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SOLUTIONS FOR THE MIXED SYLVESTER OPERATOR EQUATIONS

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Abstract. This paper is devoted to investigating some system of mixed coupled generalized Sylvester operator equations. The block operator matrix decomposition is used to find the necessary and sufficient conditions for the solvability to these systems. The solutions of the system are expressed in terms of the Moore–Penrose inverses of the coefficient operators.
Keywords: Sylvester operator equations, Matrix equations, C^*-modules

1. Introduction and Preleminaries

The generalized Sylvester matrix equations have been attracting much attention from both practical and theoretical importance. The Sylvester matrix equation $AX - XB = C$ or generalized Sylvester matrix equation $AX - YB = C$ has massive applications in control theory [16, 15], singular system control [11], and widely used in many other fields such as signal and color image processing, orbital mechanics, robust control, neural network, computer graphics.

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Wimmer in [15] gave a necessary and sufficient condition for the existence of a simultaneous solution of

$$\begin{cases} A_1X + YB_1 = C_1, \\ A_2X + YB_2 = C_2. \end{cases} \tag{1.1}$$

Kägström in [7] obtained a solution of (1.1) by using generalized Schur methods. Recently, some mixed Sylvester matrix equations have been investigated in some papers (see [12]). Lee and Vu [8] gave some solvability conditions to mixed Sylvester matrix equations

$$\begin{cases} A_1X + YB_1 = C_1, \\ A_2Z + YB_2 = C_2. \end{cases} \tag{1.2}$$

The general solution of systems of coupled generalized Sylvester matrix equations to (1.2) was established by He and Wang in [1, 2, 3, 4, 5, 13, 14].

In this paper, by using the block operator matrix decomposition, we present a new approach to find the necessary and sufficient conditions for the solvability of mixed generalized coupled Sylvester operator equations. We obtain an arbitrary solutions of these systems that it is expressed in terms of the Moore–Penrose inverses of the coefficient operators.

Throughout this paper, we use $\mathcal{H}$ and $\mathcal{H}_i$ for denote Hilbert spaces. Also, $\mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$ instate the set of all bounded Linear operators from $\mathcal{H}_i$ to $\mathcal{H}_j$. For any $A \in \mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$, the null and the range space of A are denoted by $\ker(A)$ and ran(A), respectively. In the case $\mathcal{H}_i = \mathcal{H}_j$, $\mathcal{L}(\mathcal{H}_i, \mathcal{H}_i)$ which is abbreviated to $\mathcal{L}(\mathcal{H}_i)$. The identity operator on $\mathcal{H}$ is denoted by $1_{\mathcal{H}}$ or 1 if there is no ambiguity.

Definition 1.1. Let $\mathcal{H}$ be Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. The Moore-Penrose inverse $A^\dagger$ of A is an element $X \in \mathcal{L}(\mathcal{H})$ which satisfies

$$(1) AXA = A, \quad (2) XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

From the definition of Moore-Penrose inverse, it can be proved that the Moore-Penrose inverse of an operator (if it exists) is unique and $A^\dagger A$ and $AA^\dagger$ are orthogonal projections, in the sense that they are self adjoint and idempotent operators. More precisely $A \in \mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$ have a closed range. Then $AA^\dagger$ is the orthogonal projection from $\mathcal{H}_j$ onto ran(A) and $A^\dagger A$ is the orthogonal projection from $\mathcal{H}_i$ onto $\mathrm{ran}(\mathrm{A}^*)$.

Clearly, A is Moore-Penrose invertible if and only if A^* is Moore-Penrose invertible, and in this case $(A^*)^\dagger = (A^\dagger)^*$. By Definition 1.1, it is concluded $\mathrm{ran(A)} = \mathrm{ran}(\mathrm{AA}^\dagger)$, $\mathrm{ran}(\mathrm{A}^\dagger) = \mathrm{ran}(\mathrm{A}^\dagger\mathrm{A}) = \mathrm{ran}(\mathrm{A}^*)$, $\ker(A) = \ker(A^\dagger A)$ and $\ker(A^\dagger) = \ker(AA^\dagger) = \ker(A^*)$. For more related results, we refer the interested readers to [6] and [9] and references therein.

2. Solutions for the mixed Sylvester operator equations

In this section, by using some block matrix technique we find the conditions for solvability of the linear system equations (1.2) where A_i, B_i ($i \in \{1, 2\}$) are given matrices, X, Y and Z be arbiterary. First, we establish necessary and sufficient conditions for the solvability of (1.2) and the expression of the general solutions to the system when it is solvable.

When A_i, B_i ($i \in \{1,2\}$) are invertible operators. It can straightforward be seen that the proof of the following Theorem is valid in rings with involution.

So let A_i, B_i ($i \in \{1,2\}$) be Moore-Penrose invertible operators.

Theorem 2.1. *Suppose that $\{\mathcal{H}_i\}_{i=1}^4$ are Hilbert spaces and $B_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $A_i \in B(\mathcal{H}_4, \mathcal{H}_3)$; $i \in \{1,2\}$ are invertible operators and $C_1, C_2 \in B(\mathcal{H}_1, \mathcal{H}_3)$. Then the following statements are equivalent:*

(a) There exists solutions $X, Z \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ of the system (1.2),

(b) $C_1 = C_2 B_2^{-1} B_1$.

In which case, the general solutions X, Y, Z to the system (1.2) are of the form

$$X = \frac{1}{2}(A_1^{-1}C_1 + Z_1 B_1), \tag{2.1}$$

$$Y = \frac{1}{2}(C_2 B_2^{-1} + A_2 Z_2^*), \tag{2.2}$$

$$Z = \frac{1}{2}(A_2^{-1}C_2 - Z_2^* B_2), \tag{2.3}$$

where $Z_1 \in B(\mathcal{H}_2, \mathcal{H}_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$ satisfy $Z_2 = -Z_1^* A_1^* (A_2^*)^{-1}$.

Proof. $(a) \Rightarrow (b)$ It is clear.

$(b) \Rightarrow (a)$: By matrix representations, the system (1.2) become into the following form

$$\begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} \begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} = \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}.$$

Let $X, Z \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ be the general solutions to the system (1.2). Then

$$\begin{aligned} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \\ &+ \left(\frac{1}{2} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \right. \\ &- \left. \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} \right) \\ &\times \begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{1}{2}\left(\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right)\right. \\
&- \left.\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}\right)\begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \\
&+ \frac{1}{2}\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} - \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}\right) \\
&\times \begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix}\begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 & A_1^{-1}C_1 + Z_1B_1 \\ (B_2^*)^{-1}C_2^* + Z_2A_2^* & 0 \end{bmatrix}.
\end{aligned}
$$

Where, Z_1, Z_2 take in the following matrix

$$
\begin{aligned}
\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} - \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & XB_1^{-1} - A_1^{-1}Y \\ Y^*(A_2^*)^{-1} - (B_2^*)^{-1}X^* & 0 \end{bmatrix}.
\end{aligned}
$$

Then,

$$X = \frac{1}{2}(A_1^{-1}C_1 + Z_1B_1), \tag{2.4}$$

$$Y = \frac{1}{2}(C_2B_2^{-1} + A_2Z_2^*). \tag{2.5}$$

Also,

$$
\begin{aligned}
\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} &= \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
&- \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
&= \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
&- \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix}.
\end{aligned}
$$

We have,

$$\begin{aligned}
& \begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
+ \ & \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right) \\
- \ & \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right) \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix} \\
& (\frac{1}{2}\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} + \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}\right) \\
- \ & \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}) \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix} \\
& \left(\frac{1}{2}\left(\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right)\right) \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix} \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1B_1^{-1} - A_1Z_1 \\ C_2^*(A_2^*)^{-1} - B_2^*Z_2 & 0 \end{bmatrix}.
\end{aligned}$$

Therefore,

$$Z = \frac{1}{2}(A_2^{-1}C_2 - Z_2^*B_2), \tag{2.6}$$

$$Y = \frac{1}{2}(C_1B_1^{-1} - A_1Z_1). \tag{2.7}$$

Since $C_1 = C_2B_2^{-1}B_1$ and $Z_2 = -Z_1^*A_1^*(A_2^*)^{-1}$ imply that Eqs. (1.2) and (2.7) coincide with other. This completes the proof. □

Theorem 2.2. *Let $\{\mathcal{H}_i\}_{i=1}^4$ be Hilbert spaces and $B_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $A_i \in B(\mathcal{H}_4, \mathcal{H}_3)$; $i \in \{1, 2\}$ be invertible operators and $C_1, C_2 \in B(\mathcal{H}_1, \mathcal{H}_3)$. Then the following statements are equivalent:*

(a) *There exists solutions $X \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ of the system (1.1),*

(b) *$C_1 = C_2B_2^{-1}B_1$, and $C_2 = A_2A_1^{-1}C_1$.*

If (a) or (b) is satisfied, then any solutions of the system (1.1) has the form

$$X = \frac{1}{2}(A_1^{-1}C_1 + Z_1B_1), \tag{2.8}$$

$$Y = \frac{1}{2}(C_2B_2^{-1} + A_2Z_2^*), \tag{2.9}$$

where $Z_1 \in B(\mathcal{H}_2, H_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$ satisfy $Z_2 = -Z_1^*A_1^*(A_2^*)^{-1}$ and $Z_1 = -Z_2^*B_2B_1^{-1}$.

Proof. The proof is quite similar to the proof of the previous theorem. □

Theorem 2.3. *Let $\{\mathcal{H}_i\}_{i=1}^4$ be Hilbert spaces and $A_i \in B(\mathcal{H}_4, \mathcal{H}_3)$ and $B_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ ($i \in \{1,2\}$) have closed range operators such that $\mathrm{ran}(B_1^*) = \mathrm{ran}(B_2^*)$, $\mathrm{ran}(B_1) = \mathrm{ran}(B_2)$ and $\mathrm{ran}(A_1) = \mathrm{ran}(A_2)$. If $C_1, C_2 \in B(\mathcal{H}_1, \mathcal{H}_3)$ such that $(1 - B_1^\dagger B_1)C_1B_1^\dagger = (1 - B_1^\dagger B_1)C_2B_2^\dagger$, then the following statements are equivalent:*

(a) There exists solutions $X, Z \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ of the system (1.2),

(b) $(1 - A_iA_i^\dagger)C_i(1 - B_i^\dagger B_i) = 0$ $(i \in \{1,2\})$ and $B_1^\dagger B_1C_1A_1A_1^\dagger = B_1^\dagger B_1C_2B_2^\dagger B_1$

If (a) or (b) is satisfied, then the general solutions to the system (1.2) has the form

$$\begin{aligned} X &= -\frac{1}{2}A_1^\dagger C_1B_1^\dagger B_1 + \frac{1}{2}A_1^\dagger A_1Z_1B_1 + A_1^\dagger C_1 + (1 - A_1^\dagger A_1)Z_3, \\ Y &= -\frac{1}{2}A_1A_1^\dagger C_2B_2^\dagger + \frac{1}{2}A_2Z_2^*B_1B_1^\dagger + C_2B_2^\dagger + Z_4(1 - B_1B_1^\dagger), \\ Z &= -\frac{1}{2}A_2^\dagger C_2B_2^\dagger B_2 - \frac{1}{2}A_2^\dagger A_2Z_2^*B_2 + A_2^\dagger C_2 + (1 - A_2^\dagger A_2)Z_5, \end{aligned}$$

where $Z_1 \in B(\mathcal{H}_2, \mathcal{H}_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$ satisfy

$$B_1B_1^\dagger Z_2A_1^\dagger A_1 = -B_1B_1^\dagger Z_1^*A_1^*(A_2^*)^\dagger,$$

and $Z_3, Z_5 \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Z_4 \in B(\mathcal{H}_2, \mathcal{H}_3)$ are arbitrary.

Proof. $(a) \Rightarrow (b)$ It is clear.

$(b) \Rightarrow (a)$ In view of [10, Corollary 1.2.] we can consider the matrix forms of the operators as follows

$$\begin{aligned} A_1 &= \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A_1^*) \\ \ker(A_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_1) \\ \ker(A_1^*) \end{bmatrix}, \\ A_2 &= \begin{bmatrix} A_{21} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A_2^*) \\ \ker(A_2) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_1) \\ \ker(A_1^*) \end{bmatrix}, \\ X &= \begin{bmatrix} X_{11} & X_{12} \\ X_{13} & X_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(B_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_1^*) \\ \ker(A_1) \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
Z &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{13} & Z_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_2^*)} \\ \ker(A_2) \end{bmatrix}, \\
B_1 &= \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(B_1)} \\ \ker(B_1^*) \end{bmatrix}, \\
B_2 &= \begin{bmatrix} B_{21} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(B_1)} \\ \ker(B_1^*) \end{bmatrix}, \\
Y &= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{13} & Y_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1)} \\ \ker(B_1^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1)} \\ \ker(A_1^*) \end{bmatrix},
\end{aligned}
$$

where A_{11}, A_{21}, B_{11} and B_{21} are invertible. In addition, conditions $(1 - A_iA_i^\dagger)C_i(1 - B_i^\dagger B_i) = 0$, $(i \in \{1, 2\})$ in (b) implies that $C_{14} = C_{24} = 0$. Therefore,

$$
\begin{aligned}
C_1 &= \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1)} \\ \ker(A_1^*) \end{bmatrix}, \\
C_2 &= \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1)} \\ \ker(A_1^*) \end{bmatrix}.
\end{aligned}
$$

Hence, the mixed Sylvester operator equations (1.2) obtain as follow.

$$
\begin{cases}
\begin{bmatrix} A_{11}X_{11} & A_{11}X_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{11}B_{11} & 0 \\ Y_{13}B_{11} & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & 0 \end{bmatrix}, \\[2ex]
\begin{bmatrix} A_{21}Z_{11} & A_{21}Z_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{11}B_{21} & 0 \\ Y_{13}B_{21} & 0 \end{bmatrix} = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & 0 \end{bmatrix}.
\end{cases}
$$

Then, the following relations hold.

$$
\begin{cases} A_{11}X_{11} + Y_{11}B_{11} = C_{11}, \\ A_{21}Z_{11} + Y_{11}B_{21} = C_{21}. \end{cases} \tag{2.10}
$$

$$A_{11}X_{12} = C_{12}, \tag{2.11}$$

$$A_{21}Z_{12} = C_{22}, \tag{2.12}$$

$$Y_{13}B_{11} = C_{13}, \tag{2.13}$$

$$Y_{13}B_{21} = C_{23}. \tag{2.14}$$

[10, Corollary 1.2.] implies that A_{i1}, B_{i1} for $i \in \{1, 2\}$ are invertible and also condition $B_1^\dagger B_1 C_1 A_1 A_1^\dagger = B_1^\dagger B_1 C_2 B_2^\dagger B_1$ and their matrix representations on the following forms

$$B_1^\dagger B_1 C_1 A_1 A_1^\dagger = B_1^\dagger B_1 C_2 B_2^\dagger B_1.$$

Namely,

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & 0 \end{bmatrix} \\
&\times \begin{bmatrix} B_{21}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

which is implies that $C_{11} = C_{21}B_{21}^{-1}B_{11}$.

Now, by applying Theorem 2.1, general solutions of the system (2.10) can be stated as

$$\begin{aligned} X_{11} &= \frac{1}{2}(A_{11}^{-1}C_{11} + (Z_1)_{11}B_{11}), \\ Y_{11} &= \frac{1}{2}(C_{21}B_{21}^{-1} + A_{21}(Z_2^*)_{11}), \\ Z_{11} &= \frac{1}{2}(A_{21}^{-1}C_{21} - (Z_2^*)_{11}B_{21}), \end{aligned}$$

where, $(Z_1)_{11}$ and $(Z_2)_{11}$ satisfy $(Z_2)_{11} = -(Z_1^*)_{11}A_{11}^*(A_{21}^*)^{-1}$.

Condition $B_1B_1^\dagger Z_2A_1^\dagger A_1 = -B_1B_1^\dagger Z_1^*A_1^*(A_2^*)^\dagger$ is equal to

$$(Z_2)_{11} = -(Z_1^*)_{11}A_{11}^*(A_{21}^*)^{-1},$$

where $Z_1 \in B(\mathcal{H}_2, \mathcal{H}_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$.

Since with rewrite their matrix representations on the following forms

$$B_1B_1^\dagger Z_2A_1^\dagger A_1 = -B_1B_1^\dagger Z_1^*A_1^*(A_2^*)^\dagger.$$

In fact,

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (Z_2)_{11} & (Z_2)_{12} \\ (Z_2)_{21} & (Z_2)_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (Z_1^*)_{11} & (Z_1^*)_{21} \\ (Z_1^*)_{12} & (Z_1^*)_{22} \end{bmatrix} \\ &\times \begin{bmatrix} A_{11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_{21}^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

Thus,

$$\begin{bmatrix} (Z_2)_{11} & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} (Z_1^*)_{11}A_{11}^*(A_{21}^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Eqs. (2.11) and (2.12) imply that $X_{12} = A_{11}^{-1}C_{12}$ and $Z_{12} = A_{21}^{-1}C_{22}$.

Also, the condition $(1 - B_1^\dagger B_1)C_1B_1^\dagger = (1 - B_1^\dagger B_1)C_2B_2^\dagger$ ensures that $C_{13}B_{11}^{-1} = C_{23}B_{21}^{-1}$. Therefore, Eqs. (2.13) and (2.14) are solvable and $Y_{13} = C_{13}B_{11}^{-1} = C_{23}B_{21}^{-1}$.

Hence,

$$\begin{aligned} X &= \begin{bmatrix} \frac{1}{2}(A_{11}^{-1}C_{11} + (Z_1)_{11}B_{11}) & A_{11}^{-1}C_{12} \\ X_{13} & X_{14} \end{bmatrix}, \\ Y &= \begin{bmatrix} \frac{1}{2}(C_{21}B_{21}^{-1} + A_{21}(Z_2^*)_{11}) & Y_{12} \\ C_{23}B_{21}^{-1} & Y_{14} \end{bmatrix}, \end{aligned}$$

and

$$Z = \begin{bmatrix} \frac{1}{2}(A_{21}^{-1}C_{21} - (Z_2^*)_{11}B_{21}) & A_{21}^{-1}C_{22} \\ Z_{13} & Z_{14} \end{bmatrix},$$

X_{13}, X_{14}, Y_{12}, Y_{14}, Z_{13} and Z_{14} can be taken arbitrary.

By using the matrix forms, we get

$$\begin{aligned}\frac{1}{2}(A_1^\dagger C_1 B_1^\dagger B_1 + A_1^\dagger A_1 Z_1 B_1) &= \begin{bmatrix} \frac{1}{2}(A_{11}^{-1}C_{11} + (Z_1)_{11}B_{11}) & 0 \\ 0 & 0 \end{bmatrix}, \\ A_1^\dagger C_1(1 - B_1^\dagger B_1) &= \begin{bmatrix} 0 & A_{11}^{-1}C_{12} \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

By taking $Z_3 = \begin{bmatrix} Z_{31} & Z_{32} \\ X_{13} & X_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1^*)} \\ \ker(A_1) \end{bmatrix}$ we conclude $(1 - A_1^\dagger A_1)Z_3 = \begin{bmatrix} 0 & 0 \\ X_{13} & X_{14} \end{bmatrix}$. Then

$$X = \frac{1}{2}(A_1^\dagger C_1 B_1^\dagger B_1 + A_1^\dagger A_1 Z_1 B_1) + A_1^\dagger C_1(1 - B_1^\dagger B_1) + (1 - A_1^\dagger A_1)Z_3.$$

Also,

$$\frac{1}{2}(A_1 A_1^\dagger C_2 B_2^\dagger + A_2 Z_2^* B_1 B_1^\dagger) = \begin{bmatrix} \frac{1}{2}(C_{21}B_{21}^{-1} + A_{21}(Z_2^*)_{11}) & 0 \\ 0 & 0 \end{bmatrix},$$

$$(1 - A_1 A_1^\dagger)C_2 B_2^\dagger = \begin{bmatrix} 0 & 0 \\ C_{23}B_{21}^{-1} & 0 \end{bmatrix}.$$

By taking $Z_4 = \begin{bmatrix} Z_{41} & Y_{12} \\ Z_{43} & Y_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1)} \\ \ker(B_1^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1)} \\ \ker(A_1^*) \end{bmatrix}$, we derive $Z_4(1 - B_1 B_1^\dagger) = \begin{bmatrix} 0 & Y_{12} \\ 0 & Y_{14} \end{bmatrix}$. Then

$$Y = \frac{1}{2}(A_1 A_1^\dagger C_2 B_2^\dagger + A_2 Z_2^* B_1 B_1^\dagger) + (1 - A_1 A_1^\dagger)C_2 B_2^\dagger + Z_4(1 - B_1 B_1^\dagger).$$

By using the matrix forms, we get

$$\begin{aligned}\frac{1}{2}(A_2^\dagger C_2 B_2^\dagger B_2 - A_2^\dagger A_2 Z_2^* B_2) &= \begin{bmatrix} \frac{1}{2}(A_{21}^{-1}C_{21} - (Z_2^*)_{11}B_{21}) & 0 \\ 0 & 0 \end{bmatrix}, \\ A_2^\dagger C_2(1 - B_2^\dagger B_2) &= \begin{bmatrix} 0 & A_{21}^{-1}C_{22} \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

By taking $Z_5 = \begin{bmatrix} Z_{51} & Z_{52} \\ Z_{13} & Z_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1^*)} \\ \ker(A_1) \end{bmatrix}$, we conclude $(1 - A_2^\dagger A_2)Z_5 = \begin{bmatrix} 0 & 0 \\ Z_{13} & Z_{14} \end{bmatrix}$. Then

$$Z = \frac{1}{2}(A_2^\dagger C_2 B_2^\dagger B_2 - A_2^\dagger A_2 Z_2^* B_2) + A_2^\dagger C_2(1 - B_2^\dagger B_2) + (1 - A_2^\dagger A_2)Z_5.$$

□

In the following theorem, consider the solvability and the expressions of the general solutions to the following systems of four coupled one sided Sylvester-type operator equations.

Theorem 2.4. *Suppose that $\mathcal{H}$ is Hilbert space and where $A_i, B_i, C_i \in B(\mathcal{H})$ ($i \in \{1,2,3,4\}$) are given operators such that $C_3 = A_2C_2B_3^{-1}$ and $X_1, ..., X_5 \in B(\mathcal{H})$ are unknowns operator A_i, B_i ($i \in \{1,2,3,4\}$) are invertible operators. Then the following statements are equivalent:*

(a) The system

$$\left\{\begin{array}{l} A_1X_1 + X_2B_1 = C_1, \\ A_2X_3 + X_2B_2 = C_2, \\ A_3X_4 + X_3B_3 = C_3, \\ A_4X_4 + X_5B_4 = C_4, \end{array}\right. \tag{2.15}$$

is solvable,

(b) $C_1 = C_3B_2^{-1}B_1$ and $C_4^* = C_2^*(A_3^*)^{-1}A_4^*$.

In which case, the general solution to the system (2.15) are of the form

$$\begin{array}{rcl} X_1 & = & \frac{1}{2}(A_1^{-1}C_1 + Z_1B_1), \\ X_2 & = & \frac{1}{2}(C_3B_2^{-1} + A_2Z_4^*), \\ X_3 & = & \frac{1}{2}(A_2^{-1}C_3 - Z_4^*B_2), \\ X_4 & = & \frac{1}{2}(A_3^{-1}C_2 + Z_3B_3^*), \\ X_5 & = & \frac{1}{2}(C_4B_4^{-1} + A_4Z_2^*), \end{array}$$

where $Z_1, Z_2, Z_3, Z_4 \in B(\mathcal{H})$ satisfy $Z_3 = -Z_2^*B_4B_3^{-1}$, $Z_4 = -Z_1^*A_1^*(A_2^*)^{-1}$ and $Z_3 = A_3^{-1}Z_4^*B_2$.

Proof. By taking $T_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_4^* \end{bmatrix}$, $T_2 = \begin{bmatrix} A_2 & 0 \\ 0 & B_3^* \end{bmatrix}$, $S_1 = \begin{bmatrix} A_4^* & 0 \\ 0 & B_1 \end{bmatrix}$, $S_2 = \begin{bmatrix} A_3^* & 0 \\ 0 & B_2 \end{bmatrix}$, $U_1 = \begin{bmatrix} 0 & C_1 \\ C_4^* & 0 \end{bmatrix}$ and $U_2 = \begin{bmatrix} 0 & C_2 \\ C_3^* & 0 \end{bmatrix}$ that are given operators and $X = \begin{bmatrix} 0 & X_1 \\ X_5^* & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & X_2 \\ X_4^* & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 0 & X_3 \\ X_3^* & 0 \end{bmatrix}$ are unknowns operators. Hence system (2.15) get into

$$\left\{\begin{array}{l} T_1X + YS_1 = U_1, \\ T_2Z + YS_2 = U_2, \end{array}\right. \tag{2.16}$$

Condition (b) is equal to
$\begin{bmatrix} 0 & C_1 \\ C_4^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & C_3 \\ C_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_3^*)^{-1} & 0 \\ 0 & (B_2)^{-1} \end{bmatrix} \begin{bmatrix} A_4^* & 0 \\ 0 & B_1 \end{bmatrix}$. By applying Theorem 2.1, implies that system 2.15 are solvable, then any solutions have the following form

$$\begin{aligned}
\begin{bmatrix} 0 & X_1 \\ X_5^* & 0 \end{bmatrix} &= \frac{1}{2}(\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_4^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_4^* & 0 \end{bmatrix} + W_1 \begin{bmatrix} A_4^* & 0 \\ 0 & B_1 \end{bmatrix}), \\
\begin{bmatrix} 0 & X_2 \\ X_4^* & 0 \end{bmatrix} &= \frac{1}{2}(\begin{bmatrix} 0 & C_3 \\ C_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_3^*)^{-1} & 0 \\ 0 & (B_2)^{-1} \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ 0 & B_3^* \end{bmatrix} W_2^*), \\
\begin{bmatrix} 0 & X_3 \\ X_3^* & 0 \end{bmatrix} &= \frac{1}{2}(\begin{bmatrix} A_2^{-1} & 0 \\ 0 & (B_3^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_3 \\ C_2^* & 0 \end{bmatrix} - W_2^* \begin{bmatrix} A_3^* & 0 \\ 0 & B_2 \end{bmatrix}),
\end{aligned}$$

where $W_1 = \begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix}$ and $W_2 = \begin{bmatrix} 0 & Z_3 \\ Z_4 & 0 \end{bmatrix}$.
Which is satisfy that $W_2 = -W_1^* T_1^* (T_2^*)^{-1}$ that is,
$\begin{bmatrix} 0 & Z_3 \\ Z_4 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & Z_2^* \\ Z_1^* & 0 \end{bmatrix} \begin{bmatrix} A_1^* & 0 \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & (B_3)^{-1} \end{bmatrix}$ that
$Z_3 = -Z_2^* B_4 B_3^{-1}$ and $Z_4 = -Z_1^* A_1^* (A_2^*)^{-1}$. Since, $C_3 = A_2 C_2 B_3^{-1}$ and Z_3, Z_4 satisfy $Z_3 = A_3^{-1} Z_4^* B_2$. Therefore,

$$\begin{aligned}
X_1 &= \frac{1}{2}(A_1^{-1} C_1 + Z_1 B_1), \\
X_2 &= \frac{1}{2}(C_3 B_2^{-1} + A_2 Z_4^*), \\
X_3 &= \frac{1}{2}(A_2^{-1} C_3 - Z_4^* B_2), \\
X_3^* &= \frac{1}{2}((B_3^*)^{-1} C_2^* - Z_3^* A_3^*), \\
X_4^* &= \frac{1}{2}(C_2^* (A_3^*)^{-1} + B_3 Z_3^*), \\
X_5^* &= \frac{1}{2}((B_4^*)^{-1} C_4^* + Z_2 A_4^*).
\end{aligned}$$

□

3. Conclusion

We have used the block operator matrix decomposition to find the general solutions of mixed Sylvester operator equations with three unknowns (1.2) and five unknowns (2.15) . We have provided some necessary and sufficient conditions for the existence of a solution to this system based on matrix representation. We have also derived the general solution to this system when it is solvable.

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