will be widely used in the sequel.

Lemma 1.1. *Let $f : [a,b] \to R$ be a differentiable function and $f' \in L[a,b]$. For any $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, the following equality for fractional integrals holds:*

$$\frac{f(b)(1+\alpha(1-\lambda)) + f(a)(1-\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}({}_{RL}J^{\alpha}_{a^+} f(b) + {}_{RL}J^{\alpha}_{b^-} f(a))$$
$$= \frac{b-a}{2} \int_0^1 (t^{\alpha} + \alpha(1-\lambda) - (1-t)^{\alpha}) f'(tb + (1-t)a)dt.$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 (t^{\alpha} + \alpha(1-\lambda) - (1-t)^{\alpha}) f'(tb + (1-t)a)dt \\ &= \int_0^1 (t^{\alpha} + \alpha(1-\lambda)) f'(tb + (1-t)a)dt - \int_0^1 (1-t)^{\alpha} f'(tb + (1-t)a)dt \\ &:= I_1 + I_2. \qquad (1.1) \end{aligned}$$

Integrating by parts

$$\begin{aligned} I_1 &:= \int_0^1 (t^\alpha + \alpha(1-\lambda)) f'(tb+(1-t)a)dt \\ &= \frac{f(b)(1+\alpha(1-\lambda)) - f(a)\alpha(1-\lambda)}{b-a} - \frac{\alpha}{b-a}\int_0^1 t^{\alpha-1} f(tb+(1-t)a)dt \\ &= \frac{f(b)(1+\alpha(1-\lambda)) - f(a)\alpha(1-\lambda)}{b-a} - \frac{\alpha}{(b-a)^2}\int_a^b \left(\frac{u-a}{b-a}\right)^{\alpha-1} f(u)du \\ &= \frac{f(b)(1+\alpha(1-\lambda)) - f(a)\alpha(1-\lambda)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\, {}_{RL}J^\alpha_{b^-} f(a), \end{aligned} \tag{1.2}$$

and similarly we get

$$\begin{aligned} I_2 &:= -\int_0^1 (1-t)^\alpha f'(tb+(1-t)a)dt \\ &= \frac{f(a)}{b-a} - \frac{\alpha}{(b-a)^2}\int_a^b \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)du \\ &= \frac{f(a)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\, {}_{RL}J^\alpha_{a^+} f(b). \end{aligned} \tag{1.3}$$

Submitting (1.2) and (1.3) to (1.1), we have

$$\begin{aligned} I &:= \frac{f(b)(1+\alpha(1-\lambda)) + f(a)(1-\alpha(1-\lambda))}{b-a} \\ &\quad - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}[{}_{RL}J^\alpha_{a^+} f(b) + {}_{RL}J^\alpha_{b^-} f(a)]. \end{aligned} \tag{1.4}$$

Next, by multiplying both sides by $\frac{b-a}{2}$ for (1.4), we have the conclusion. □

Lemma 1.2. *Let $f : [a,b] \to R$ be a twice differentiable function and $f'' \in L[a,b]$. For any $0 < \alpha \le 1$ and $0 < m \le 1$, the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(mb)+f(a)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}({}_{RL}J^\alpha_{a^+} f(mb) + {}_{RL}J^\alpha_{mb^-} f(a)) \\ &= \frac{(mb-a)^2}{2}\int_0^1 \frac{(1-(1-t)^{\alpha+1} - t^{\alpha+1})}{\alpha+1} f''(ta+m(1-t)b)dt. \end{aligned}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 ((1-t)^{\alpha+1} + t^{\alpha+1}) f''(ta+m(1-t)b)dt \\ &= \frac{f'(mb)-f'(a)}{mb-a} + \frac{\alpha+1}{mb-a}\int_0^1 (t^\alpha - (1-t)^\alpha) f'(ta+m(1-t)b)dt, \end{aligned} \tag{1.5}$$

By using Lemma 1.1 with $\lambda = 1$, it suffices to verify that

$$\begin{aligned} &\int_0^1 (t^\alpha - (1-t)^\alpha) f'(ta + m(1-t)b)dt \\ (1.6) \qquad = &-\frac{f(mb)+f(a)}{mb-a} + \frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}({}_{RL}J^\alpha_{a^+} f(mb) + {}_{RL}J^\alpha_{mb^-} f(a)). \end{aligned}$$

Submitting (1.6) to (1.5), we have

$$\begin{aligned} I &= \frac{f'(mb)-f'(a)}{mb-a} - \frac{(\alpha+1)(f(mb)+f(a))}{(mb-a)^2} \\ (1.7) \qquad &+ \frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha+2}}({}_{RL}J^\alpha_{a^+} f(mb) + {}_{RL}J^\alpha_{mb^-} f(a)). \end{aligned}$$

Note that

$$(1.8) \qquad f'(mb) - f'(a) = \int_a^{mb} f''(x)dx = (mb-a)\int_0^1 f''(ta + m(1-t)b)dt.$$

Submitting (1.8) to (1.7) and multiplying both sides by $\frac{(mb-a)^2}{2}$ for (1.7), we obtain the result. $\square$

In the present paper, we present some new Hermite-Hadamard's inequalities involving left-sided and right-sided Riemann-Liouville fractional integrals by using our established equalities in lemmas 1.1 and 1.2 and definitions 1.1, 1.2, 1.3 and 1.4. Finally, some applications to special means of real numbers are given as well.

2. Main results for s- and (s, m)-convex functions

Firstly, we give the inequalities for s-convex functions.

Theorem 2.1. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$. If $|f'| \in L[a, b]$ and $|f'|$ is s-convex function. Then for some $0 < \alpha \le 1$ and $0 < \lambda \le 1$, the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}({}_{RL}J^\alpha_{b^-} f(a) + {}_{RL}J^\alpha_{a^+} f(b))\right| \\ \le\ &\frac{b-a}{2}(\varphi(\alpha,\lambda,s)|f'(b)| + \phi(\alpha,\lambda,s)|f'(a)|), \end{aligned}$$

where

$$\varphi(\alpha,\lambda,s) = \frac{1}{\alpha+s+1} + \frac{\alpha(1-\lambda)}{s+1} - \frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)},$$

and

$$\phi(\alpha,\lambda,s) = \frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)} + \frac{\alpha(1-\lambda)}{s+1} - \frac{1}{\alpha+s+1}.$$

Proof. Using Lemma 1.1 via $f' \in L[a,b]$ and $|f'|$ is s-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}({}_{RL}J^{\alpha}_{b^-} f(a) + {}_{RL}J^{\alpha}_{a^+} f(b))\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 (t^{\alpha} + \alpha(1-\lambda) - (1-t)^{\alpha})|f'(tb+(1-t)a)|dt \\
\leq\ & \frac{b-a}{2}\int_0^1 (t^{\alpha} + \alpha(1-\lambda) - (1-t)^{\alpha})(t^s|f'(b)| + (1-t)^s|f'(a)|)dt \\
=\ & \frac{b-a}{2}(\varphi(\alpha,\lambda,s)|f'(b)| + \phi(\alpha,\lambda,s)|f'(a)|).
\end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let $f : [a,b] \to R$ be differentiable. If $|f'|^q \in L[a,b]$ and $|f'|^q (q > 1)$ is s-convex function. Then for some $0 < \alpha \leq 1, 0 < \lambda \leq 1$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}({}_{RL}J^{\alpha}_{b^-} f(a) + {}_{RL}J^{\alpha}_{a^+} f(b))\right| \\
\leq\ & \frac{b-a}{2}\psi(\alpha,\lambda,p)\left(\frac{|f'(a)|^q + |f'(b)|^q}{1+sq}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\psi(\alpha,\lambda,p) = \alpha(1-\lambda) + 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}$.

Proof. Using Lemma 1.1, Hölder's inequality via $|f'|^q \in L[a,b]$ and $|f'|^q$ is s-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}({}_{RL}J^{\alpha}_{b^-} f(a) + {}_{RL}J^{\alpha}_{a^+} f(b))\right| \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 (t^{\alpha} + \alpha(1-\lambda))|f'(tb+(1-t)a)|dt + \int_0^1 (1-t)^{\alpha}|f'(tb+(1-t)a)|dt\right) \\
\leq\ & \frac{b-a}{2}\Bigg[\left(\int_0^1 (t^{\alpha} + \alpha(1-\lambda))^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}} \\
& + \left(\int_0^1 (1-t)^{\alpha p} dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\Bigg] \\
\leq\ & \frac{b-a}{2}\left[\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}} + \left(\int_0^1 \alpha^p (1-\lambda)^p dt\right)^{\frac{1}{p}} + \left(\int_0^1 (1-t)^{\alpha p} dt\right)^{\frac{1}{p}}\right] \\
& \times\left[\left(\int_0^1 (t^{qs}|f'(b)|^q dt)\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{qs}|f'(a)|^q dt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{b-a}{2}\psi(\alpha,\lambda,p)\left(\frac{|f'(b)|^q + |f'(a)|^q}{1+qs}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use the following fact due to Minkowski's inequality for $p \geq 1$:

$$\left(\int_0^1 (t^\alpha + \alpha(1-\lambda))^p dt\right)^{\frac{1}{p}} \leq \left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}} + \left(\int_0^1 (\alpha(1-\lambda))^p dt\right)^{\frac{1}{p}},$$

and

$$\left(\int_0^1 |f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}} \leq \left(\int_0^1 t^{qs}|f'(b)|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{qs}|f'(a)|^q dt\right)^{\frac{1}{q}}.$$

This completes the proof. □

Secondly, we give the inequalities for (s,m)-convex functions.

Theorem 2.3. *Let $f:[a,b]\to R$ be twice differentiable on (a,b) with $a \geq 0$, $f'' \in L[a,b]$ and $a < mb < b$. If $|f''|$ is (s,m)-convex function, then for any $0 < \alpha \leq 1$ we have*

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left({}_{RL}J^\alpha_{mb^-} f(a) + {}_{RL}J^\alpha_{a^+} f(mb)\right)\right| \\
\leq\ & \frac{(mb-a)^2}{2(\alpha+1)}\left[\omega(\alpha,s)(|f''(a)| - m|f''(b)|) + \frac{m\alpha}{\alpha+2}|f''(b)|\right],
\end{aligned}$$

where

$$\omega(\alpha,s) = \frac{1}{s+1} - \frac{1}{\alpha+s+2} - \frac{\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+3)}.$$

Proof. Using Lemma 1.2 via $|f''|$ is (s,m)-convex function, we have

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left({}_{RL}J^\alpha_{mb^-} f(a) + {}_{RL}J^\alpha_{a^+} f(mb)\right)\right| \\
\leq\ & \frac{(mb-a)^2}{2}\int_0^1 \frac{(1-(1-t)^{\alpha+1} - t^{\alpha+1})}{\alpha+1}|f''(ta+m(1-t)b|dt \\
\leq\ & \frac{(mb-a)^2}{2(\alpha+1)}\int_0^1 (1-(1-t)^{\alpha+1} - t^{\alpha+1})(t^s|f''(a)| + m(1-t^s)|f''(b)|)dt \\
\leq\ & \frac{(mb-a)^2}{2(\alpha+1)}\Big[(|f''(a)| - m|f''(b)|)\int_0^1 (t^s - t^s(1-t)^{\alpha+1} - t^{\alpha+s+1})dt \\
& + m\int_0^1 (1-(1-t)^{\alpha+1} - t^{\alpha+1})dt\Big] \\
=\ & \frac{(mb-a)^2}{2(\alpha+1)}\left[\omega(\alpha,s)(|f''(a)| - m|f''(b)|) + \frac{m\alpha}{\alpha+2}|f''(b)|\right].
\end{aligned}$$

This completes the proof. □

Theorem 2.4. *Let $f:[a,b]\to R$ be twice differentiable on (a,b) with $a\geq 0, f''\in L[a,b]$ and $a<mb<b$. If $|f''|^q(q>1)$ is (s,m)-convex function then*

$$\left|\frac{f(mb)+f(a)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}({}_{RL}J^{\alpha}_{mb^-}f(a)+{}_{RL}J^{\alpha}_{a^+}f(mb))\right|$$
$$\leq \frac{(mb-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\psi(s,q,m),$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\psi(s,q,m)=\left(\frac{|f''(a)|^q}{qs+1}\right)^{\frac{1}{q}}+\left(\frac{qsm^q|f''(b)|^q}{qs+1}\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1.2, Holder's inequality and the fact that $|f''|^q$ is $(s.m)$-convex function, we have

$$\left|\frac{f(mb)+f(a)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}({}_{RL}J^{\alpha}_{mb^-}f(a)+{}_{RL}J^{\alpha}_{a^+}f(mb))\right|$$
$$\leq \frac{(mb-a)^2}{2}\int_0^1\frac{(1-(1-t)^{\alpha+1}-t^{\alpha+1})}{\alpha+1}|f''(ta+m(1-t)b|dt$$
$$\leq \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1(t^s|f''(a)|+m(1-t^s)|f''(b)|)^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)})dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1 t^{qs}|f''(a)|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 m^q|f''(b)|^q(1-t^s)^q dt\right)^{\frac{1}{q}}\right]$$
$$= \frac{(mb-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q}{qs+1}\right)^{\frac{1}{q}}+\left(\frac{qsm^q|f''(b)|^q}{qs+1}\right)^{\frac{1}{q}}\right].$$

Here we use the following fact due to Minkowski's inequality for $q>1$:

$$\left(\int_0^1(t^s|f''(a)|+m(1-t^s)|f''(b)|)dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^1 t^{qs}|f''(a)|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 m^q|f''(b)|^q(1-t^s)^q dt\right)^{\frac{1}{q}}.$$

This completes the proof. □

3. Main results for s- and (s,m)-Godunova-Levin functions

We first give the inequalities for s-Godunova-Levin functions.

Theorem 3.1. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$ and $|f'|$ is s-Godunova-Levin function, then for some $0 < \alpha \le 1$ and $0 < \lambda \le 1$, the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}({}_{RL}J^\alpha_{b^-} f(a) + {}_{RL}J^\alpha_{a^+} f(b))\right|$$
$$\le \frac{(b-a)}{2}(\Psi_1(\alpha,\lambda,s)|f'(b)| + \Psi_2(\alpha,\lambda,s)|f'(a)|),$$

where

$$\Psi_1(\alpha,\lambda,s) = \frac{1}{\alpha-s+1} + \frac{\alpha(1-\lambda)}{1-s} - \frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)},$$
$$\Psi_2(\alpha,\lambda,s) = \frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)} + \frac{\alpha(1-\lambda)}{1-s} - \frac{1}{\alpha-s+1}.$$

Proof. Using Lemma 1.1 via $f' \in L[a,b]$ and $|f'|$ is s-Godunova-Levin function, we have

$$\begin{aligned}
&\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}({}_{RL}J^\alpha_{b^-} f(a) + {}_{RL}J^\alpha_{a^+} f(b))\right| \\
\le\ & \frac{b-a}{2}\int_0^1 (t^\alpha + \alpha(1-\lambda) - (1-t)^\alpha)|f'(tb+(1-t)a)|dt \\
\le\ & \frac{b-a}{2}\int_0^1 (t^\alpha + \alpha(1-\lambda) - (1-t)^\alpha)\left[\frac{|f'(b)|}{t^s} + \frac{|f'(a)|}{(1-t)^s}\right]dt \\
=\ & \frac{b-a}{2}\left(|f'(b)|\int_0^1 \frac{t^\alpha+\alpha(1-\lambda)-(1-t)^\alpha}{t^s}dt + |f'(a)|\int_0^1 \frac{t^\alpha+\alpha(1-\lambda)-(1-t)^\alpha}{(1-t)^s}dt\right) \\
=\ & \frac{(b-a)}{2}(\Psi_1(\alpha,\lambda,s)|f'(b)| + \Psi_2(\alpha,\lambda,s)|f'(a)|).
\end{aligned}$$

This complete the proof. □

Theorem 3.2. *Let $f : [a,b] \to R$ be a differentiable. If $|f'|^q \in L[a,b]$ and $|f'|^q (q > 1)$ is s-Godunova-Levin function, then for some $0 < \alpha \le 1$ and $0 < \lambda \le 1$, the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)(1-\alpha(1-\lambda)) + f(b)(1+\alpha(1-\lambda))}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}({}_{RL}J^\alpha_{b^-} f(a) + {}_{RL}J^\alpha_{a^+} f(b))\right|$$
$$\le \frac{b-a}{2}\psi(\alpha,\lambda,p)\left(\frac{|f'(a)|^q + |f'(b)|^q}{1-sq}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\psi(\alpha,\lambda,p) = \alpha(1-\lambda) + 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}$.

Proof. Using Lemma 1.1, Holder's inequality via $|f'|^q \in L[a,b]$ and $|f'|^q$ is s-Godunova-Levin function, we have

$$\begin{aligned}
&\left|\frac{f(a)(1-\alpha(1-\lambda))+f(b)(1+\alpha(1-\lambda))}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}({}_{RL}J^\alpha_{b^-}f(a)+{}_{RL}J^\alpha_{a^+}f(b))\right|\\
\le\ & \frac{b-a}{2}\Big(\int_0^1(t^\alpha+\alpha(1-\lambda))|f'(tb+(1-t)a)|dt+\int_0^1(1-t)^\alpha|f'(tb+(1-t)a)|dt\Big)\\
\le\ & \frac{b-a}{2}\Big[\left(\int_0^1(t^\alpha+\alpha(1-\lambda))^pdt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tb+(1-t)a)|^qdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tb+(1-t)a)|^qdt\right)^{\frac{1}{q}}\Big]\\
\le\ & \frac{b-a}{2}\Big[\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}+\left(\int_0^1\alpha^p(1-\lambda)^pdt\right)^{\frac{1}{p}}+\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\Big]\\
&\times\left(\int_0^1\left(\frac{|f'(b)|^q}{t^s}+\frac{|f'(a)|^q}{(1-t)^s}\right)dt\right)^{\frac{1}{q}}\\
=\ & \frac{b-a}{2}\psi(\alpha,\lambda,p)\left(\frac{|f'(a)|^q+|f'(b)|^q}{1-sq}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use the following fact due to Minkowski's inequality for $p \ge 1$:

$$\left(\int_0^1(t^\alpha+\alpha(1-\lambda))^pdt\right)^{\frac{1}{p}}\le\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}+\left(\int_0^1(\alpha(1-\lambda))^pdt\right)^{\frac{1}{p}}.$$

This completes the proof. □

Next, we give the inequalities for (s,m)-Godunova-Levin functions.

Theorem 3.3. *Let $f:[a,b]\to R$ be twice differentiable on (a,b) with $a\ge 0$, $f''\in L[a,b]$ and $a<mb\le b$. If $|f''|$ is (s,m)-Godunova-Levin function, then for any $0<\alpha\le 1$ we have*

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}({}_{RL}J^\alpha_{mb^-}f(a)+{}_{RL}J^\alpha_{a^+}f(mb))\right|\\
\le\ & \frac{(mb-a)^2}{2}\left(\Phi_1(\alpha,s)|f''(a)|+\Phi_2(\alpha,s)\frac{|f''(b)|}{m}\right),
\end{aligned}$$

where

$$\Phi_1(\alpha,s)=\frac{1}{1-s}-\frac{1}{\alpha-s+2}-\frac{\Gamma(\alpha+2)\Gamma(1+s)}{\Gamma(\alpha-s+3)},$$

and

$$\Phi_2(\alpha,s)=\frac{s+2}{s+1}-\frac{1}{a+2}-\frac{\Gamma(1+s)\Gamma(\alpha+2)}{\Gamma(\alpha+s+3)}-\frac{2(\alpha+2)+s}{(\alpha+2)(\alpha+s+2)}.$$

Proof. Using Lemma 1.2 via $|f''|$ is (s,m)-Godunova-Levin function, we have

$$\begin{aligned}
&\left|\frac{f(mb)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}({}_{RL}J^{\alpha}_{mb^-}f(a)+{}_{RL}\,J^{\alpha}_{a^+}f(mb))\right|\\
\leq\;& \frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}|f''(ta+m(1-t)b)|dt\\
\leq\;& \frac{(mb-a)^2}{2(\alpha+1)}\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})\left(\frac{|f''(a)|}{t^s}+\frac{|f''(b)|}{m(1-t^s)}\right)dt\\
=\;& \frac{(mb-a)^2}{2}\left(\Phi_1(\alpha,s)|f''(a)|+\Phi_2(\alpha,s)\frac{|f''(b)|}{m}\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $f:[a,b]\to R$ be twice differentiable on (a,b) with $a\geq 0$, $f''\in L[a,b]$ and $a<mb<b$. If $|f''|^q$ $(q>1)$ is (s,m)-Godunova-Levin function, then for any $0<\alpha\leq 1$ we have*

$$\begin{aligned}
&\left|\frac{f(mb)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}({}_{RL}J^{\alpha}_{mb^-}f(a)+{}_{RL}\,J^{\alpha}_{a^+}f(mb))\right|\\
\leq\;& \frac{(mb-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q}{1-s}+\frac{(s+2)|f''(b)|^q}{m^q(1+s)}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.2, Holder's inequality and the fact that $|f''|^q$ is $(s.m)$-Godunva-Levin function, we have

$$\begin{aligned}
&\left|\frac{f(mb)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}({}_{RL}J^{\alpha}_{mb^-}f(a)+{}_{RL}\,J^{\alpha}_{a^+}f(mb))\right|\\
\leq\;& \frac{(mb-a)^2}{2}\int_0^1\frac{(1-(1-t)^{\alpha+1}-t^{\alpha+1})}{\alpha+1}|f''(ta+m(1-t)b)|dt\\
\leq\;& \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(ta+m(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
\leq\;& \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)})dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\frac{|f''(a)|^q}{t^s}+\frac{|f''(b)|^q}{m^q(1-t^s)}\right)dt\right)^{\frac{1}{q}}\\
=\;& \frac{(mb-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q}{1-s}+\frac{(s+2)|f''(b)|^q}{m^q(1+s)}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is complete. □

4. Applications to special means

Consider the following special means (see Pearce and Pečarić [15]) for arbitrary real numbers $\alpha, \beta, \alpha \neq \beta$ as follows:

$$
\begin{aligned}
H(\alpha,\beta) &= \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},\ \alpha,\ \beta \in R\setminus\{0\},\\
A(\alpha,\beta) &= \frac{\alpha+\beta}{2},\ \alpha,\beta \in R,\\
L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},\ |\alpha|\neq|\beta|,\ \alpha\beta\neq 0,\\
L_n(\alpha,\beta) &= \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\ n\in Z\setminus\{-1,0\},\ \alpha,\beta\in R,\ \alpha\neq\beta.
\end{aligned}
$$

Now, using the theory results in Section 3 Section and 4, we give some applications to special means of real numbers.

Theorem 4.1. *For some $s \in (0,1]$, $n \in Z\setminus\{-1,0\}$, $0 \le a < b$, the following inequality for fractional integrals holds:*

$$\left|A(a^2,b^2)-L_n^n(a^2,b^2)\right| \le 2(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{a^q+b^q}{1+sq}\right)^{\frac{1}{q}}.$$

where

$$\frac{1}{p}+\frac{1}{q}=1,\ 1<q<\infty.$$

Proof. Applying Theorem 2.2 for $f(x)=x^2, \alpha=1, \lambda=1$ one can obtain the result immediately. □

Theorem 4.2. *For some $s \in (0,1]$, $0 \le a < b$, the following inequality for fractional integrals holds:*

$$\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right| \le (b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{b^{2q}+a^{2q}}{a^{2q}b^{2q}(1+sq)}\right)^{\frac{1}{p}}.$$

where $\frac{1}{p}+\frac{1}{q}=1, 1<q<\infty$.

Proof. Applying Theorem 2.2 for $f(x)=\frac{1}{x}, \alpha=1, \lambda=1$ one can obtain the result immediately. □

Theorem 4.3. *For some $s, m \in (0,1]$, $n \in Z\setminus\{-1,0\}$, $0 \le a < b$, the following inequality for fractional integrals holds:*

$$\left|A(a^2,m^2b^2)-L_n^n(a^2,m^2b^2)\right| \le \frac{(mb-a)^2}{2}\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}\left(\left(\frac{1}{1+sq}\right)^{\frac{1}{q}}+\frac{qsm^q}{1+sq}\right)^{\frac{1}{q}}.$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<q<\infty$.

Proof. Applying Theorem 2.4 for $f(x)=x^2, \alpha=1, \lambda=1$ one can obtain the result immediately. □

Theorem 4.4. *For some* $s,m\in(0,1]$, $0\le a<b$, *the following inequality for fractional integrals holds:*

$$\left|\frac{1}{H(a,mb)}-\frac{1}{L(a,mb)}\right|\le\frac{(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{a^{2q}+b^{2q}}{a^{2q}b^{2q}(1-sq)}\right)^{\frac{1}{q}}.$$

where

$$\frac{1}{p}+\frac{1}{q}=1,\ 1<q<\infty.$$

Proof. Applying Theorem 3.2 for $f(x)=\frac{1}{x}, \alpha=1, \lambda=1$ one can obtain the result immediately. □

Theorem 4.5. *For some* $s,m\in(0,1]$, $n\in Z\setminus\{-1,0\}$, $0\le a<b$, *the following inequality for fractional integrals holds:*

$$\left|A(a^2,m^2b^2)-L_n^n(a^2,m^2b^2)\right|\le\frac{(mb-a)^2}{2}\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{1-s}+\frac{s+2}{m^q(1+s)}\right)^{\frac{1}{q}}.$$

where

$$\frac{1}{p}+\frac{1}{q}=1,\ 1<q<\infty.$$

Proof. Applying Theorem 3.4 for $f(x)=x^2, \alpha=1, \lambda=1$ one can obtain the result immediately. □

Theorem 4.6. *For some* $s,m\in(0,1]$, $0\le a<b$, *the following inequality for fractional integrals holds:*

$$\left|\frac{1}{H(a,mb)}-\frac{1}{L(a,mb)}\right|\le\frac{(mb-a)^2}{2}\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{a^{3q}(1-s)}+\frac{s+2}{b^{3q}m^q(1+s)}\right)^{\frac{1}{q}}.$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<q<\infty$.

Proof. Applying Theorem 3.4 for $f(x)=\frac{1}{x}, \alpha=1, \lambda=1$ one can obtain the result immediately. □

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REFERENCES

1. J. Cal, J. Carcamob, L. Escauriaza: *A general multidimensional Hermite-Hadamard type inequality*. J. Math. Anal. Appl., 356 (2009), 659-663.
2. M. Avci, H. Kavurmaci, M. E. Özdemir: *New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications*. Appl. Math. Comput., 217 (2011), 5171-5176.
3. M. E. Özdemir, M. Avci, H. Kavurmaci: *Hermite-Hadamard-type inequalities via* (α, m)*-convexity*. Comput. Math. Appl., 61 (2011), 2614-2620.
4. S. S. Dragomir: *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*. Linear Algebra Appl., 436 (2012), 1503-1515.
5. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak: *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*. Math. Comput. Model., 57 (2013), 2403-2407.
6. M. Bessenyei: *The Hermite-Hadamard inequality in Beckenbach's setting*. J. Math. Anal. Appl., 364 (2010), 366-383.
7. K. Tseng, S. Hwang, K. Hsu: *Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications*. Comput. Math. Appl., 64 (2012), 651-660.
8. C. P. Niculescu: *The Hermite-Hadamard inequality for log-convex functions*. Nonlinear Anal.:TMA, 75 (2012), 662-669.
9. J. Wang, J. Deng, M. Fečkan: *Hermite-Hadamard type inequalities for r-convex functions via Riemann-Liouville fractional integrals*. Ukrainian Math. J., 65 (2013), 193-211.
10. R. Bai, F. Qi, B. Xi: *Hermite-Hadamard type inequalities for the m- and* (α, m)*-logarithmically convex functions*. Filomat, 27 (2013), 1-7.
11. H. Hudzik, L. Maligranda: *Some remarks on s-convex functions*. Aequationes Math., 48 (1994), 100-111.
12. M. K. Bakula, M. E. Özdemir, J. Pečrić: *Hadamard type inequalities for m-convex and* (α, m)*-convex functions*. J. Inequal. Pure Appl. Math., 9 (2008), Article 96.
13. M. A. Noor, K. I. Noor, M. U. Awan, S. Khan: *Fractional Hermite-Hadamard Inequalities for some New Classes of Godunova-Levin functions*. Appl. Math. Inf. Sci., 8 (2014), 2865-2872.
14. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo: *Theory and applications of fractional differential equations*. Elsevier Science B.V., 2006.
15. C. E. M. Pearce, J. E. Pearce, Pečarić: *Inequalities for differentiable mappings with application to special means and quadrature formula*. Appl. Math. Lett., 13 (2000), 51-55.

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