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SOME CURVATURE PROPERTIES ON PARACONTACT METRIC (k, μ)-MANIFOLDS WITH RESPECT TO THE SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. The object of the present paper is to characterize paracontact metric (k, μ)-manifolds satisfying certain semisymmetry curvature conditions with respect to the Schouten-van Kampen connection.
Key words: Paracontact metric $(k; \mu)$-manifolds; Schouten-van Kampen connection; h-projective semisymmetric; ϕ-projective semisymmetric.

1. Introduction

Paracontact metric structures have been introduced in [5], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds have been studied by many authors in the recent years, particularly since the appearance of [19]. An important class among paracontact metric manifolds is that of the (k, μ)-manifolds, which satisfies the nullity condition [2]

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1.1}$$

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for all X, Y vector fields on M, where k and μ are constants and $h = \frac{1}{2}\mathcal{L}_\xi \phi$. This class includes the para-Sasakian manifolds [5, 19], the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all X, Y [20].

Among the geometric properties of manifolds symmetry is an important one. From the local point view it was introduced by Shirokov as a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where ∇ is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was introduced by Cartan in 1927. A manifold is called semisymmetric if the curvature tensor R satisfies $R(X,Y) \cdot R = 0$, where $R(X,Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Semisymmetric manifolds were locally classified by Szabó [16]. Also in [17] and [18], Yildiz and De studied h-Weyl semisymmetric, ϕ-Weyl semisymmetric, h-projectively semisymmetric and ϕ-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifolds and paracontact metric (k,μ)-manifolds respectively. Recently Mandal and De have studied certain curvature conditions on paracontact (k,μ)-spaces [6].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional semi-Riemannian manifold with metric g. The Ricci operator Q of (M,g) is defined by $g(QX,Y) = S(X,Y)$, where S denotes the Ricci tensor of type $(0,2)$ on M. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.2}$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor.

In fact M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

A paracontact metric (k,μ)-manifold is said to be an Einstein manifold if the Ricci tensor satisfies $S = \lambda_1 g$, and an η-Einstein manifold if the Ricci tensor satisfies $S = \lambda_1 g + \lambda_2 \eta \otimes \eta$, where λ_1 and λ_2 are constants.

On the other hand, the Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [1, 4, 10]. Solov'ev has investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [12, 13, 14, 15]. Then Olszak has studied the Schouten-van Kampen connection to adapted to an almost (para) contact metric structure [8]. He has characterized some classes of almost (para) contact metric manifolds with the Schouten-van Kampen connection and he has finded certain curvature properties of this connection on these manifolds

In the present paper we have studied certain curvature properties of a paracontact metric (k,μ)-space. The outline of the article goes as follows: After introduction, in Section 2, we recall basic facts which we will need throughout the paper. Section 3 deals with some basic results of paracontact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution with respect to the Schouten-van Kampen connection. In section 4, we characterize paracontact metric (k,μ)-manifolds satisfying some semisymmetry curvature conditions. We prove that a h-projectively semisymmetric and ϕ-projectively semisymmetric paracontact metric (k,μ)-manifold with respect to the Schouten-van Kampen connection is an η-Einstein manifold with respect to the Levi-Civita connection, respectively. In the all cases we assume that $k \neq -1$.

2. Preliminaries

An $(2n+1)$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions:

- (i) $\eta(\xi)=1,\ \phi^2 = I - \eta\otimes\xi,$
- (ii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D}=\ker(\eta)$, i.e. the ±1-eigendistributions, $\mathcal{D}^{\pm}=\mathcal{D}_\phi(\pm1)$ of ϕ have equal dimension n.

From the definition it follows that $\phi\xi=0$, $\eta\circ\phi=0$ and the endomorphism ϕ has rank $2n$. The Nijenhius torsion tensor field $[\phi,\phi]$ is given by

$$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]. \tag{2.1}$$

When the tensor field $N_\phi=[\phi,\phi]-2d\eta\otimes\xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{2.2}$$

for all $X,Y\in\Gamma(TM)$, then we say that (M,ϕ,ξ,η,g) is an *almost paracontact metric manifold.* Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1,n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,\xi\}$, such that $g(X_i,X_j)=\delta_{ij}$, $g(Y_i,Y_j)=-\delta_{ij}$, $g(X_i,Y_j)=0$, $g(\xi,X_i)=g(\xi,Y_j)=0$, and $Y_i=\phi X_i$, for any $i,j\in\{1,\ldots,n\}$. Such basis is called a ϕ-basis.

We can now define the fundamental form of the almost paracontact metric manifold by $\theta(X,Y)=g(X,\phi Y)$. If $d\eta(X,Y)=g(X,\phi Y)$, then (M,ϕ,ξ,η,g) is said to be paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator $h=\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [19] that h anti-commutes with ϕ and satisfies $h\xi = 0$, $trh = trh\phi = 0$ and

$$\nabla_X \xi = -\phi X + \phi hX, \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) - g(hX, \phi Y), \tag{2.4}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g). Let R be Riemannian curvature operator

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.5}$$

Moreover $h = 0$ if and only if ξ is Killing vector field. In this case (M, ϕ, ξ, η, g) is said to be a K-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Also, in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{2.6}$$

3. Paracontact metric (k, μ)-manifolds with respect to the Schouten-van Kampen connection

Let (M, ϕ, ξ, η, g) be a paracontact manifold. The (k, μ)-nullity distribution of a (M, ϕ, ξ, η, g) for the pair (k, μ) is a distribution

$$\begin{aligned} N(k,\mu) \ &: \ p \to N_p(k,\mu) \\ &= \left\{ \begin{array}{c} Z \in T_pM \mid R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \\ +\mu(g(Y,Z)hX - g(X,Z)hY) \end{array} \right\}, \end{aligned} \tag{3.1}$$

for some real constants k and μ. If the characteristic vector field ξ belongs to the (k, μ)-nullity distribution we have (3.1). [2] is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (the condition (3.1), for some real numbers k and μ).

Lemma 3.1. [2] *Let M be a paracontact metric (k, μ)-manifold of dimension $2n+1$. Then the following holds:*

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X \ &= \ -(1+k)(2g(X,\phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X) \\ &\quad +(1-\mu)(\eta(X)\phi hY - \eta(Y)\phi hX), \end{aligned} \tag{3.2}$$

$$\begin{aligned} (\nabla_X \phi h)Y - (\nabla_Y \phi h)X \ &= \ (1+k)(\eta(X)Y - \eta(Y)X) \\ &\quad +(\mu-1)(\eta(X)hY - \eta(Y)hX), \end{aligned} \tag{3.3}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi + g(hX, Y)\xi + \eta(Y)X - \eta(X)Y, \qquad k \neq -1, \tag{3.4}$$

for any vector fields X, Y on M.

Lemma 3.2. [2] *In any* $(2n+1)$*-dimensional paracontact metric* (k,μ)*-manifold* (M,ϕ,ξ,η,g) *such that* $k\neq -1$*, the Ricci operator* Q *is given by*

$$Q=(2(1-n)+n\mu)I+(2(n-1)+\mu)h+(2(n-1)+n(2k-\mu))\eta\otimes\xi. \tag{3.5}$$

On the other hand, we have two naturally defined distribution in the tangent bundle TM of M as follows:

$$H=ker\eta, \qquad V=span\{\xi\}. \tag{3.6}$$

Then we have $TM=H\oplus V$, $H\cap V=\{0\}$ and $H\perp V$. This decomposition allows one to define the Schoutenvan Kampen connection $\widetilde{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\widetilde{\nabla}$ on an almost (para) contact metric manifold with respect to Levi-Civita connection ∇ is defined by [12]

$$\widetilde{\nabla}_XY=\nabla_XY-\eta(Y)\nabla_X\xi+(\nabla_X\eta)(Y)\xi. \tag{3.7}$$

Thus with the help of the Schouten-van Kampen connection (3.7), many properties of some geometric objects connected with the distributions H, V can be characterized [12, 13, 14]. For example g, ξ and η are parallel with respect to $\widetilde{\nabla}$, that is, $\widetilde{\nabla}\xi=0, \widetilde{\nabla}g=0, \widetilde{\nabla}\eta=0$. Also, the torsion $\widetilde{T}$ of $\widetilde{\nabla}$ is defined by

$$\widetilde{T}(X,Y)=\eta(X)\nabla_Y\xi-\eta(Y)\nabla_X\xi+2d\eta(X,Y)\xi. \tag{3.8}$$

Now we consider a paracontact metric (k,μ)-manifold with respect to the Schouten-van Kampen connection. Firstly, using (2.3) and (2.4) in (3.7), we get

$$\widetilde{\nabla}_XY=\nabla_XY-\eta(Y)\phi X-\eta(Y)\phi hX+g(X,\phi Y)\xi-g(hX,\phi Y)\xi. \tag{3.9}$$

Let R and $\widetilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\widetilde{\nabla}$,

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}, \qquad \widetilde{R}(X,Y)=[\widetilde{\nabla}_X,\widetilde{\nabla}_Y]-\widetilde{\nabla}_{[X,Y]}. \tag{3.10}$$

If we substitute equation (3.7) in the definition of the Riemannian curvature tensor

$$\widetilde{R}(X,Y)Z=\widetilde{\nabla}_X\widetilde{\nabla}_YZ-\widetilde{\nabla}_Y\widetilde{\nabla}_XZ-\widetilde{\nabla}_{[X,Y]}Z. \tag{3.11}$$

Using (3.9) in (3.11), we have

$$\begin{aligned}\widetilde{R}(X,Y)Z \;=\;& \widetilde{\nabla}_X(\nabla_YZ-\eta(Z)\phi Y-\eta(Z)\phi hY\\ &+g(Y,\phi Z)\xi-g(hY,\phi Z)\xi)\\ &-\widetilde{\nabla}_Y(\nabla_XZ-\eta(Z)\phi X-\eta(Z)\phi hX\\ &+g(X,\phi Z)\xi-g(hX,\phi Z)\xi)\\ &-(\nabla_{[X,Y]}Z+\eta(Z)\phi[X,Y]-\eta(Z)\phi h[X,Y]\\ &+g([X,Y],\phi Z)\xi-g(h[X,Y],\phi Z)\xi).\end{aligned} \tag{3.12}$$

Using (3.2), (3.3) and (3.4) in (3.12), we obtain the following formula connecting R and $\widetilde{R}$ on M

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + g(hY,\phi Z)\phi X \\ &-g(hX,\phi Z)\phi Y + g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY \\ &+g(hX,\phi Z)\phi hY - g(hY,\phi Z)\phi hX \\ &+(k+1)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \\ &+k(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\ &+(\mu-1)(g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi) \\ &+\mu(\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX).\end{aligned} \tag{3.13}$$

Now taking the inner product in (3.13) with a vector field W, we have

$$\begin{aligned}g(\widetilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\ &+g(hY,\phi Z)g(\phi X,W) - g(hX,\phi Z)g(\phi Y,W) \\ &+g(Y,\phi Z)g(\phi hX,W) - g(X,\phi Z)g(\phi hY,W) \\ &+g(hX,\phi Z)g(\phi hY,W) - g(hY,\phi Z)g(\phi hX,W) \\ &+(k+1)(g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ &+k(g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)) \\ &+(\mu-1)(g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)) \\ &+\mu(g(hY,W)\eta(X)\eta(Z) - g(hX,W)\eta(Y)\eta(Z)).\end{aligned} \tag{3.14}$$

If we take $X = W = e_i$, $\{i = 1, ..., 2n+1\}$, in (3.14), where$\{ei\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$\begin{aligned}\widetilde{S}(Y,Z) &= S(Y,Z) - (k+2)g(Y,Z) \\ &+(k+2-2nk)\eta(Y)\eta(Z) - (\mu-1)g(hY,Z),\end{aligned} \tag{3.15}$$

where $\widetilde{S}$ and S denote the Ricci tensor of the connections $\widetilde{\nabla}$ and ∇, respectively. As a consequence of (3.15), we get for the Ricci operator $\widetilde{Q}$

$$\widetilde{Q}Y = QY - (k+2)Y + (k+2-2nk)\eta(Y)\xi - (\mu-1)hY, \tag{3.16}$$

Also if we take $Y = Z = e_i$, $\{i = 1, ..., 2n+1\}$, in (3.16), we get

$$\widetilde{r} = r - 4n(k+1), \tag{3.17}$$

where $\widetilde{r}$ and r denote the scalar curvatures of the connections $\widetilde{\nabla}$ and ∇, respectively.

4. Some semisymmetry curvature conditions on paracontact metric (k, μ)-manifolds

In this section we study some semisymmetry curvature conditions on paracontact metric (k, μ)-manifolds with respect to the Schouten-van Kampen connection. Firstly we give the following:

Definition 4.1. A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be h-projectively *semisymmetric if*

$$P(X,Y)\cdot h = 0, \tag{4.1}$$

holds on M.

Let M be a h-projectively semisymmetric paracontact metric (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then above equation is equivalent to

$$\widetilde{P}(X,Y)hZ - h\widetilde{P}(X,Y)Z = 0. \tag{4.2}$$

for any $X, Y, Z \in \chi(M)$. Thus we write

$$\begin{aligned} &\widetilde{R}(X,Y)hZ - h\widetilde{R}(X,Y)Z \\ &-\frac{1}{2n}\{\widetilde{S}(Y,hZ)X - \widetilde{S}(X,hZ)Y - \widetilde{S}(Y,Z)hX + \widetilde{S}(X,Z)hY\} = 0. \end{aligned} \tag{4.3}$$

Using (3.13) in (4.3), we have

$$\begin{aligned} &R(X,Y)hZ - hR(X,Y)Z + g(X,\phi hZ)\phi Y - g(Y,\phi hZ)\phi X \\ &-g(hY,h\phi Z)\phi X + g(hX,h\phi Z)\phi Y + g(Y,\phi hZ)\phi hX \\ &-g(X,\phi hZ)\phi hY - g(hX,h\phi Z)\phi hY + g(hY,h\phi Z)\phi hX \\ &+(k+1)\{g(X,hZ)\eta(Y)\xi - g(Y,hZ)\eta(X)\xi\} \\ &+(\mu-1)\{g(hX,hZ)\eta(Y)\xi - g(hY,hZ)\eta(X)\xi\} \\ &-g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X - g(hY,\phi Z)h\phi X \\ &+g(hX,\phi Z)h\phi Y - g(Y,\phi Z)h\phi hX + g(X,\phi Z)h\phi hY \\ &-g(hX,\phi Z)h\phi hY + g(hY,\phi Z)h\phi hX \\ &-k\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\} \\ &-\mu\{\eta(X)\eta(Z)h^2Y - \eta(Y)\eta(Z)h^2X\} \\ &-\frac{1}{2n}\{S(Y,hZ)X - S(X,hZ)Y - S(Y,Z)hX + S(X,Z)hY \\ &-(k+2)[g(Y,hZ)X - g(X,hZ)Y + g(Y,Z)hX - g(X,Z)hY] \\ &+(\mu-1)[g(hX,hZ)Y - g(hY,hZ)X + g(hY,Z)hX - g(hX,Z)hY] \\ &+(k+2-2nk)[\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX]\} = 0. \end{aligned} \tag{4.4}$$

Yıldız and De [18] proved that

$$\begin{aligned} R(X,Y)hZ - hR(X,Y)Z \;=\; &\mu(k+1)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ &+k\{g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi \\ &+\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\ &+g(\phi Y,Z)\phi hX - g(\phi X,Z)\phi hY\} \\ &+(\mu+k)\{g(\phi hX,Z)\phi Y - g(\phi hY,Z)\phi X\} \\ &+2\mu g(\phi X,Y)\phi hZ. \end{aligned} \tag{4.5}$$

Again using (4.5) in (4.4), we get

$$
\begin{aligned}
&\mu(k+1)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}\\
&+k\{g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi + \eta(X)\eta(Z)hY\\
&-\eta(Y)\eta(Z)hX - g(\phi Y,Z)h\phi X + g(\phi X,Z)h\phi Y\}\\
&-(\mu+k)\{g(h\phi X,Z)\phi Y - g(h\phi Y,Z)\phi X\}\\
&-2\mu g(\phi X,Y)h\phi Z - g(X,h\phi Z)\phi Y + g(Y,h\phi Z)\phi X\\
&-(k+1)[g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y - g(X,\phi Z)h\phi Y\\
&+g(Y,\phi Z)h\phi X - g(X,hZ)\eta(Y)\xi + g(Y,hZ)\eta(X)\xi]\\
&+g(Y,h\phi Z)h\phi X - g(X,h\phi Z)h\phi Y\\
&+(\mu-1)(k+1)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
&-g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X - g(hY,\phi Z)h\phi X + g(hX,\phi Z)h\phi Y\\
&+(k+1)[g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y + g(hX,\phi Z)\phi Y - g(hY,\phi Z)\phi X]\\
&-k\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\} - \mu(k+1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}\\
&-\frac{1}{2n}\{S(Y,hZ)X - S(X,hZ)Y - S(Y,Z)hX + S(X,Z)hY\\
&-(k+2)[g(Y,hZ)X - g(X,hZ)Y + g(X,Z)hY - g(Y,Z)hX]\\
&+(\mu-1)(k+1)[g(X,Z)Y - \eta(X)\eta(Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X]\\
&-(k+2-2nk)[\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY]\\
&+(\mu-1)[g(hY,Z)hX + g(hX,Z)hY] = 0,
\end{aligned}
\tag{4.6}
$$

which gives to

$$
\begin{aligned}
&\mu\{g(h\phi Y,Z)g(\phi X,W) - g(h\phi X,Z)g(\phi Y,W) + 2(X,\phi Y)g(h\phi Z,W)\}\\
&+(k+1)\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)\}\\
&+g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)\\
&-\frac{1}{2n}\{S(Y,hZ)g(X,W) - S(X,hZ)g(Y,W)\\
&+S(X,Z)g(hY,W) - S(Y,Z)g(hX,W)\\
&-(k+2)[g(Y,hZ)g(X,W) - g(X,hZ)g(Y,W)\\
&+g(X,Z)g(hY,W) - g(Y,Z)g(hX,W)]\\
&-(\mu-1)(k+1)[g(Y,Z)g(X,W) - g(X,W)\eta(Y)\eta(Z)\\
&+g(Y,W)\eta(X)\eta(Z) - g(X,Z)g(Y,W)]\\
&-(k+2-2nk)[g(hX,W)\eta(Y)\eta(Z) - g(hY,W)\eta(X)\eta(Z)]\\
&+(\mu-1)[g(hY,Z)g(hX,W) + g(hX,Z)g(hY,W)] = 0.
\end{aligned}
\tag{4.7}
$$

Putting $X = W = e_i$ in (4.7), we get

$$
\begin{aligned}
&\mu(k+1)g(hZ,Y) + \mu(k+1)\{g(Y,Z) - \eta(Y)\eta(Z)\} - g(hY,Z)\\
&-\frac{1}{2n}\{(2n+1)[S(hY,Z) - (k+2)g(Y,hZ)
\end{aligned}
$$

$$
\begin{aligned}
(4.8) \qquad & -(\mu-1)(k+1)(g(Y,Z)-\eta(Y)\eta(Z))] \\
& +(k+2)g(Y,hZ)+2(\mu-1)(k+1)[g(Y,Z)-\eta(Y)\eta(Z)] \\
& -(k+2)g(hY,Z)\} = 0.
\end{aligned}
$$

Again putting $Y = hY$ in (4.8) and using $h^2 = (k+1)\phi^2$, we obtain

$$
\begin{aligned}
& (k+1)\{[2n\mu(k+1)-2n+(2n+1)(k+2)]g(Y,Z) \\
& -[2n\mu(k+1)-2n+(2n+1)(k+2)-(2n+1)2nk]\eta(Y)\eta(Z) \\
(4.9) \qquad & +[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]g(hY,Z) \\
& -(2n+1)S(Y,Z)\} = 0.
\end{aligned}
$$

As well known that

$$
\begin{aligned}
g(hY,Z) &= \frac{1}{(2(n-1)+\mu)}S(Y,Z)-\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(Y,Z) \\
(4.10) \qquad & \quad -\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z).
\end{aligned}
$$

Hence using (4.10) in (4.9), we get

$$
\begin{aligned}
& (k+1)\{[2n\mu(k+1)-2n+(2n+1)(k+2)]g(Y,Z) \\
& -[2n\mu(k+1)-2n+(2n+1)(k+2)-(2n+1)2nk]\eta(Y)\eta(Z) \\
(4.11) \qquad & +[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\{\frac{1}{(2(n-1)+\mu)}S(Y,Z) \\
& -\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(Y,Z)-\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z)\} \\
& -(2n+1)S(Y,Z)\} = 0.
\end{aligned}
$$

Hence one can write

$$
(4.12) \qquad S(Y,Z) = \frac{A_1}{A}g(Y,Z)+\frac{A_2}{A}\eta(Y)\eta(Z),
$$

where

$$
\begin{aligned}
A_1 &= 2n\mu(k+1)-2n+(2n+1)(k+2) \\
& \quad -[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}, \\
A_2 &= -2n\mu(k+1)+2n+(2n+1)(k+2)+(2n+1)2nk \\
& \quad -[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}, \\
A &= 2n+1-[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\frac{1}{(2(n-1)+\mu)}.
\end{aligned}
$$

Therefore from (4.12) it follows that the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 4.1. *Let M be a $(2n+1)$-dimensional h-projectively semisymmetric paracontact (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection provided $\mu \neq 2(1-n)$.*

Definition 4.2. A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be ϕ-projectively *semisymmetric if*

$$P(X,Y)\cdot\phi = 0 = 0, \tag{4.13}$$

holds on M for all $X, Y \in \chi(M)$.

Let M be a ϕ-projectively semisymmetric paracontact metric (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then above equation is equivalent to

$$\widetilde{P}(X,Y)\phi Z - \phi\widetilde{P}(X,Y)Z = 0, \tag{4.14}$$

for any $X, Y, Z, W \in \chi(M)$. Thus we have

$$\begin{aligned} &\widetilde{R}(X,Y)\phi Z - \phi\widetilde{R}(X,Y)Z \\ &-\frac{1}{2n}\{\widetilde{S}(Y,\phi Z)X - \widetilde{S}(X,\phi Z)Y - \widetilde{S}(Y,Z)\phi X + \widetilde{S}(X,Z)\phi Y\} = 0, \end{aligned} \tag{4.15}$$

Using (3.13) in (4.15), we get

$$\begin{aligned} &R(X,Y)\phi Z - \phi R(X,Y)Z + g(X,Z)\phi Y - \eta(X)\eta(Z)\phi Y \\ &-g(Y,Z)\phi X + \eta(Y)\eta(Z)\phi X + g(hY,Z)\phi X - g(hX,Z)\phi Y \\ &+g(Y,Z)\phi hX - \eta(Y)\eta(Z)\phi hX - g(X,Z)\phi hY + \eta(X)\eta(Z)\phi hY \\ &+g(hX,Z)\phi hY - g(hY,Z)\phi hX \\ &+(k+1)\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\} \\ &+(\mu-1)\{g(hX,\phi Z)\eta(Y)\xi - g(hY,\phi Z)\eta(X)\xi\} \\ &-g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X - g(Y,\phi Z)\eta(X)\xi \\ &-g(hY,\phi Z)X + g(hY,\phi Z)\eta(X)\xi + g(hX,\phi Z)Y - g(hX,\phi Z)\eta(Y)\xi \\ &-g(Y,\phi Z)hX + g(X,\phi Z)hY - g(hX,\phi Z)hY + g(hY,\phi Z)hX \\ &-k\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\} - \mu\{\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\} \\ &-\frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y - S(Y,Z)\phi X + S(X,Z)\phi Y \\ &-(k+2)[g(Y,\phi Z)X - g(X,\phi Z)Y + g(X,Z)\phi Y - g(Y,Z)\phi X] \\ &-(\mu-1)[g(hY,\phi Z)X - g(hX,\phi Z)Y] \\ &-(k+2-2nk)[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y] \\ &+(\mu-1)[g(hY,Z)\phi X - g(hX,Z)\phi Y]\} = 0. \end{aligned} \tag{4.16}$$

In [18], Yıldız and De proved that

$$
\begin{aligned}
R(X,Y)\phi Z - \phi R(X,Y)Z \;=\;& g(X,\phi Z)Y - g(Y,\phi Z)X + g(Y,Z)\phi X \\
& - g(X,Z)\phi Y - g(X,\phi Z)hY + g(Y,\phi Z)hX \\
& + g(hY,\phi Z)X - g(hX,\phi Z)Y - g(Y,Z)\phi hX \\
& + g(X,Z)\phi hY - g(hY,Z)\phi X + g(hX,Z)\phi Y \\
& + \frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)hX - g(hX,\phi Z)hY - g(hY,Z)\phi hX \\
& + g(hX,Z)\phi hY\} - \frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,Z)\phi hY - g(hY,Z)\phi hX \\
& + g(hY,\phi Z)hX - g(hX,\phi Z)hY\} \\
& + (k+1)\{g(\phi X,Z)\eta(Y)\xi - g(\phi Y,Z)\eta(X)\xi \\
& + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\} \\
& + (\mu-1)\{g(\phi hX,Z)\eta(Y)\xi - g(\phi hY,Z)\eta(X)\xi \\
& + \eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\}.
\end{aligned} \tag{4.17}
$$

Using (4.17) in (4.16), we obtain

$$
\begin{aligned}
& g(hX,Z)g(\phi hY,W) - g(hY,Z)g(\phi hX,W) + \eta(X)\eta(Z)g(\phi hY,W) \\
& -\eta(Y)\eta(Z)g(\phi hX,W) + g(X,\phi Z)\eta(Y)\eta(W) - g(Y,\phi Z)\eta(X)\eta(W) \\
& + g(hY,\phi Z)g(hX,W) - g(hX,\phi Z)g(hY,W) \\
& + \frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)g(hX,W) - g(hX,\phi Z)g(hY,W) \\
& - g(hY,Z)g(\phi hX,W) + g(hX,Z)g(\phi hY,W)\} \\
& - \frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,Z)g(\phi hY,W) - g(hY,Z)g(\phi hX,W) \\
& + g(hY,\phi Z)g(hX,W) - g(hX,\phi Z)g(hY,W)\} \\
& + (\mu-1)\{g(hX,\phi Z)\eta(Y)\xi - g(hY,\phi Z)\eta(X)\xi\} \\
& - \frac{1}{2n}\{S(Y,\phi Z)g(X,W) - S(X,\phi Z)g(Y,W) + S(X,Z)g(\phi Y,W) \\
& - S(Y,Z)g(\phi X,W) - (k+2)[g(Y,\phi Z)g(X,W) - g(X,\phi Z)g(Y,W) \\
& + g(X,Z)g(\phi Y,W) - g(Y,Z)g(\phi X,W)] + (\mu-1)[g(hY,Z)g(\phi X,W) \\
& - g(hX,Z)g(\phi Y,W) + g(hY,\phi Z)g(X,W) - g(hX,\phi Z)g(Y,W)] \\
& - (k+2-2nk)[\eta(Y)\eta(Z)g(\phi X,W) - \eta(X)\eta(Z)g(\phi Y,W)]\} = 0,
\end{aligned} \tag{4.18}
$$

If we put $Y = \phi Y$ in (4.18), we have

$$
\begin{aligned}
& g(h\phi Y,Z)g(hX,\phi W) - g(X,hZ)g(h\phi^2 Y,W) - g(h\phi^2 Y,W)\eta(X)\eta(Z) \\
& - g(\phi Y,\phi Z)\eta(X)\eta(W) + g(h\phi Y,\phi Z)g(hX,W) - g(X,h\phi Z)g(h\phi Y,W) \\
& + \frac{-1-\frac{\mu}{2}}{k+1}\{g(h\phi Y,\phi Z)g(hX,W) - g(X,h\phi Z)g(h\phi Y,W)
\end{aligned}
$$

$$
\begin{aligned}
&+g(h\phi Y,Z)g(hX,\phi W)-g(X,hZ)g(h\phi^2 Y,W)\} \\
&-\frac{-k+\frac{\mu}{2}}{k+1}\{-g(X,hZ)g(h\phi^2 Y,W)+g(h\phi Y,Z)g(hX,\phi W) \\
&-g(\phi h\phi Y,Z)g(hX,W)-g(X,h\phi Z)g(h\phi Y,W)\} \\
&-(\mu-1)g(h\phi Y,\phi Z)\eta(X)\eta(W) \\
&-\frac{1}{2n}\{S(\phi Y,\phi Z)g(X,W)-S(X,\phi Z)g(\phi Y,W)+S(X,Z)g(\phi^2 Y,W) \\
&-S(\phi Y,Z)g(\phi X,W)-(k+2)[g(\phi Y,\phi Z)g(X,W)-g(X,\phi Z)g(\phi Y,W) \\
&+g(X,Z)g(\phi^2 Y,W)-g(\phi Y,Z)g(\phi X,W)]+(\mu-1)[g(h\phi Y,Z)g(\phi X,W) \\
&-g(hX,Z)g(\phi^2 Y,W)+g(h\phi Y,\phi Z)g(X,W)-g(hX,\phi Z)g(\phi Y,W)] \\
&+(k+2-2nk)\eta(X)\eta(Z)g(\phi^2 Y,W)\}=0.
\end{aligned}
\tag{4.19}
$$

Putting $X=W=e_i$, $\{i=1,...,2n+1\}$, in (4.19), we obtain

$$
\begin{aligned}
S(Y,Z) \;=\; & \frac{2n}{2n-1}[\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\}g(Y,Z) \\
& +\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\}\eta(Y)\eta(Z) \\
& -(\mu-1)\{1+\frac{2n-1}{2n}\}g(hY,Z)].
\end{aligned}
\tag{4.20}
$$

Using (4.10) in (4.20), we obtain

$$
\begin{aligned}
S(Y,Z) \;=\; & \frac{2n}{2n-1}[\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\}g(Y,Z) \\
& -\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\}\eta(Y)\eta(Z) \\
& -\{(\mu-1)(1+\frac{2n-1}{2n})\}\{\frac{1}{(2(n-1)+\mu)}S(Y,Z) \\
& -\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(Y,Z)-\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z)\}],
\end{aligned}
$$

which gives

$$
\begin{aligned}
& \{1+[(\mu-1)(1+\frac{2n-1}{2n})][\frac{1}{(2(n-1)+\mu)}]\}S(Y,Z) \\
= \; & \{\frac{2n}{2n-1}\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\} \\
& +\{(\mu-1)(1+\frac{2n-1}{2n})\}\{\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}\}g(Y,Z) \\
& -\{\frac{2n}{2n-1}\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\} \\
& +\{(\mu-1)(1+\frac{2n-1}{2n})\}\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z).
\end{aligned}
\tag{4.21}
$$

Hence one can write

$$S(Y,Z) = \frac{B_1}{B} g(Y,Z) + \frac{B_2}{B}\eta(Y)\eta(Z), \tag{4.22}$$

where

$$\begin{aligned} B_1 &= \frac{2n}{2n-1}\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\} \\ &\quad +\{(\mu-1)(1+\frac{2n-1}{2n})\}\{\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}\}, \\ B_2 &= -\{\frac{2n}{2n-1}\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\} \\ &\quad +\{(\mu-1)(1+\frac{2n-1}{2n})\}\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}, \\ B &= 1+(\mu-1)(1+\frac{2n-1}{2n})\frac{1}{(2(n-1)+\mu)}. \end{aligned}$$

Therefore from (4.22) it follows that the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 4.2. *Let M be a $(2n+1)$-dimensional ϕ-projectively semisymmetric paracontact (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection provided $\mu \neq 2(1-n)$.*

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