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WEYL TYPE THEOREMS FOR ALGEBRAICALLY CLASS p-$wA(s,t)$ OPERATORS

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Abstract. In this paper, we study Weyl type theorems for $f(T)$, where T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and f is an analytic function defined on an open neighborhood of the spectrum of T. Also we show that if $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then generalized Weyl's theorem , a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB})$, where d_{AB} denote the generalized derivation $\delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\delta_{AB}(X) = AX - XB$ or the elementary operator $\Delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\Delta_{AB}(X) = AXB - X$.
Keywords: class p-$wA(s,t)$ operator, polaroid operator, Bishop's property (beta), Weyl type theorems, elementary operator.

1. Introduction and Preliminaries

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space $\mathcal{H}$. Throughout this paper $R(T)$, $\ker(T)$, $\sigma(T)$ denotes range, null space and spectrum of $T \in B(\mathcal{H})$ respectively. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel

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condition $\ker U = \ker |T|$, then this decomposition is called the polar decomposition, which is one of the most important results in operator theory. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker U = \ker |T|$. An operator $T \in B(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. The Aluthge transformation introduced by Aluthge[5] is defined by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ where $T = U|T|$ be the polar decomposition of $T \in B(H)$. The generalized Aluthge transformation $T(s,t)$ ($s,t > 0$) is given by $T(s,t) = |T|^s U|T|^t$. Recall that an operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ $(0 < p \leq 1)$, *w-hyponormal* if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$, *class A* if $|T^2| \geq |T|^2$, *class* $A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ ([13]) and *class* $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([16]). Prasad and Tanahashi [19] introduced *class* *p-*$wA(s,t)$ operators as follows:

Definition 1.1. ([19]) Let $T = U|T|$ be the polar decomposition of T and let $s,t > 0$ and $0 < p \leq 1$. T is called class *p-*$wA(s,t)$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \quad \text{and} \quad (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}.$$

In general the following inclusions hold:
p-hyponormal $\subseteq$ w-hyponormal $\subseteq$ class $wA(s,t)$ $\subseteq$ class p-$wA(s,t)$.

Many interesting results for class *p-*$wA(s,t)$ has been studied in [10, 11, 19, 20, 21, 22, 24].

Let $\alpha(T)$ and $\beta(T)$ denote the nullity and the deficiency of $T \in B(\mathcal{H})$, defined by $\alpha(T)$= dim(ker(T)) and $\beta(T)$=dim(ker(T^*)). An operator T is said to be *upper semi-Fredholm* (resp.,*lower semi- Fredholm*) if $R(T)$ of $T \in B(\mathcal{H})$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$). Let $SF_+(\mathcal{H})$ (resp., $SF_-(\mathcal{H})$) denote the semigroup of upper semi-Fredholm (resp., lower semi-Fredholm) operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *semi-Fredhom*, $T \in SF(\mathcal{H})$, if $T \in SF_+(\mathcal{H}) \cup SF_-(\mathcal{H})$ and *Fredholm*, $T \in F(\mathcal{H})$, if $T \in SF_+(\mathcal{H}) \cap SF_-(\mathcal{H})$. The index of semi-Fredholm operator T is defined by ind $(T) = \alpha(T) - \beta(T)$. Recall[14], the *ascent* of an operator $T \in B(\mathcal{H})$, $a(T)$, is the smallest non negative integer $\mathtt{p}$ such that $\ker(T^{\mathtt{p}}) = \ker(T^{(\mathtt{p}+1)})$. Such $\mathtt{p}$ does not exist, then $\mathtt{p}(T) = \infty$. The *descent* of $T \in B(\mathcal{H})$, $d(T)$, is defined as the smallest non negative integer $\mathtt{q}$ such that $R(T^{\mathtt{q}}) = R(T^{(\mathtt{q}+1)})$. An operator $T \in B(\mathcal{H})$ is *Weyl*, $T \in W(\mathcal{H})$ it is Fredholm of index zero and *Browder* if T is Fredholm of finite ascent and descent. The Weyl spectrum of T, denoted by $\sigma_W(T)$, is given by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathcal{H})\}.$$

We say that $T \in B(\mathcal{H})$ satisfies *Weyl's theorem* if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T).$$

where $E_0(T)$ denote the set of eigenvalues of T of finite geometric multiplicity isolated in $\sigma(T)$. Let $SF_+^-(\mathcal{H}) = \{T \in SF_+(\mathcal{H}) : \text{ind(T)} \leq 0\}$. essential approximate point spectrum $\sigma_{SF_+^-}(T)$ of T is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathcal{H})\}.$$

Let $\sigma_a(T)$ denote the approximate point spectrum of $T \in B(\mathcal{H})$. An operator $T \in B(\mathcal{H})$ holds *a-Weyl's theorem* if,

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T),$$

where $E_0^a(T) = \{\lambda \in \mathbb{C} : \lambda \in \text{iso } \sigma_a(T) \text{ and } 0 < \alpha(T - \lambda) < \infty\}$. We say that an operator $T \in B(\mathcal{H})$ satisfies *a-Browder's theorem* if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$, where $\Pi_0^a(T)$ denote the set the left poles of T of finite rank. An operator $T \in B(\mathcal{H})$ is called *B-Fredholm*, $T \in BF(\mathcal{H})$ if there exist a non negative integer n for which the induced operator

$$T_{[n]} : R(T_{[n]}) \to R(T_{[n]})(\text{ in particular } T_{[0]} = T).$$

is Fredholm in the usual sense (see [7]). An operator $T \in B(\mathcal{H})$ is called *B-Weyl*, $T \in BW(\mathcal{H})$, if it is B-Fredholm with $\text{ind}(T_{[n]}) = 0$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin BW(\mathcal{H})\}$ (see [7]). Let $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$. We say that T satisfies *generalized Weyl's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$. A bounded operator $T \in B(\mathcal{H})$ is said to satisfy *generalized Browders's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$, where $\Pi(T)$ is the set of poles of T (See [8]). We refer the readers to [1], where Weyl type theorems are extensively treated.

Recall that an operator $T \in B(\mathcal{H})$ is said to have the *single-valued extension property* (SVEP) if for every open subset U of $\mathbb{C}$ and any analytic function $f : U \to H$ such that $(T - z)f(z) \equiv 0$ on U, we have $f(z) \equiv 0$ on U. A Hilbert space operator $T \in B(\mathcal{H})$ satisfies *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and every sequence $f_n : U \longrightarrow \mathcal{H}$ of analytic functions with $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of U, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of U. For $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, the local resolvent set of T at x $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, which verifies $(T - z)f(z) = x$. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x. For each subset F of $\mathbb{C}$, the local spectral subspace of T, $\mathcal{H}_T(F)$, is given by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

Bishop's property $(\beta) \Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

See [1, 17] for more details.

Weyl's theorem for class p-$wA(s,t)$ has been studied in [22]. In this paper, we focus Weyl type theorems for algebraically class p-$wA(s,t)$ operators and elementary operator with class p-$wA(s,t)$ operator entries.

2. algebraically class p-$wA(s,t)$ operators and Weyl type theorem

We say that $T \in B(\mathcal{H})$ is *algebraically class p-$wA(s,t)$ operator* with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ if there exists a non- constant complex polynomial q such that $q(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$.

In general, the following inclusions hold:

p-hyponormal $\subset$ class p-$wA(s,t)$ $\subset$ algebraically class p-$wA(s,t)$.

Lemma 2.1. *[20] Let $T \in B(\mathcal{H})$ be a class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be a quasinilpotent algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T is nilpotent.*

Proof. Suppose $T \in B(\mathcal{H})$ is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then there exists a non- constant complex polynomial q such that $q(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Since $\sigma(q(T)) = q(\sigma(T))$ and $\sigma(T) = \{0\}$, the operator $q(T) - q(0)$ is quasinilpotent. By Lemma 2.1, $\sigma(q(T) - q(0)) = \{0\}$ implies that $q(T) - q(0) = 0$. Hence it follows that,

$$0 = q(T) - q(0) = cT^m(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)$$

where $m \geq 1$. Since $T - \lambda_i I$ is invertible for every $\lambda_i \neq 0$, we must have $T^m = 0$. □

It is well known that if both ascent and descent of T are finite then they are equal (see, [14, Proposition 38.3]). Moreover, $0 < a(T - \mu I) = d(T - \mu I) < \infty$ precisely when μ is a pole of the resolvent of T (see, [14, Proposition 50.2]).

An operator $T \in B(H)$ is *polaroid* if the isolated points of the spectrum of T are poles of the resolvent T. Evidently, T is polaroid implies T is isoloid (ie., every isolated point of $\sigma(T)$ is an eigenvalue of T).

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T is polaroid.*

Proof. Assume that $T \in B(\mathcal{H})$ is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and let μ be an isolated point of $\sigma(T)$. To prove that T is polaroid, it is enough to show that $a(T - \mu I) < \infty$ and $d(T - \mu I) < \infty$. Let E_μ denote the spectral projection associated with λ. Then the Riesz idempotent E of T with respect to z is defined by

$$E_\mu := \frac{1}{2\pi i} \int_{\partial D} (zI - T)^{-1} dz,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. We can represent T on $\mathcal{H} = R(E_\mu) \oplus \ker(E_\mu)$ as follows

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $\sigma(A) = \{\mu\}$ and $\sigma(B) = \sigma(T) \setminus \{\mu\}$.

Since $T \in B(\mathcal{H})$ is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, $q(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ for some non constant complex polynomial q. Thus, $\sigma(q(A)) = q(\sigma(A)) = q(\mu)$. Therefore, $q(A) - q(\mu)$ is quasi nilpotent. Then by Lemma 2.1, $q(A) - q(\mu) = 0$. Put $r(z) = q(A) - q(\mu)$, then $r(A) = 0$ and so A is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Since $\sigma(A) = \{\mu\}$, it follows from Theorem 2.1 that $A - \mu I$ is nilpotent and so $a(A - \mu I) < \infty$ and $d(A - \mu I) < \infty$. Also, $a(B - \mu I) < \infty$ and $d(B - \mu I) < \infty$ follows from the invertibility of $B - \mu I$. Consequently, $T - \mu I$ has finite ascent and descent. This completes the proof. □

Theorem 2.3. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T satisfies generalized Weyl's theorem.*

Proof. Suppose that T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. From Theorem 2.2, T is polaroid . Since T is algebraically class p-$wA(s,t)$ with $s,t \leq 1$, $p(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ for some nonconstant polynomial q, it follows that $q(T)$ has Bishop's property (β) by [24, Theorem 2.4] or [22]. Therefore, $q(T)$ has SVEP. Then by [17, Theorem 3.3.9] T has SVEP . Hence the required result follows from [3, Theorem 4.1]. □

Corollary 2.1. *Let $T \in B(\mathcal{H})$ be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T satisfies Weyl's theorem.*

According to Berkani and Koliha [8], an operator $T \in B(\mathcal{H})$ is said to be Drazin invertible if T has finite ascent and descent. The Drazin spectrum of $T \in B(\mathcal{H})$, denoted by $\sigma_D(T)$, is defined $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}$ (See, [7]). Let $H(\sigma(T))$ denote the set of analytic functions which are defined on an open neighborhood of $\sigma(T)$.

Theorem 2.4. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then the equality $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in H(\sigma(T))$.*

Proof. Since T is algebrically class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, T has SVEP. Hence, $f(T)$ satisfies generalized Browder's theorem. Then by [12, Theorem 2.1] we have

$$\sigma_{BW}(f(T)) = \sigma_D(f(T)).$$

By [12, Theorem 2.7]), $\sigma_D(f(T)) = f(\sigma_D(T))$ and hence $\sigma_{BW}(f(T)) = f(\sigma_D(T))$. Since T is algebraically class p-$wA(s,t)$ with $0 < s,t,s+t \leq 1$, T satisfies generalized Weyl's theorem. Thus, T satisfies generalized Browder's theorem and so $f(\sigma_D(T)) = f(\sigma_{BW}(T))$. Therefore, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. This completes the proof. □

Theorem 2.5. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then $f(T)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. Suppose T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ s. Since the equality $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in H(\sigma(T))$ by Theorem 2.4, it follows that $f(T)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$. □

Theorem 2.6. *Let T^* be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then a-Weyl's theorem holds for T.*

Proof. Since T^* is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, $q(T^*)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ for some nonconstant polynomial q. It follows from [22] that $q(T^*)$ has SVEP. Therefore, T^* has SVEP by [17, Theorem 3.3.9]. By Theorem 2.2, T^* is polaroid. Since T^* is polaroid, T is polaroid. By applying [4, Theorem 3.10], it follows that a-Weyl's theorem holds for T. □

Theorem 2.7. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$ for every $f \in H(\sigma(T))$.*

Proof. Let $f \in H(\sigma(T))$. Recall that for every $T \in B(\mathcal{H})$, the following inclusion

$$\sigma_{SF_+^-}(f(T)) \subseteq f(\sigma_{SF_+^-}(T))$$

is always true. Now it suffices to show that $\sigma_{SF_+^-}(f(T)) \supseteq f(\sigma_{SF_+^-}(T))$. Let $\lambda \notin \sigma_{SF_+^-}(f(T))$. Then $f(T) - \lambda I \in SF_-^+(\mathcal{H})$. Let

$$f(T) - \lambda I = c(T - \mu_1)(T - \mu_2)....., (T - \mu_n)g(T), \tag{2.1}$$

where $c, \mu_1, \mu_2...., \mu_n \in \mathbb{C}$ and $g(T)$ is invertible. Since T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, T has SVEP. It follows from [1, Corollary 3.19] that $\text{ind}(T - \mu) \leq 0$ for all μ for which $T - \mu$ is Fredholm, $T - \mu_i$ is Fredholm of index zero for each $i = 1, 2, .., n$. Therefore, $\mu_i \notin \sigma_{SF_+^-}(T)$ for all $1 \leq i \leq n$. Hence,

$$\lambda = f(\mu_i) \notin f(\sigma_{SF_+^-}(T)).$$

This completes the theorem. □

Recall that an operator $T \in B(\mathcal{H})$ is said to be a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Evidently, if T is a-isoloid, then it is isoloid.

Theorem 2.8. *Let T^* be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. From Theorem 2.6, a-Weyl's theorem holds for T. Hence, T satisfies a-Browder's theorem. Since T^* is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, T^* has SVEP. If $f \in H(\sigma(T))$, then by [17, Theorem 3.3.9], $f(T)$, or $f(T)$ satisfies the SVEP. Applying [18, Theorem 2.4], it follows that $f(T)$ satisfies a- Browder's theorem. To prove a-Weyl's theorem holds for $f(T)$ it is enough to show that $E_0^a(f(T)) = \Pi_0^a(f(T))$. The inclusion $\Pi_0^a(f(T)) \subseteq E_0^a(f(T))$ is trivial. To prove the reverse inclusion let $\lambda \in E_0^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $\alpha(f(T)-\lambda I) < \infty$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\mu_i \in \sigma_a(T)$, then μ_i is an isolated point of $\sigma_a(T)$ by (2.1). That is, T is a-isoloid. Thus, $0 < \alpha(f(T) - \mu_i I) < \infty$ for each $i = 1, 2, ..., n$. Since T satisfies a-Weyl's theorem , $T - \mu_i I \in SF_+^-(\mathcal{H})$for each $i = 1, 2, ..., n$. Therefore $f(T) - \lambda I \in SF_+(\mathcal{H})$ and

$$\operatorname{ind}(f(T) - \lambda I) = \sum_{i=1}^{n} \operatorname{ind}(f(T) - \mu_i I) \leq 0.$$

Hence, $\lambda \notin \sigma_{SF_+^-}(f(T))$. Since $f(T)$ satisfies a-Browder's theorem, $\lambda \in \Pi_0^a(f(T))$. This completes the proof. □

Theorem 2.9. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then Weyl's theorem holds for $T + R$ for any finite rank operator $F \in B(\mathcal{H})$ commuting with T.*

Proof. Suppose T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then from Theorem 2.2, isolated point of spectrum of T are eigenvalues. By Theorem 2.1, T satisfies Weyl's theorem. Then it follows that Weyl's theorem holds for $T + R$ for any finite rank operator $R \in B(\mathcal{H})$ by [15, Theorem 3.3],. □

Theorem 2.10. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then for any function $f \in H(\sigma(T))$ and any finite rank operator $R \in B(\mathcal{H})$ commuting with T, Weyl's theorem holds for $f(T) + R$.*

Proof. Suppose T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then T is polaroid by Theorem 2.2 and hence T is isoloid. Therefore, $f(T)$ is isoloid for any function f analytic on a neighborhood of $\sigma(T)$ by [15, Lemma 3.6]. Then $f(T)$ obeys generalized Weyl theorem for any function $f \in H(\sigma(T))$ by Theorem 2.5. Then from [15, Theorem 3.3], it follows that Weyl's theorem holds for $f(T) + R$ for any finite rank operator R. □

3. elementary operator d_{AB} and Weyl type theorem

Let d_{AB} denote the *generalized derivation* $\delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\delta_{AB}(X) = AX - XB$ or the *elementary operator* $\Delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\Delta_{AB}(X) =$

$AXB - X$. In this section we show that if $A, B^* \in B(H)$ are class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then generalized Weyl's theorem , a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem holds for $f(d_{AB})$ for every $f \in H\sigma(d_{AB})$. Recall that an operator $T \in B(H)$ is said to have the property (δ) if for every open covering (U, V) of $\mathbb{C}$, we have $\mathcal{H} = \mathcal{H}_T(\bar{U}) + \mathcal{H}_T(\bar{V})$.

Lemma 3.1. *Let $A, B \in B(\mathcal{H})$. If A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then d_{AB} has SVEP.*

Proof. Suppose that A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then A and B^* satisfies Bishop's property (β) by [24, Theorem 2.4] or [22]. Hence B satisfies property (δ) by [17, Theorem 2.5.5]. Since both AX and XB satisfies property (C) by Corollary 3.6.11of [17]. Then SVEP holds for both $AX - XB$ and $AXB - X$ by [17, Theorem 3.6.3] and [17, Note 3.6.19]. Then, d_{AB} has SVEP.

□

Lemma 3.2. *Let $A, B \in B(\mathcal{H})$. If A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then d_{AB} is polaroid.*

Proof. Since A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, A and B^* are polaroid by Proposition 2.2. It is known that if B^* is polaroid then B is polaroid. Hence the required result follows by [26, Lemma 4.1] □

Theorem 3.1. *If $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then generalized Weyl's theorem holds for d_{AB}.*

Proof. Since A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, d_{AB} has SVEP by Lemma 3.1. By Lemma 3.2, d_{AB} is polaroid. Then by applying [4, theorem 3.10], it follows that generalized Weyl's theorem holds for d_{AB} □

Theorem 3.2. *If $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$.*

Proof. Since A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, d_{AB} has SVEP by Lemma 3.1. By Lemma 3.2 the operator d_{AB} is polaroid and so d_{AB} is isoloid. Then by applying [25, theorem 2.2], it follows that generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H\sigma(d_{AB})$. □

We say that $T \in B(\mathcal{H})$ possesses property (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ [23]. In Theorem 2.8 of [2], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in B(\mathcal{H})$ possesses property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gw) has been introduced and studied in [6]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [6] that an operator possessing property (gw) possesses property (w) but the converse is not true in general.

Theorem 3.3. *Let $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem hold for every $f \in H(\sigma(d_{AB}))$.*

Proof. By Lemma 3.1, the operator d_{AB} has SVEP. The operator d_{AB} is polaroid by Lemma 3.2,. Then by applying [4, theorem 3.12], it follows that a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem hold for every $f \in H(\sigma(d_{AB}))$. □

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