FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 1 (2022), 41–50
https://doi.org/10.22190/FUMI201203005B
Original Scientific Paper

QUASI NORMS INTEGRAL INEQUALITIES RELATED TO LAPLACE TRANSFORMATION

Bouharket Benaissa[1] and Kadda Maazouz[2]

[1] **Faculty of Material Sciences, Department of Physics**
University of Tiaret-Algeria, Laboratory of Informatics and Mathematics
[2] **Faculty of Mathematics and Informatics**
Department of Mathematics, University of Tiaret-Algeria

Abstract. In [4], the authors have proved the theorems 2.4 and 2.5 related to some integral inequalities via the Laplace transformation with the parameter $p > 1$. In this manuscript, we propose new extension for integral inequalities related to Laplace transformation with two parameters p, q and using the weight functions w, ϕ. We deduce some new inequalities linked to the Laplace transformation.
Key words: Laplace transformation, integral inequalities, weight functions.

1. Introduction

In this article, we seek to extend certain integral inequalities of Moazzen and Lashkaripour, which involve an exponent $p > 1$ corresponding to a Lebesgue norm, by considering their behavior when this parameter is taken equal to $0 < p < 1$.

As it is well known in Functional Analysis, this difference fundamentally modifies the behavior of the associated Lebesgue spaces. The conjugate Hölder exponent is now negative, leading to a phenomenon of inverse Hölder inequality, the spaces are not more normalized but quasi-normalized, and the functions in these spaces no longer need to be locally integrable, so that they fail to be distributions and we cannot make them fit into the traditional framework of Sobolev's inequalities and

Received December 3, 2020, accepted: January 19, 2022
Communicated by Dijana Mosić
Corresponding Author: Bouharket Benaissa, Faculty of Material Sciences, Department of Physics, University of Tiaret-Algeria, Laboratory of Informatics and Mathematics | E-mail: bouharket.benaissa@univ-tiaret.dz
2010 *Mathematics Subject Classification.* Primary 26D10; Secondary 44A10, 26D15

the Fourier transform which is important for many applications for these purposes. Hardy spaces are generally considered to be the natural analogue of L_p with $p \geq 1$.

Historically, the case of $0 < p < 1$ has been rather widely treated as an arithmetic or functional analytical curiosity, rather than something of immediate interest to applications, but in recent years the utility of inequalities involving these small exponents has become apparent in a number of situations where the problems of interest are naturally formulated in contexts of low regularity. (See [3]).

Moazzen and Lashkaripour [4] have proved the Theorem 2.4:

Theorem 1.1. *Let f be non-negative integrable function on $(0,\infty)$ with its Laplace transformation exists and $p > 1$, $r > \frac{p-1}{p}$. Define*

$$f_r(x) = \int_0^x (x-\tau)^{r-1} f(\tau) d\tau.$$

Then

$$\int_0^\infty \left(\frac{f_r(x)}{x^r}\right)^p dx < \beta(rp-p+1, p-1) \left(\int_0^\infty L_f(s) ds\right)^{p-1} \left(\int_0^\infty f(\tau) d\tau\right) \tag{1.1}$$

where L_f is the Laplace transformation of f.

For $r = 1$, its result

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \frac{1}{p-1} \left(\int_0^\infty L_f(s) ds\right)^{p-1} \left(\int_0^\infty f(\tau) d\tau\right), \tag{1.2}$$

where F is the Hardy operator defined by $F(x) = \int_0^x f(\tau) d\tau$.
The authors also proved Theorem 2.5 :

Theorem 1.2. *Let f be non-negative integrable function on $(0,\infty)$ and $p > 1$, $r > \frac{p-1}{p}$. Define*

$$f^r(x) = \int_x^\infty (x-\tau)^{r-1} f(\tau) d\tau.$$

Then

$$\int_0^\infty (f^r(x))^p dx < \frac{1}{rp-p+1} \left(\int_0^\infty (\tau f(\tau))^{rp-p+1} d\tau\right) \left(\int_0^\infty (f(\tau))^{p'-rp'+1} d\tau\right)^{p-1}. \tag{1.3}$$

For $r = 1$, thus

$$\int_0^\infty F^p(x) dx < \left(\int_0^\infty \tau f(\tau) d\tau\right) \left(\int_0^\infty f(t) dt\right)^{p-1}. \tag{1.4}$$

2. Notations and properties

Definition 2.1. [5] Suppose that f is a real- or complex-valued function of the (time) variable $\tau > 0$ and s is a real or complex parameter. We define the Laplace transform of f as

$$\mathcal{L}_f(s) = \int_0^\infty e^{-s\tau} f(\tau) d\tau, \tag{2.1}$$

whenever the integral exists (as a finite number).

Remark 2.1. [4] For all $s, \tau > 0$

$$\int_0^\infty \frac{f(\tau)}{\tau} d\tau = \int_0^\infty \mathcal{L}_f(\mu) d\mu, \tag{2.2}$$

Proof. For all $s, \tau > 0$

$$\begin{aligned}\int_0^\infty \mathcal{L}_f(s) ds &= \int_0^\infty \left(\int_0^\infty e^{-s\tau} f(\tau) d\tau \right) ds \\ &= \int_0^\infty \left(\int_0^\infty e^{-s\tau} ds \right) f(\tau) d\tau \\ &= \int_0^\infty \frac{f(\tau)}{\tau} d\tau.\end{aligned}$$

□

We introduce the following characteristic function χ properties [4],

$$\text{For all } 0 < p < 1: \quad \left(\chi_{(0,\,x]}(\tau)\right)^p = \chi_{(0,\,x]}(\tau),$$

and

$$\text{For all } x,\, \tau \in (0,\,\infty): \quad \chi_{[x,\,\infty)}(\tau) = \chi_{(0,\,\tau]}(x). \tag{2.3}$$

We give the following Theorem of the reverse Minkowski's inequality [3], [1] which will be used frequently in the proof.

Theorem 2.1. *Let $0 < p < 1, -\infty \le a < b \le +\infty$ and $-\infty \le c < d \le +\infty$. Suppose that $f : (a,b) \times (c,d) \longrightarrow \mathbb{R}$ is positive measurable function and $f(.,y) \in L_p(a,b)$ for almost all $y \in (c,d)$, then*

$$\| \int_c^d f(x,y) dy \|_{L_p(a,b)} \ge \int_c^d \|f(x,y)\|_{L_p(a,b)} dy, \tag{2.4}$$

if left-hand side is finite.

We rewrite the inequality (2.4) in the form

$$\left(\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \geq \int_c^d \left(\int_a^b |f(x,y)|^p\, dx\right)^{\frac{1}{p}} dy,$$

therefore

$$\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx \geq \left(\int_c^d \left(\int_a^b |f(x,y)|^p\, dx\right)^{\frac{1}{p}} dy\right)^p. \tag{2.5}$$

3. Main Results

We consider the weighted Hardy operator $\mathcal{F}$ of the form

$$\mathcal{F}(x) = \int_0^x w(\tau)f(\tau)d\tau, \tag{3.1}$$

and its dual

$$\widetilde{\mathcal{F}}(x) = \int_x^\infty w(\tau)f(\tau)d\tau, \tag{3.2}$$

where $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a weight function (positive and measurable).

The goal of the present work is to establish some extensions of the inequalities (1.1) and (1.3) by using w and $\Phi(x) = \int_0^x \phi(\tau)d\tau$,where ϕ is a weight function. Also, we give some new integral inequalities related to the Laplace transformation for $0 < p < 1$. All integrals throughout are assumed to exist and are finite. We begin by proving the following Theorem.

Theorem 3.1. *Let f a non-negative integrable function on $(0,\infty)$ and w, ϕ be a weight function on $(0,\infty)$. Define*

$$\Phi(x) = \int_0^x \phi(\tau)d\tau, \quad \widetilde{\mathcal{F}}(x) = \int_x^\infty w(\tau)f(\tau)d\tau.$$

If $0 < \phi \leq 1$, then for all $0 < p$, $q < 1$

$$\int_0^\infty \frac{\widetilde{\mathcal{F}}^p(x)}{\Phi^q(x)}dx \geq \frac{1}{1-q}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi^{(p'-1)(q-1)}(\tau)}d\tau\right)^{p-1}\left(\int_0^\infty w(\tau)f(\tau)d\tau\right). \tag{3.3}$$

$$\int_0^\infty \frac{\widetilde{\mathcal{F}}^p(x)}{\Phi^q(x)}dx \geq \frac{1}{1-q}\left(\int_0^\infty w(\tau)f(\tau)\Phi^{1-q}(\tau)d\tau\right)\left(\int_0^\infty w(\tau)f(\tau)d\tau\right)^{p-1}. \tag{3.4}$$

Proof. By using the inequalities (2.5) and (2.3), we get

$$\begin{aligned}
\int_0^\infty \frac{\widetilde{F}^p(x)}{\Phi^q(x)}dx &= \int_0^\infty \left(\int_x^\infty \frac{w(\tau)f(\tau)}{\Phi^{\frac{q}{p}}(x)}dt\right)^p dx \\
&\geq \left[\int_x^\infty \left(\int_0^\infty \frac{1}{\Phi^q(x)}dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&= \left[\int_0^\infty \left(\int_0^\infty \chi_{(x,\infty)}(\tau)\Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&= \left[\int_0^\infty \left(\int_0^\infty \chi_{(0,\tau)}(x)\Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&= \left[\int_0^\infty \left(\int_0^\tau \Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p.
\end{aligned}$$

Since $0 < \phi \leq 1$ and $1-q > 0$, we get

$$\begin{aligned}
\int_0^\infty \frac{\widetilde{F}^p(x)}{\Phi^q(x)}dx &\geq \left[\int_0^\infty \left(\int_0^\tau \phi(x)\Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&= \frac{1}{1-q}\left[\int_0^\infty \Phi^{\frac{1-q}{p}}(\tau)w(\tau)f(\tau)d\tau\right]^p,
\end{aligned}$$

let $\frac{1}{p}+\frac{1}{p'}=1$, rewriting the last integral on the forms

$$\int_0^\infty \left(\frac{w(\tau)f(\tau)}{\Phi^{(p'-1)(q-1)}(\tau)}\right)^{\frac{1}{p'}} (w(\tau)f(\tau))^{\frac{1}{p}}\, d\tau,$$

or

$$\int_0^\infty \left(w(\tau)f(\tau)\Phi^{1-q}(\tau)\right)^{\frac{1}{p}} (w(\tau)f(\tau))^{\frac{1}{p'}}\, d\tau,$$

by using the reverse Hölder integral inequality on the last integral to finish the proof. □

If we take $w(x) = 1$, we get the following Corollary.

Corollary 3.1. *Let f a non-negative integrable function on $(0,\infty)$ and ϕ be a weight function on $(0,\infty)$. Define*

$$\Phi(x) = \int_0^x \phi(\tau)d\tau, \quad \widetilde{F}(x) = \int_x^\infty f(\tau)d\tau.$$

If $0<\phi\leq 1$, *then for all* $0<p,\, q<1$

$$\int_0^\infty \frac{\widetilde{F}^p(x)}{\Phi^q(x)}dx \geq \frac{1}{1-q}\left(\int_0^\infty \frac{f(\tau)}{\Phi^{(p'-1)(q-1)}(\tau)}d\tau\right)^{p-1}\left(\int_0^\infty f(\tau)d\tau\right). \tag{3.5}$$

$$\int_0^\infty \frac{\widetilde{F}^p(x)}{\Phi^q(x)}dx \geq \frac{1}{1-q}\left(\int_0^\infty f(\tau)\Phi^{1-q}(\tau)d\tau\right)\left(\int_0^\infty f(\tau)d\tau\right)^{p-1}. \tag{3.6}$$

If we take $q=p$, we get the following Lemma.

Lemma 3.1. *Let f a non-negative integrable function on $(0,\infty)$ and w, ϕ be a weight functions on $(0,\infty)$. Define*

$$\Phi(x)=\int_0^x \phi(\tau)d\tau, \quad \widetilde{\mathcal{F}}(x)=\int_x^\infty w(\tau)f(\tau)d\tau.$$

If $0<\phi\leq 1$, *then for all* $0<p<1$

$$\int_0^\infty \left(\frac{\widetilde{\mathcal{F}}(x)}{\Phi(x)}\right)^p dx \geq \frac{1}{1-p}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi(\tau)}d\tau\right)^{p-1}\left(\int_0^\infty w(\tau)f(\tau)d\tau\right). \tag{3.7}$$

$$\int_0^\infty \left(\frac{\widetilde{\mathcal{F}}(x)}{\Phi(x)}\right)^p dx \geq \frac{1}{1-p}\left(\int_0^\infty w(\tau)f(\tau)\Phi^{1-p}(\tau)d\tau\right)\left(\int_0^\infty w(\tau)f(\tau)d\tau\right)^{p-1}. \tag{3.8}$$

Remark 3.1. If we take $w(\tau)=\phi(\tau)=1$ in the Lemma 3.1 and using (2.2), then for all $0<p<1$ the following inequalities hold

$$\int_0^\infty \left(\frac{\widetilde{F}(x)}{x}\right)^p dx \geq \frac{1}{1-p}\left(\int_0^\infty \mathcal{L}_f(s)ds\right)^{p-1}\left(\int_0^\infty f(\tau)d\tau\right). \tag{3.9}$$

$$\int_0^\infty \left(\frac{\widetilde{F}(x)}{x}\right)^p dx \geq \frac{1}{1-p}\left(\int_0^\infty \tau^{1-p}f(\tau)d\tau\right)\left(\int_0^\infty f(\tau)d\tau\right)^{p-1}. \tag{3.10}$$

Remark 3.2. Hence the adjoint Hardy operator [2] is defined by

$$\mathcal{H}^\star(x)=\int_x^\infty \frac{f(\tau)}{\tau}d\tau.$$

If we take $w(\tau)=\frac{1}{\tau}$ and $\phi(\tau)=1$ in the Lemma 3.1 and using (2.2), then for all $0<p<1$ the following inequalities holds

$$\int_0^\infty \left(\frac{\mathcal{H}^\star(x)}{x}\right)^p dx \geq \frac{1}{1-p}\left(\int_0^\infty \frac{f(\tau)}{\tau^2}d\tau\right)^{p-1}\left(\int_0^\infty \mathcal{L}_f(s)ds\right). \tag{3.11}$$

$$\int_0^\infty \left(\frac{\mathcal{H}^\star(x)}{x}\right)^p dx \geq \frac{1}{1-p}\left(\int_0^\infty \frac{f(\tau)}{\tau^p}d\tau\right)\left(\int_0^\infty \mathcal{L}_f(s)ds\right)^{p-1}. \tag{3.12}$$

Remark 3.3. For all $s > 0$, if we take $w(\tau) = e^{-s\tau}$ and $\phi(\tau) = 1$ in the inequality (3.7), then

$$\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi(\tau)}d\tau \leq \int_0^\infty \frac{f(\tau)}{\tau}d\tau,$$

hence

$$\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi(\tau)}d\tau\right)^{p-1} \geq \left(\int_0^\infty \mathcal{L}_f(s)ds\right)^{p-1}.$$

Thus, for all $s > 0$

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty e^{-s\tau}f(\tau)d\tau\right)^p dx \geq \frac{1}{1-p}\mathcal{L}_f(s)\left(\int_0^\infty \mathcal{L}_f(s)ds,\right)^{p-1}. \tag{3.13}$$

Theorem 3.2. *Let f a non-negative integrable function on $(0,\infty)$, $0 < p < 1$ and w, ϕ be a weight function on $(0,\infty)$. Define*

$$\Phi(x) = \int_0^x \phi(\tau)d\tau, \quad \mathcal{F}(x) = \int_0^x w(\tau)f(\tau)d\tau.$$

If $\phi \leq 1$ and $\Phi(\infty) = \infty$, then for all $q > 1$

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^q(x)}dx \geq \frac{1}{q-1}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi^{q-1}(\tau)}d\tau\right)\left(\int_0^\infty w(\tau)f(\tau)d\tau\right)^{p-1}. \tag{3.14}$$

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^q(x)}dx \geq \frac{1}{q-1}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi^{(p'-1)(q-1)}(\tau)}d\tau\right)^{p-1}\left(\int_0^\infty w(\tau)f(\tau)d\tau\right). \tag{3.15}$$

Proof. Applying (2.5) and (2.3) , we get

$$\begin{aligned}
\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^q(x)}dx &= \int_0^\infty \left(\int_0^x \frac{w(\tau)f(\tau)}{\Phi^{\frac{q}{p}}(x)}d\tau\right)^p dx \\
&\geq \left[\int_0^x \left(\int_0^\infty \frac{1}{\Phi^q(x)}dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&\geq \left[\int_0^\infty \left(\int_0^\infty \chi_{(0,x)}(\tau)\Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&= \left[\int_0^\infty \left(\int_0^\infty \chi_{(\tau,\infty)}(x)\Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p \\
&= \left[\int_0^\infty \left(\int_\tau^\infty \Phi^{-q}(x)dx\right)^{\frac{1}{p}} w(\tau)f(\tau)d\tau\right]^p.
\end{aligned}$$

Since $\phi \le 1$, $\Phi(\infty)=\infty$ and $1-q<0$ therefore

$$\int_\tau^\infty \Phi^{-q}(x)dx \ge \int_\tau^\infty \phi(x)\Phi^{-q}(x)dx = \frac{1}{q-1}\Phi^{1-q}(\tau),$$

then

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^q(x)}dx \ge \frac{1}{q-1}\left[\int_0^\infty \Phi^{-\frac{q-1}{p}}(\tau)w(\tau)f(\tau)d\tau\right]^p. \tag{3.16}$$

We rewrite the integral in the right hands side of (3.3) on the forms

$$\int_0^\infty \left(w(\tau)f(\tau)\Phi^{-q+1}(\tau)\right)^{\frac{1}{p}}\left(w(\tau)f(\tau)\right)^{\frac{1}{p'}}d\tau,$$

or

$$\int_0^\infty \left(\frac{w(\tau)f(\tau)}{\Phi^{(p'-1)(q-1)}(\tau)}\right)^{\frac{1}{p'}}\left(w(\tau)f(\tau)\right)^{\frac{1}{p}}d\tau,$$

finally by using the reverse Hölder integral inequality, we get the result. □

If we take $w(x)=1$, we get the following Corollary.

Corollary 3.2. *Let f a non-negative integrable function on $(0,\infty)$, $0<p<1$ and ϕ be a weight function on $(0,\infty)$. Define*

$$\Phi(x)=\int_0^x \phi(\tau)d\tau, \quad F(x)=\int_0^x f(\tau)d\tau.$$

If $\phi\le 1$ and $\Phi(\infty)=\infty$, then for all $q>1$

$$\int_0^\infty \frac{F^p(x)}{\Phi^q(x)}dx \ge \frac{1}{q-1}\left(\int_0^\infty \frac{f(\tau)}{\Phi^{q-1}(\tau)}d\tau\right)\left(\int_0^\infty f(\tau)d\tau\right)^{p-1}. \tag{3.17}$$

$$\int_0^\infty \frac{F^p(x)}{\Phi^q(x)}dx \ge \frac{1}{q-1}\left(\int_0^\infty \frac{f(\tau)}{\Phi^{(p'-1)(q-1)}(\tau)}d\tau\right)^{p-1}\left(\int_0^\infty f(\tau)d\tau\right). \tag{3.18}$$

If we take $q=\frac{1}{p}$, we get the following Lemma.

Lemma 3.2. *Let f a non-negative integrable function on $(0,\infty)$ and w, ϕ be a weight function on $(0,\infty)$. Define*

$$\Phi(x)=\int_0^x \phi(\tau)d\tau, \quad \mathcal{F}(x)=\int_0^x w(\tau)f(\tau)d\tau.$$

If $\phi\le 1$ and $\Phi(\infty)=\infty$, then for all $0<p<1$

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^{\frac{1}{p}}(x)}dx \ge \frac{p}{1-p}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi^{\frac{1-p}{p}}(\tau)}d\tau\right)\left(\int_0^\infty w(\tau)f(\tau)d\tau\right)^{p-1}. \tag{3.19}$$

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^{\frac{1}{p}}(x)}dx \geq \frac{p}{1-p}\left(\int_0^\infty w(\tau)f(\tau)\Phi^{\frac{1}{p}}(\tau)d\tau\right)^{p-1}\left(\int_0^\infty w(\tau)f(\tau)d\tau\right). \tag{3.20}$$

If we take $q = 1 + p$, we get the following Lemma.

Lemma 3.3. *Let f a non-negative integrable function on $(0, \infty)$ and w, ϕ be a weight functions on $(0, \infty)$. Define*

$$\Phi(x) = \int_0^x \phi(\tau)d\tau, \quad \mathcal{F}(x) = \int_0^x w(\tau)f(\tau)d\tau.$$

If $\phi \leq 1$ and $\Phi(\infty) = \infty$, then for all $0 < p < 1$

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^{p+1}(x)}dx \geq \frac{1}{p}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi^p(\tau)}d\tau\right)\left(\int_0^\infty w(\tau)f(\tau)d\tau\right)^{p-1}. \tag{3.21}$$

$$\int_0^\infty \frac{\mathcal{F}^p(x)}{\Phi^{p+1}(x)}dx \geq \frac{1}{p}\left(\int_0^\infty \frac{w(\tau)f(\tau)}{\Phi^{p'}(\tau)}d\tau\right)^{p-1}\left(\int_0^\infty w(\tau)f(\tau)d\tau\right). \tag{3.22}$$

Remark 3.4. If we take $w(\tau) = \phi(\tau) = 1$ in the Lemma 3.3, then for all $0 < p < 1$ the following inequalities holds

$$\int_0^\infty \frac{F^p(x)}{x^{p+1}}dx \geq \frac{1}{p}\left(\int_0^\infty \frac{f(\tau)}{\tau^p}d\tau\right)\left(\int_0^\infty f(\tau)d\tau\right)^{p-1}. \tag{3.23}$$

$$\int_0^\infty \frac{F^p(x)}{x^{p+1}}dx \geq \frac{1}{p}\left(\int_0^\infty \frac{f(\tau)}{\tau^{p'}}d\tau\right)^{p-1}\left(\int_0^\infty f(\tau)d\tau\right). \tag{3.24}$$

Remark 3.5. Let $\widetilde{\mathcal{H}^\star}$ the dual of the adjoint Hardy operator defined by

$$\widetilde{\mathcal{H}^\star}(x) = \int_0^x \frac{f(\tau)}{\tau}d\tau,$$

if we take $w(\tau) = \frac{1}{\tau}$ and $\phi(\tau) = 1$ in the Lemma 3.3 and using (2.2), then for all $0 < p < 1$ the following inequalities holds

$$\int_0^\infty \frac{(\widetilde{\mathcal{H}^\star}(x))^p}{x^{p+1}}dx \geq \frac{1}{p}\left(\int_0^\infty \frac{f(\tau)}{\tau^{p+1}(\tau)}d\tau\right)\left(\int_0^\infty \mathcal{L}_f(s)ds\right)^{p-1}. \tag{3.25}$$

$$\int_0^\infty \frac{(\widetilde{\mathcal{H}^\star}(x))^p}{x^{p+1}}dx \geq \frac{1}{p}\left(\int_0^\infty \frac{f(\tau)}{\tau^{1+p'}}d\tau\right)^{p-1}\left(\int_0^\infty \mathcal{L}_f(s)ds\right). \tag{3.26}$$

Acknowledgements

The authors would like to thank the reviewers for their careful proofreading of the original manuscript of this work and their fruitful comments, important remarks, which enabled us to correct and improve the final version. This work was supported by the General Directorate of Scientific Research and Technological Development (DGRSDT) of Algeria.

REFERENCES

1. B. Benaissa: *On the Reverse Minkowski's Integral Inequality*. Kragujevac. J. Math. Vol. **46 (3)** (2022), 407–416.
2. B. Benaissa: *Some new extension of Levinson's integral inequalities*. Journal of Mathematical Extension. Vol. **15 (4)** (2021), 1–17. https://doi.org/10.30495/JME.2021.1711
3. G. Hardy, J. Littlewood, and G. Pólya: *Inequalities*. Cambridge University Press, (1952).
4. A. R. Moazzen and R. Lashkaripour: *Some Inequalities Involving Laplace Transformation*. J. Math. Extension, Vol. **8 (3)** (2014), 1-15.
5. Joel L. Schiff: *The Laplace Transform: Theory and Applications*. Springer (1988).