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ON POINTWISE $(f, \lambda)-$ STATISTICAL CONVERGENCE OF ORDER α AND STRONG POINTWISE $(V, f, \lambda)-$ SUMMABILITY OF ORDER α OF SEQUENCES OF FUZZY MAPPINGS

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Abstract. In this paper, we introduce pointwise $(f, \lambda)-$statistical convergence of order α and strong pointwise $(V, f, \lambda)-$summability of order α of sequences of fuzzy mappings. In addition, we examined some inclusion theorems among these concepts.
Keywords: statistical convergence, pointwise summability, fuzzy mappings.

1. Introduction, Definitions and Preliminaries

The idea of statistical convergence was introduced by Steinhaus [41] and Fast [24] and later reintroduced by Schoenberg [39]. Over the years, and under different names, statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altin, Et and Tripathy [2], Altin, Et and Basarir [3], Altinok and Kasap [4], Altinok, Et and Altin [5], Bhardwaj and Dhawan ([7],[8]), Çakalli [10], Cinar, Karakas and Et [11], Caserta, Di Maio and Kocinac [12], Çolak and Bektas [16], Connor [17], Et [22], Et, Çolak and Altin [23], Tripathy and Dutta [45], Srivastava and Et [40], Fridy [25], Isik and Akbas [28], Isik and Et [29], Mursaleen [35], Salat [37], Tripathy [43] and many others. Statistical convergence of order α and strong Cesàro summability of order α were given by Çolak [14].

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Let $x = (x_k)$ be any sequence in $\mathbb{R}$ (or $\mathbb{C}$). The sequence (x_k) is named statistically convergent to the number L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0,$$

where $\mathbb{N}$ is the set of positive integers and $\delta(A) = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} \mathcal{X}_A(k)$ provided the limit exists, is the natural density of the set $A \subseteq \mathbb{N}$ and $\mathcal{X}_A$ is the characteristic function of A. In this case we write $S - \lim x_k = L$. The set of all statistically convergent sequences will be denoted by S. , where $|A|$ denotes the number of elements of the enclosed set.

The concept of modulus function was formally introduced by Nakano [34]. A mapping $f : [0, \infty) \to [0, \infty)$ is said to be a modulus if

i) $f(x) = 0$ iff $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

iii) f is increasing,

iv) f is right-continuous at 0.

The continuity of f everywhere on $[0, \infty)$ follows from above definition. A modulus function can be bounded or unbounded. For example $f(x) = x^p, (0 < p \leq 1)$ is unbounded and $f(x) = \frac{x}{1+x}$ is bounded. From (ii) we easily get the inequality $f(nx) \leq nf(x)$ and so that $f(n) \leq nf(1)$ for every positive integer n and real $x \geq 0$.

Aizpuru, Listan-Garcia and Barreno [1] defined the $f-$density of a subset A of $\mathbb{N}$ by using an unbounded modulus function. After then, Bhardwaj and Dhawan ([7],[8]) introduced $f-$statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus function f for real sequences.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. By Λ we denote the class of all such sequences. The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$. A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number L if $t_n(x) \to L$ as $n \to \infty$.

The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but Altin, Et and Tripathy [2], Altin, Et and Basarir [3], Altinok and Kasap [4], Altinok, Et and Altin [5], Altinok, Et and Çolak [6], Burgin [9], Tripathy and Dutta [45], Et,Tripathy and Dutta [19], Tripathy and Sen [46], Tripathy and Ray [44] extended the idea to apply to sequences of fuzzy numbers.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [47] a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \longrightarrow [0,1]$. The function u itself is often used for the fuzzy set.

Let $C(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}^n$. If $A, B \in C(\mathbb{R}^n)$, then the distance between A and B is defined by the Hausdorff metric

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \parallel a - b \parallel, \sup_{b\in B} \inf_{a\in A} \parallel a - b \parallel\},$$

where $\parallel . \parallel$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space ([18], [31]).

Denote $L(\mathbb{R}^n) = \{u : \mathbb{R}^n \longrightarrow [0,1] : u$ satisfies $(i) - (iv)$ below$\}$,

$i)$ u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

$ii)$ u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \leq \mu \leq 1, u(\mu x + (1-\mu)y) \geq \min[u(x), u(y)]$;

$iii)$ u is upper semicontinuous;

$iv)$ The closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathbb{R}^n)$, then u is called a fuzzy number, and $L(\mathbb{R}^n)$ is said to be a fuzzy number space.

For $0 < \beta \leq 1$, the β-level set $[u]^\beta$ is defined by

$$[u]^\beta = \{x \in \mathbb{R}^n : u(x) \geq \beta\}.$$

Define, for each $1 \leq q < \infty$,

$$d_q(u, v) = \left(\int_0^1 \left[\delta_\infty([u]^\beta, [v]^\beta)\right]^q d\beta \right)^{1/q}$$

and $d_\infty(u, v) = \sup_{0\leq\beta\leq1} \delta_\infty([u]^\beta, [v]^\beta)$, where δ_∞ is the Hausdorff metric. Clearly

$$d_\infty(u, v) = \lim_{q\to\infty} d_q(u, v)$$

with $d_q \leq d_s$ if $q \leq s$ ([18], [31]). For simplicity of notation, throughout the paper d will denote the notation d_q with $1 \leq q \leq \infty$.

A fuzzy mapping X is a mapping from a set $T\,(\subset \mathbb{R}^n)$ to the set of all fuzzy numbers. A sequence of fuzzy mappings is a function whose domain is the set of positive integers and whose range is a set of fuzzy numbers. We denote a sequence of fuzzy mappings by (X_k). If (X_k) is a sequence of fuzzy mappings then $(X_k\,(t))$ is a sequence of fuzzy numbers for every $t \in T$. Corresponding to a number t in the domain of each of terms of the sequence of fuzzy mappings (X_k), there is a sequence of fuzzy numbers $(X_k(t))$. If $(X_k(t))$ converges for each number t in a set T and we get $\lim_k X_k(t) = X(t)$, then we say that (X_k) converges pointwise to X on T [33]. By $B(A_F)$ we denote the class of all bounded sequences of fuzzy mappings on T.

2. Pointwise $(f,\lambda)-$ Statistical Convergence of Order α

In this section, we introduce the concept of pointwise $(f,\lambda)-$statistical convergence of order α of sequences of fuzzy mappings. In Theorem 2.4 we give the inclusion relations between the sets of pointwise $(f,\lambda)-$statistically convergent sequences of order α of fuzzy mappings for different α's and μ's.

Before giving the inclusion relations we will give a new definition.

Definition 2.1. *Let $\lambda = (\lambda_n) \in \Lambda$, $\alpha \in (0,1]$ be an unbounded modulus. A sequence (X_k) of fuzzy mappings is said to be pointwise $(f,\lambda)-$statistically convergent of order α (or pointwise $S_{f\lambda}^{\alpha}-$statistically convergent) to X on a set T if, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n^{\alpha})} f\left(\left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon\ \}\right|\right) = 0,$$

for every $t \in T$, where $|A|$ denotes the number of elements of the enclosed set, λ_n^{α} denotes the α^{th} power $(\lambda_n)^{\alpha}$ of λ_n, that is $\lambda^{\alpha} = (\lambda_n^{\alpha}) = (\lambda_1^{\alpha},\ \lambda_2^{\alpha},\ ...,\ \lambda_n^{\alpha},\ ...)$. In this case we write $S_{\lambda f}^{\alpha} - \lim X_k(t) = X(t)$ on T.

The set of all pointwise $(f,\lambda)-$statistically convergent sequences of fuzzy mappings of order α will be denoted by $S_{f\lambda}^{\alpha}(F)$. In case of $\alpha = 1$, we shall write $S_{f\lambda}(F)$ instead of $S_{f\lambda}^{\alpha}(F)$, for $\lambda_n = n$ we shall write $S_f^{\alpha}(F)$ instead of $S_{f\lambda}^{\alpha}(F)$, and for $f(x) = x$ we shall write $S_{\lambda}^{\alpha}(F)$ instead of $S_{f\lambda}^{\alpha}(F)$. In the special cases $\lambda_n = n, f(x) = x$ and $\alpha = 1$ we shall write $S(F)$ instead of $S_{f\lambda}^{\alpha}(F)$. In case of $X = \bar{0}$, we'll write $S_{0f\lambda}^{\alpha}(F)$ instead of $S_{f\lambda}^{\alpha}(F)$, where

$$\bar{0}(t) = \begin{cases} 1, & \text{for } t = (0,0,0,...,0) \\ 0, & \text{otherwise} \end{cases}$$

The proof of the following theorem is easy, so we state it without proof.

Theorem 2.1. *Let $\lambda \in \Lambda$, α be a real number such that $\alpha \in (0,1]$, f be an unbounded modulus and (X_k) and (Y_k) be two sequences of fuzzy mappings.*

(*i*) *If $S_{f\lambda}^{\alpha} - \lim X_k(t) = X_0(t)$ and $c \in \mathbb{R}$, then $S_{f\lambda}^{\alpha} - \lim cX_k(t) = cX_0(t)$,*

(*ii*) *If $S_{f\lambda}^{\alpha} - \lim X_k(t) = X_0(t)$ and $S_{f\lambda}^{\alpha} - \lim Y_k(t) = Y_0(t)$, then $S_{f\lambda}^{\alpha} - \lim(X_k(t) + Y_k(t)) = X_0(t) + Y_0(t)$.*

Theorem 2.2. *Let $\lambda \in \Lambda$, α be a real number such that $\alpha \in (0,1]$ and f be an unbounded modulus. Then every pointwise convergent sequence of fuzzy mappings is pointwise $(f, \lambda)-$statistically convergent of order α.*

Proof. Let $X = (X_k)$ be any pointwise convergent sequence and assume that $X_k(t) \longrightarrow X(t)$. Then for each $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}$$

is finite. Say

$$|\{k \in \mathbb{N} : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}| = M.$$

Since

$$\begin{aligned} & \{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\} \\ \subset \quad & \{k \in \mathbb{N} : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\} \end{aligned}$$

and f is increasing we have

$$\begin{aligned} & f(|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}|) \\ \leq \quad & f(|\{k \in \mathbb{N} : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}|) \\ = \quad & f(M) \end{aligned}$$

and so

$$\frac{f(|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \ \}|)}{f(\lambda_n^\alpha)} \leq \frac{f(M)}{f(\lambda_n^\alpha)}.$$

Since $f(M)$ is a constant, taking limit as $n \to \infty$, on the both sides, we get

$$\lim_{n \to \infty} \frac{f(|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \ \}|)}{f(\lambda_n^\alpha)} = 0.$$

This means that $X = (X_k)$ is pointwise $(f, \lambda)-$statistically convergent of order α. $\square$

Theorem 2.3. *Let $\lambda \in \Lambda$, α be a real number such that $\alpha \in (0,1]$. If*

$$\liminf_{n \to \infty} \frac{\lambda_n^\alpha}{n^\alpha} > 0$$

then $S^\alpha(F) \subseteq S_\lambda^\alpha(F)$.

Proof. Let $X = (X_k)$ be any pointwise statistically convergent sequence of fuzzy mappings of order α, then for each $\varepsilon > 0$, we have

$$\begin{aligned} & \{k \leq n : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\} \\ \supset \quad & \{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}, \end{aligned}$$

and therefore we may write

$$\begin{aligned} & \frac{1}{n^{\alpha}} f\left|\left\{k \leq n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon \text{ for every } t \in T\right\}\right| \\ \geq\ & \frac{1}{n^{\alpha}} f\left|\left\{k \in I_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon \text{ for every } t \in T\right\}\right| \\ =\ & \frac{\lambda_n^{\alpha}}{n^{\alpha}} \frac{1}{\lambda_n^{\alpha}} f\left|\left\{k \in I_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon \text{ for every } t \in T\right\}\right| \end{aligned}$$

Taking limit as $n \to \infty$, on the both sides, we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} f\left|\left\{k \in I_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon \text{ for every } t \in T\right\}\right| = 0.$$

This means that $X = (X_k)$ is pointwise (f, λ) −statistically convergent of order α. □

Lemma 2.1. (Pehlivan and Fisher [36]) *Let f be a modulus and let $0 < \delta < 1$. Then, for each $x \geq \delta$, we have $f(x) \leq 2f(1)\delta^{-1}x$.*

Theorem 2.4. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$, f be an unbounded modulus and α, β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.*

(i) If

$$\liminf_{n\to\infty} \frac{f\left(\lambda_n^{\alpha}\right)}{f\left(\mu_n^{\beta}\right)} > 0 \tag{1}$$

then $S_{f\mu}^{\beta}(F) \subseteq S_{f\lambda}^{\alpha}(F)$,

(ii) If

$$\lim_{n\to\infty} \frac{\mu_n}{\lambda_n^{\beta}} = 1 \tag{2}$$

then $S_{f\lambda}^{\alpha}(F) \subseteq S_{f\mu}^{\beta}(F)$.

Proof. (i) Omitted.

(ii) Let $S_{f\lambda}^{\alpha}-\lim X_k(t) = X(t)$ on T and (2) be satisfied and $J_n = [n - \mu_n + 1, n]$. Since $I_n \subset J_n$, for $\varepsilon > 0$ we may write

$$\begin{aligned} & \frac{1}{\mu_n^{\beta}} \left|\left\{k \in J_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon\ \right\}\right| \\ =\ & \frac{1}{\mu_n^{\beta}} \left|\left\{n - \mu_n + 1 \leq k \leq n - \lambda_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon\ \right\}\right| \\ & \quad + \frac{1}{\mu_n^{\beta}} \left|\left\{k \in I_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon\ \right\}\right| \\ \leq\ & \left(\frac{\mu_n}{\lambda_n^{\beta}} - 1\right) + \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : d\left(X_k\left(t\right), X\left(t\right)\right) \geq \varepsilon\ \right\}\right| \end{aligned}$$

for every $t\in T$ and so, from definition of modulus function and Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{f\left(\mu_n^{\beta}\right)} f\left(\left|\left\{k\in J_n : d\left(X_k\left(t\right),X\left(t\right)\right)\geq\varepsilon\ \right\}\right|\right)\\
\leq\ & f\left(\frac{\mu_n}{\lambda_n^{\beta}}-1\right)+\frac{1}{f\left(\lambda_n^{\alpha}\right)} f\left(\left|\left\{k\in I_n : d\left(X_k\left(t\right),X\left(t\right)\right)\geq\varepsilon\ \right\}\right|\right)\\
\leq\ & 2f\left(1\right)\delta^{-1}\left(\frac{\mu_n}{\lambda_n^{\beta}}-1\right)+\frac{1}{f\left(\lambda_n^{\alpha}\right)} f\left(\left|\left\{k\in I_n : d\left(X_k\left(t\right),X\left(t\right)\right)\geq\varepsilon\ \right\}\right|\right).
\end{aligned}$$

for every $t\in T$. Since $\lim\limits_{n}\frac{\mu_n}{\lambda_n^{\beta}}=1$ by (2) the first term goes to 0 since $S_{f\lambda}^{\alpha}-\lim X_k\left(t\right)=X\left(t\right)$ on T, the second term of right hand side of above inequality tends to 0 as $n\to\infty$ (note that $\frac{\mu_n}{\lambda_n^{\beta}}-1\geq 0$). This implies that $S_{\lambda}^{\alpha}\left(F\right)\subseteq S_{\lambda}^{\beta}\left(F\right)$. □

From Theorem 2.4 (i) we have the following, by taking $\beta=\alpha$, $\beta=1$ and $\beta=\alpha=1$, respectively.

Corollary 2.1. *Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ and f be an unbounded modulus. If (1) holds then,*

i) $S_{f\mu}^{\alpha}\left(F\right)\subseteq S_{f\lambda}^{\alpha}\left(F\right)$ *for each* $\alpha\in(0,1]$,

ii) $S_{f\mu}\left(F\right)\subseteq S_{f\lambda}^{\alpha}\left(F\right)$ *for each* $\alpha\in(0,1]$,

iii) $S_{f\mu}\left(F\right)\subseteq S_{f\lambda}\left(F\right)$.

From Theorem 2.4 (ii) we have the following, with taking $\beta=\alpha$, $\beta=1$ and $\beta=\alpha=1$, respectively.

Corollary 2.2. *Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ and f be an unbounded modulus. If (2) holds then,*

i) $S_{f\lambda}^{\alpha}\left(F\right)\subseteq S_{f\mu}^{\alpha}\left(F\right)$ *for each* $\alpha\in(0,1]$,

ii) $S_{f\lambda}^{\alpha}\left(F\right)\subseteq S_{f\mu}\left(F\right)$ *for each* $\alpha\in(0,1]$,

iii) $S_{f\lambda}\left(F\right)\subseteq S_{f\mu}\left(F\right)$.

3. Strong Pointwise $(V,f,\lambda)-$ Summability of Order α

In this section, we introduce the concept of strong pointwise $(V,f,\lambda)-$summability of order α of sequences of fuzzy mappings.

Definition 3.1. *Let the sequence $\lambda=(\lambda_n)$ be as above, $\alpha\in(0,1]$ and f be an unbounded modulus. A sequence (X_k) of fuzzy mappings is said to be strongly pointwise $(V,f,\lambda)-$summable (or $w_{f\lambda}^{\alpha}\left(F\right)-$summable) of order α to X on a set T if, for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n,t\in T} f\left(d\left(X_k\left(t\right),X\left(t\right)\right)\right)=0.$$

In this case we write $w^{\alpha}_{f\lambda}-\lim X_k(t) = X(t)$ on T. The set of all $w^{\alpha}_{f\lambda}(F)$ −summable sequences of fuzzy mappings will be denoted by $w^{\alpha}_{f\lambda}(F)$. In case of $\alpha = 1$, we shall write $w_{f\lambda}(F)$ instead of $w^{\alpha}_{f\lambda}(F)$, for $\lambda_n = n$ we shall write $w^{\alpha}_{f}(F)$ instead of $w^{\alpha}_{f\lambda}(F)$, and for $f(x) = x$ we shall write $w^{\alpha}_{\lambda}(F)$ instead of $w^{\alpha}_{f\lambda}(F)$. In the special cases $\lambda_n = n, f(x) = x$ and $\alpha = 1$ we'll write $w(F)$ instead of $w^{\alpha}_{f\lambda}(F)$. If $X = \bar{0}$, we shall write $w^{\alpha}_{0f\lambda}(F)$ instead of $w^{\alpha}_{f\lambda}(F)$

Theorem 3.1. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n, f$ be an unbounded modulus and α, β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.*

(i) If (1) holds then $w^{\beta}_{f\mu}(F) \subseteq w^{\alpha}_{f\lambda}(F)$,

(ii) If (2) holds then $B(A_F) \cap w^{\alpha}_{f\lambda}(F) \subset w^{\beta}_{f\mu}(F)$.

Proof. (i) Omitted.

(ii) Let $(X_k) \in B(A_F) \cap w^{\alpha}_{f\lambda}(F)$ and suppose that (2) holds. Since $(X_k) \in B(A_F)$ then there exists some positive integer K such that $d(X_k(t), X(t)) \leq K$ for all $k \in \mathbb{N}$ and $t \in T$. Since $\lambda_n \leq \mu_n$ and $I_n \subset J_n$, and $f(d(X_k(t), X(t))) \leq f(K) \leq K f(1) = M$ we may write

$$\begin{aligned}
&\frac{1}{\mu_n^{\beta}} \sum_{k\in J_n, t\in T} f(d(X_k(t), X(t))) \\
=\ & \frac{1}{\mu_n^{\beta}} \sum_{k\in J_n - I_n, t\in T} f(d(X_k(t), X(t))) \\
+\ & \frac{1}{\mu_n^{\beta}} \sum_{k\in I_n, t\in T} f(d(X_k(t), X(t))) \\
\leq\ & \left(\frac{\mu_n - \lambda_n}{\mu_n^{\beta}}\right) M + \frac{1}{\mu_n^{\beta}} \sum_{k\in I_n, t\in T} f(d(X_k(t), X(t))) \\
\leq\ & \left(\frac{\mu_n - \lambda_n^{\beta}}{\lambda_n^{\beta}}\right) M + \frac{1}{\mu_n^{\beta}} \sum_{k\in I_n, t\in T} f(d(X_k(t), X(t))) \\
\leq\ & \left(\frac{\mu_n}{\lambda_n^{\beta}} - 1\right) M + \frac{1}{\lambda_n^{\alpha}} \sum_{k\in I_n, t\in T} f(d(X_k(t), X(t))).
\end{aligned}$$

Therefore $B(A_F) \cap w^{\alpha}_{f\lambda}(F) \subset w^{\beta}_{f\mu}(F)$. □

Corollary 3.1. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n, f$ be an unbounded modulus. If (1) holds then,*

$i)$ $w^{\alpha}_{f\mu}(F) \subseteq w^{\alpha}_{f\lambda}(F)$ for each $\alpha \in (0, 1]$,

$ii)$ $w_{f\mu}(F) \subseteq w^{\alpha}_{f\lambda}(F)$ for each $\alpha \in (0, 1]$,

$iii)$ $w_{f\mu}(F) \subseteq w_{f\lambda}(F)$.

Corollary 3.2. *Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n, f$ be an unbounded modulus and α,β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$. If (2) holds then,*

i) $B(A_F)\cap w_{f\lambda}^{\alpha}(F)\subseteq w_{f\mu}^{\beta}(F)$ for each $\alpha\in(0,1]$,

ii) $B(A_F)\cap w_{f\lambda}^{\alpha}(F)\subseteq w_{f\mu}(F)$ for each $\alpha\in(0,1]$,

iii) $B(A_F)\cap w_{f\lambda}(F)\subseteq w_{f\mu}(F)$.

4. Relations Between Pointwise $(f,\lambda)-$ Statistical Convergence of order α and Strong Pointwise $(V,f,\lambda)-$ Summability of Order α

In this section, we give some relations between strong pointwise $(V,f,\lambda)-$summability of order α and pointwise $(f,\lambda)-$statistical convergence of order β for sequences of fuzzy mappings.

Theorem 4.1. *Let $\lambda=(\lambda_n)\in\Lambda$ and α,β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$ and asuume that f is an unbounded modulus function such that $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0$. Then if a sequence of fuzzy mappings is $(V,f,\lambda)-$strongly pointwise summable of order α then it is $(f,\lambda)-$pointwise statistically convergent of order β.*

Proof. $\inf\limits_{t\in(0,\infty)}\frac{f(t)}{t}=\lim\limits_{t\longrightarrow\infty}\frac{f(t)}{t}$ by Maddox [32]. Suppose that $v=\inf\limits_{t\in(0,\infty)}\frac{f(t)}{t}>0$. Then we have $\frac{f(t)}{t}>v$ and so that $vt\leq f(t)$ for every $t\in(0,\infty)$. Now if $(X_k(t))$ is $(f,\lambda)-$strongly summable of order α to $X(t)$ then, since f is modulus and

$$|\{k\in I_n: d(X_k(t),X(t))\geq\varepsilon\}|$$

is a positive integer

$$f(|\{k\in I_n: d(X_k(t),X(t))\geq\varepsilon\}|)\leq|\{k\in I_n: d(X_k(t),X(t))\geq\varepsilon\}|\,f(1)$$

is satisfied and so that we may write

$$\begin{aligned}
&\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n}f(d(X_k(t),X(t)))\\
\geq\;& v\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n}d(X_k(t),X(t))\\
\geq\;& v\frac{1}{\lambda_n^{\beta}}\sum_{k\in I_n,d(X_k(t),X(t))\geq\varepsilon}d(X_k(t),X(t))\\
\geq\;& v\frac{1}{\lambda_n^{\beta}}(|\{k\in I_n: d(X_k(t),X(t))\geq\varepsilon\}|)\,\varepsilon\\
\geq\;& \frac{f(|\{k\in I_n: d(X_k(t),X(t))\geq\varepsilon\}|)}{f\left(\lambda_n^{\beta}\right)}\frac{f\left(\lambda_n^{\beta}\right)}{\lambda_n^{\beta}}\frac{v}{f(1)}\varepsilon.
\end{aligned}$$

for every $t \in T$ and any $\varepsilon > 0$. Taking the limit on both sides as $n \longrightarrow \infty$, we obtain that $X(t) \in w^{\alpha}_{f\lambda}(X)$ implies $X(t) \in S^{\beta}_{f\lambda}(X)$, that is $w^{\alpha}_{f\lambda}(X) \subseteq S^{\beta}_{f\lambda}(X)$ since $\lim_{t \longrightarrow \infty} \frac{f(t)}{t} > 0$. Here is the proof. □

Taking special cases of (λ_n), $0 < \alpha \leq \beta \leq 1$ and unbounded modulus f in Theorem 4.1 we get the next conclusion.

Corollary 4.1. *Let $\lambda = (\lambda_n) \in \Lambda$ and α, β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and asuume that f is an unbounded modulus function such that $\lim_{t \to \infty} \frac{f(t)}{t} > 0$. Then we have*

$$
\begin{aligned}
i)&\quad w^{\alpha}_{f}(F) \subseteq S^{\beta}_{f}(F) \ \textit{for}\ \lambda_n = n,\\
ii)&\quad w^{\alpha}_{\lambda}(F) \subseteq S^{\beta}_{\lambda}(F) \ \textit{for}\ f(t) = t,\\
iii)&\quad w^{\alpha}_{f\lambda}(F) \subseteq S^{\alpha}_{f\lambda}(F) \ \textit{for}\ \beta = \alpha\\
iv)&\quad w_{f\lambda}(F) \subseteq S_{f\lambda}(F) \ \textit{for}\ \beta = \alpha = 1,\\
v)&\quad w_{f}(F) \subseteq S_{f}(F) \ \textit{for}\ \beta = \alpha = 1, \lambda_n = n,\\
vi)&\quad w(F) \subseteq S(F) \ \textit{for}\ \beta = \alpha = 1, \lambda_n = n, f(t) = t.
\end{aligned}
$$

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