A D-PEARSON EQUATION FOR DUNKL-CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. In this paper, we show that a monic orthogonal polynomial sequence is a Dunkl-classical sequence if and only if it belongs to a particular family of D-semiclassical polynomial sequence of class less or equal to two. In this case, the distributional equation fulfilled by the linear functionals corresponding to these polynomials is given. Some well known results in the literature are generalized.

Keywords: Orthogonal polynomials, Dunkl-classical polynomials, Regular linear functionals, D-semiclassical polynomials.

1. Introduction

A monic orthogonal polynomial sequence (MOPS, for shorter) \( \{P_n\}_{n \geq 0} \) is called Dunkl-classical polynomial sequence (the associated linear functional is called Dunkl-classical linear functional) if \( \{T_{\mu}P_n\}_{n \geq 1} \) is an orthogonal polynomial sequence, where \( T_{\mu} \) is the Dunkl operator \( \[6\] : T_{\mu} = D + 2\mu H_{-1}, \ \mu > -\frac{1}{2}, D \) (resp. \( H_{-1} \)) denotes the derivative operator \( D = \frac{d}{dx} \) (resp. the Hahn operator given by \( (H_{-1}f)(x) = f(x) - f(-x) \)).

Y. Ben Cheikh and his coworker \[1\] introduced the notion of Dunkl-classical orthogonal polynomials and proved that the only symmetric Dunkl-classical orthogonal polynomials are the generalized Hermite polynomials and the generalized Gegenbauer polynomials. Note that both of them are D-semiclassical sequences of class less or equal to two (see \[2\]|\[4\]). Later on, M. Sghaier \[10\] find a non-symmetric sequence of Dunkl-classical polynomials. This sequence is also D-semiclassical, since it is obtained by multiplying the generalized Gegenbauer linear functional by a polynomial of first degree \[5\].

It is natural, then, to ask if all Dunkl-classical orthogonal polynomials are D-semiclassical.

The aim of this paper is to answer this question. Namely, we prove the following result:

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Theorem 1.1. Let $u_0$ be a regular linear functional and let $\{P_n \}_{n \geq 0}$ be its corresponding MOPS. Then $u_0$ is a Dunkl-classical form if and only if there exist two polynomials $\Phi$ (monic) and $B$ with $\deg B = 1 + \deg \Phi \leq 3$ such that

$$
\begin{align*}
D \left( x^2 \Phi(x) + 2 \mu x B(x) \right) u_0 + \left( 2 \mu^2 - \mu - 2 \right) x \Phi(x) - (2 \mu^2 + 3 \mu) B(x) \\
+ \frac{1 - 4 \mu^2}{K} x^2 \Psi(x) \right) u_0 = 0,
\end{align*}
$$

(1.1)

$$
\begin{align*}
\Psi'(0) + \frac{1}{2} \Phi''(0) + \frac{1}{2} \frac{1}{1 - 4 \mu^2} (4 \mu^2 [n] - n) + \frac{1}{3} \frac{1}{(1 - 4 \mu^2)} \mu ([n] - n) \neq 0,
\end{align*}

(1.2)

$$
\begin{align*}
x \Phi(x) u_0 = h_{-1} (B(x) u_0),
\end{align*}

(1.3)

where

$$
\begin{align*}
\Psi(x) &= \frac{1 + 2 \mu}{\gamma_1} P_1, \\
K &= \frac{1 + 2 \mu}{\langle u_0, \Phi \rangle}.
\end{align*}

(1.4)

(1.5)

The structure of this paper is as follows: Section 2 is devoted to preliminary results and notations to be used in the sequel. In Section 3, we prove the main theorem. In Section 4, we illustrate 1.1 by analyzing some examples of D-semiclassical linear functionals which are Dunkl-classical.

2. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $\langle u, x^n \rangle$, $n \geq 0$, the moments of $u$. Let $h_u, g_u$ and $D_u = u'$ linear functionals defined by duality

$$
\begin{align*}
\langle h_u, f(x) \rangle &= \langle u, (h_u f)(x) \rangle = \langle u, f(ax) \rangle, f \in \mathcal{P}, a \in \mathbb{C} \setminus \{0\}, \\
\langle g_u, f(x) \rangle &= \langle u, g(x) f(x) \rangle, \langle D_u, f(x) \rangle = - \langle u, f'(x) \rangle, f, g \in \mathcal{P}.
\end{align*}
$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product $uf$ is the polynomial

$$
(uf)(x) = \langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \rangle.
$$

The division of a linear functional by a polynomial of first degree is given by

$$
\langle (x - c)^{-1} u, f \rangle = \langle u, \theta_c f \rangle, c \in \mathbb{C}, f \in \mathcal{P}, u \in \mathcal{P}',
$$

where

$$
(\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}.
$$
It is easy to see that
\( (2.1) \quad fD u = D(fu) - f'u, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}' \)

and
\( (2.2) \quad x^{-1}(xu) = u - (u)_{0} \delta_{0}, \quad u \in \mathcal{P}' \)

where \( \delta_{c}, \quad c \in \mathbb{C} \) is the Dirac linear functional defined by
\[ \langle \delta_{c}, f \rangle = f(c), \quad f \in \mathcal{P}. \]

Let \( \{P_{n}\}_{n \geq 0} \) be a sequence of monic polynomials with \( \text{deg}P_{n} = n, \quad n \geq 0 \) and let \( \{u_{n}\}_{n \geq 0} \) be its dual sequence, \( u_{n} \in \mathcal{P}' \) and defined by \( \langle u_{n}, P_{m} \rangle = \delta_{n,m}, \quad n, m \geq 0 \). Let us recall some result [7]

**Lemma 2.1.** For any \( u \in \mathcal{P}' \) and any integer \( m \geq 1 \), the following statements are equivalent

(i) \( \langle u, P_{m-1} \rangle \neq 0, \quad \langle u, P_{n} \rangle = 0, \quad n \geq m. \)

(ii) \( \exists \lambda_{v} \in \mathbb{C}, \quad 0 \leq v \leq m - 1, \lambda_{m-1} \neq 0 \) such that \( u = \sum_{v=0}^{m-1} \lambda_{v} u_{v} \).

The linear functional \( u \) is called regular if there exists a polynomial sequence (PS, in short) \( \{P_{n}\}_{n \geq 0} \) such that [4]:
\( (2.3) \quad \langle u, P_{m} P_{n} \rangle = r_{n} \delta_{n,m}, \quad n, m \geq 0, \quad r_{n} \neq 0, \quad n \geq 0. \)

The sequence \( \{P_{n}\}_{n \geq 0} \) is then called orthogonal with respect to \( u \). In this case, we have \( u_{n} = r_{n}^{-1} P_{n} u_{0}, \quad n \geq 0 \). According to the previous lemma, we have \( u = \lambda u_{0} \), where \( (u)_{0} = \lambda \neq 0 \). In what follows all regular linear functionals \( u \) will be taken normalized i.e, \( (u)_{0} = 1 \).

According to Favard’s theorem, a monic orthogonal polynomial sequence \( \{P_{n}\}_{n \geq 0} \) is characterized by the following three-term recurrence relation [4]:
\( (2.4) \quad P_{0}(x) = 1, \quad P_{1}(x) = x - \beta_{0}, \quad P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1} P_{n}(x), \quad n \geq 0, \)

with \( (\beta_{n}, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}, \quad n \geq 0. \)

The first associated of \( \{P_{n}\}_{n \geq 0} \) is the MOPS \( \{P_{n}^{(1)}\}_{n \geq 0} \) defined by
\( (2.5) \quad P_{0}^{(1)}(x) = 1, \quad P_{1}^{(1)}(x) = x - \beta_{1}, \quad P_{n+2}^{(1)}(x) = (x - \beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2} P_{n}^{(1)}(x), \quad n \geq 0. \)

**Definition 2.1.** (see [4][8]) A linear functional \( u \) is called D-semiclassical of class \( s \) if it is regular and the following statement holds: There exist two polynomials \( \Psi \) of degree \( p \geq 1 \) and \( \Phi \) of degree \( t \geq 0 \), such that
\( (2.6) \quad (\Phi u)' + \Psi u = 0, \)
\[(2.7) \quad \prod_{c \in \mathbb{Z}_\Phi} \left( |\Psi(c) + \Phi'(c)| + \left| \left( u, \theta, \Psi + \theta \Phi \right) \right| \right) > 0, \]

where \( Z_\Phi \) is the set of zeros of \( \Phi \).
The class of \( u \) is given by \( s = \max(p - 1, t - 2) \). The sequence \( \{P_n\}_{n \geq 0} \) corresponding to \( u \) is called D-semiclassical of class \( s \).
When \( s = 0 \), the linear functional \( u \) (or the sequence \( \{P_n\}_{n \geq 0} \)) is said to be D-classical.

Let us introduce the Dunkl’s operator
\[
(T_\mu f)(x) = f'(x) + 2\mu (H_{-1} f)(x), \quad f \in \mathcal{P},
\]
where
\[
(H_{-1} f)(x) = \frac{f(x) - f(-x)}{2x}.
\]
We define the operator \( T_\mu \) from \( \mathcal{P}' \) to \( \mathcal{P}' \) as follows
\[
<T_\mu u, f(x) >= - < u, (T_\mu f)(x) >, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'.
\]
In particular, this yields
\[
(T_\mu u)_n = -\mu_n (u)_{n-1}, \quad n \geq 0,
\]
where
\[
(u)_{-1} = 0, \quad \mu_n = n + 2\mu[n], \quad [n] = \frac{1 - (-1)^n}{2}, \quad n \geq 0.
\]
It is easy to see that
\[
T_\mu u = Du + 2\mu H_{-1} u,
\]
where
\[
<T_{-1} u, f(x) >= - < u, (H_{-1} f)(x) >.
\]
Now, consider a MOPS \( \{P_n\}_{n \geq 0} \) as above and let
\[
P_n^{[1]}(x) = \frac{1}{\mu_{n+1}} (T_\mu P_{n+1})(x), \quad \mu \neq -n - \frac{1}{2}, \quad n \geq 0.
\]
Let denote by \( \{n_n^{[1]}\}_{n \geq 0} \), the dual sequence of \( \{P_n^{[1]}\}_{n \geq 0} \).

**Lemma 2.2.** [10]
\[(2.8) \quad T_\mu u_n^{[1]} = -\mu_{n+1} u_{n+1}, \quad n \geq 0.\]

### 3. Proof of the main theorem

For the proof, we need the following lemma:
Proof of the main theorem

From the assumption we have

(3.1) \[ xT_\mu u = xDu - \mu(u + h_{-1}u), \quad u \in \mathcal{P}. \]

(3.2) \[ T_\mu(fu) = fT_\mu u + f'u + 2\mu(H_{-1}f)(h_{-1}u), \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \]

(3.3) \[ T_\mu(fu) = fT_\mu u + (T_\mu f)u + 2\mu(H_{-1}f)(h_{-1}u - u), \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \]

Proof of the lemma

From the definition of the operator \( T_\mu \) on \( \mathcal{P}' \), we have

\[
< xT_\mu u, f(x) > = < xDu, f(x) > + 2\mu < xH_{-1}u, f(x) > \\
= < xDu, f(x) > - 2\mu < u, \frac{xf(x) + xf(-x)}{2x} > \\
= < xDu, f(x) > - \mu < u, f(x) + f(-x) > \\
= < xDu, f(x) > - \mu \left( < u, f(x) > + < h_{-1}u, f(x) > \right) \\
= < xDu, f(x) > - \mu < u + h_{-1}u, f(x) > ,
\]

hence (3.1) follows.

For the proof of (3.2), let \( g \in \mathcal{P} \). We have

\[
< T_\mu(fu), g(x) > = - < u, f(x)g'(x) + \mu f(x)\frac{g(x) - g(-x)}{x} > \\
= - < u, (fg)'(x) - f'(x)g(x) + \mu f(x)\frac{g(x) - g(-x)}{x} > \\
= - < u, T_\mu(fg)(x) - f'(x)g(x) - \mu g(-x)\frac{f(x) - f(-x)}{x} > \\
= < fT_\mu u + f'u + 2\mu(H_{-1}f)(h_{-1}u), g(x) > .
\]

Thus, we obtain (3.2). From which we derive (3.3)

Proof of the main theorem

First of all, notice that for \( \mu = 0 \) we get the D-classical orthogonal polynomial sequences, which are D-semiclassical of class zero. Henceforth, we will suppose that \( \mu \neq 0 \).

From the assumption we have

(3.4) \[ u_n = r_n^{-1}p_nu_0, \quad n \geq 0 \]

and

(3.5) \[ u^{[1]}_n = (r^{[1]}_n)^{-1}p^{[1]}_nu^{[1]}_0, \quad n \geq 0. \]

Substitution of (3.4) and (3.5) in (2.8) gives

(3.6) \[ T_\mu(p^{[1]}_nu^{[1]}_0) = -\chi_n p_{n+1}u_0, \quad n \geq 0, \]

where

(3.7) \[ \chi_n = \mu_n + r^{[1]}_n r_n^{-1}l^{[1]}_{n+1}, \quad n \geq 0. \]
Using formula (3.3), equation (3.6) becomes

\[
T_\mu u_0^{[1]} + (T_\mu P_n^{[1]}) u_0^{[1]} + 2\mu (H_{-1} P_n^{[1]}) (h_{-1} u_0^{[1]} - u_0^{[1]}) = -\chi_0 P_{n+1} u_0, \quad n \geq 0.
\]

For \(n = 0\), equation (3.8) becomes

\[
T_\mu u_0^{[1]} = -\chi_0 P_1 u_0 = -\frac{1 + 2\mu}{\gamma_1} P_1 u_0.
\]

For \(n = 1\), equation (3.8) becomes

\[
P_1^{[1]} T_\mu u_0^{[1]} + u_0^{[1]} + 2\mu h_{-1} u_0^{[1]} = -2\gamma_1 P_2 u_0.
\]

Substitution of (3.9) in (3.10) gives

\[
u_0^{[1]} + 2\mu h_{-1} u_0^{[1]} = K \Phi u_0,
\]

where

\[
K \Phi = \frac{1 + 2\mu}{\gamma_1} P_1^{[1]} - 2\gamma_1 P_2^{[1]},
\]

\((K\text{ is a constant to make } \Phi \text{ monic}).\n
Applying the operator \(h_{-1}\) to (3.11), we get

\[
2\mu u_0^{[1]} + h_{-1} u_0^{[1]} = Kh_{-1}(\Phi u_0).
\]

Multiplying (3.13) by 2\(\mu\) and subtracting the result from (3.11), we get

\[
u_0^{[1]} = \frac{K}{1 - 4\mu^2} (\Phi u_0 - 2\mu h_{-1}(\Phi u_0)).
\]

Substitution of (3.14) in (3.9) gives

\[
T_\mu \left(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\right) + \frac{1 - 4\mu^2}{K} \Psi u_0 = 0.
\]

From (1.4), (3.15) and the regularity of \(u_0\), we have

\[
0 = < T_\mu \left(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\right) + \frac{1 - 4\mu^2}{K} \Psi u_0, P_1 >

= \frac{1 - 4\mu^2}{K} (- < u_0, \Phi > + \frac{1 + 2\mu}{K}).
\]

Thus, (1.5) follows.
Now, putting \( n = 2 \) in (3.8), we obtain
\[
\frac{p_1^{(1)}(T_\mu u_0^{(1)})}{2} + (T_\mu p_1^{(1)})u_0^{(1)} + 2\mu H_{-1} p_2^{(1)}\left(h_{-1}u_0^{(1)} - u_0^{(1)}\right) = -\chi_2 P_3 u_0.
\]
(3.16)

Taking into account (3.9) and (3.14), we get
\[
-2\mu K \left( T_\mu p_2^{(1)} - (1 + 2\mu) H_{-1} p_2^{(1)} \right) h_{-1}(\Phi u_0) = \\
\left( \frac{1 + 2\mu}{1 - 4\mu^2} p_1 p_2^{(1)} - \frac{K}{1 - 4\mu^2} \Phi T_\mu p_2^{(1)} + \frac{2\mu K}{1 - 2\mu} \Phi H_{-1} p_2^{(1)} - \chi_2 P_3 \right) u_0.
\]
(3.17)

Applying the operator \( h_{-1} \) to the last equation and taking into account the fact that
\[
(T_\mu p_2^{(1)})(x) - ((1 + 2\mu) H_{-1} p_2^{(1)})(x) = 2x
\]
and the formulas:
\[
h_{-1}(xv) = -x h_{-1} v
\]
and
\[
h_{-1}(h_{-1} v) = v, \quad v \in \mathcal{P}',
\]
we obtain (1.3), where
\[
B(x) = \frac{1 - 4\mu^2}{4\mu K} \left( \frac{1 + 2\mu}{1 - 4\mu^2} p_1(x) p_2^{(1)}(x) - \frac{K}{1 - 4\mu^2} \Phi(x)(T_\mu p_2^{(1)})(x) \right.
\]
\[
\left. + \frac{2\mu K}{1 - 2\mu} \Phi(x)(H_{-1} p_2^{(1)})(x) - \chi_2 P_3(x) \right).
\]
(3.18)

Multiplying (3.14) by \( x \) and taking into account (1.3), we get
\[
x u_0^{(1)} = \frac{K}{1 - 4\mu^2} (x\Phi(x) + 2\mu B(x)) u_0.
\]
(3.19)

Applying the operator \( h_{-1} \) to the last equation and using again (1.3), we obtain
\[
x h_{-1} u_0^{(1)} = -\frac{K}{1 - 4\mu^2} (B(x) + 2\mu x\Phi(x)) u_0.
\]
(3.20)

On the other hand, from (3.2) we have
\[
x T_\mu(u_0^{(1)}) = T_\mu(x u_0^{(1)}) - u_0^{(1)} - 2\mu h_{-1} u_0^{(1)}.
\]
(3.21)

Multiplying (3.20) by \( x \) and taking into account (3.18) and (3.19), we get
\[
x^2 T_\mu(u_0^{(1)}) = \frac{K}{1 - 4\mu^2} \left( x T_\mu(\Phi(x) + 2\mu B(x)) u_0 \right) - (1 - 4\mu^2)x\Phi(x) u_0.
\]
From (3.1) and (1.3) we get

\[(3.22) \quad x^2 T_\mu(u_0^{[1]}) = \frac{K}{1 - 4\mu^2} \left( xD(x\Phi(x) + 2\mu B(x))u_0 \right) + 
\left( (2\mu^2 - \mu - 1)x\Phi(x) - (2\mu^2 + \mu)B(x) \right)u_0, \]

or, equivalently,

\[(3.23) \quad x^2 T_\mu(u_0^{[1]}) = \frac{K}{1 - 4\mu^2} \left( D\left(x(x\Phi(x) + 2\mu B(x))u_0 \right) + 
\left( (2\mu^2 - \mu - 1)x\Phi(x) - (2\mu^2 + 3\mu)B(x) \right)u_0. \]

According to (3.9) and (1.4), from (3.23) we get (1.1).

Notice that \(x^2\Phi(x) + 2\mu B(x) \neq 0\). Indeed, if not then \(x^2\Phi(x) + 2\mu B(x) = 0\). Therefore, (3.18) becomes \(xu_0^{[1]} = 0\). This contradicts the regularity of \(u_0^{[1]}\). Thus, \(u_0\) (or \(\{P_n\}_{n \geq 0}\)) is D-semiclassical. Furthermore, by examination of the degrees of polynomials \(\Phi, \Psi\) and \(B\) in (3.12), (1.4) and (3.17) respectively, we can easily see that the class of \(\{P_n\}_{n \geq 0}\) is less or equal to two.

Conversely, suppose that \(u_0\) is a linear functional such that (1.1)-(1.5) hold. Using Lemma 2.2, we get (3.9). Substituting (3.9) in (1.1), we obtain (3.22). Putting

\[(3.24) \quad v = \frac{K}{1 - 4\mu^2} \left( \Phi u_0 - 2\mu h_{-1}(\Phi u_0) \right) \]

and using (3.1) and (1.3), we obtain

\[x^2 T_\mu v = \frac{K}{1 - 4\mu^2} \left( xT_\mu (x\Phi(x) + 2\mu B(x))u_0 \right) - (1 - 4\mu^2)x\Phi(x)u_0. \]

Therefore, equation (3.22) becomes

\[(3.25) \quad x^2 T_\mu u_0^{[1]} = x^2 T_\mu v. \]

Multiplying (3.25) by \(x^{-1}\) and using (2.2), we get

\[(3.26) \quad xT_\mu u_0^{[1]} - \left( xT_\mu u_0^{[1]} \right)_0 = xT_\mu v - \left( xT_\mu v \right)_0. \]

But, on the one hand we have

\[\left( xT_\mu u_0^{[1]} \right)_0 = -(1 + 2\mu) \]

and, on the other hand, from (3.24) and (1.5), we have

\[\left( xT_\mu v \right)_0 = -(1 + 2\mu) \]
then, \((3.26)\) becomes
\[(3.27)\]  
\( xT_\mu u_0^{[1]} = xT_\mu v. \)

In a similar way, multiplying \((3.27)\) by \(x^{-1}\) and using \((2.2)\), we get
\[(3.28)\]  
\[ T_\mu u_0^{[1]} = T_\mu v. \]

Hence,
\[ v = u_0^{[1]}. \]

Therefore, equation \((3.9)\) becomes
\[(3.29)\]  
\[ T_\mu v + \Psi u_0 = 0. \]

Let us prove that the sequence \(\{P_n\}_{n \geq 0}\) is orthogonal with respect to \(v\). Let \(m \leq n - 1\). From \((3.2)\), we have
\[ \left\langle v, P_m(x)P_n^{[1]} \right\rangle = -\frac{1}{\mu_{n+1}} \left\langle T_\mu (P_m v), P_{n+1}(x) \right\rangle = -\frac{1}{\mu_{n+1}} \left\langle P_m T_\mu v + P_m' v + 2\mu H_{-1} P_m h_{-1} v, P_{n+1} \right\rangle. \]

Taking into account \((3.29)\) and the fact that \(\{P_n\}_{n \geq 0}\) is orthogonal with respect to \(u_0\), we get
\[ \left\langle v, P_m(x)P_n^{[1]} \right\rangle = -\frac{1}{\mu_{n+1}} \left\langle v, P_{n+1}(x)P_m'(x) + 2\mu (H_{-1} P_m)(x)P_{n+1}(-x) \right\rangle. \]

Using \((3.24)\), the orthogonality of \(\{P_n\}_{n \geq 0}\) with respect to \(u_0\) and the fact that \(deg \Phi \leq 2\), we obtain
\[ \left\langle v, P_m(x)P_n^{[1]} \right\rangle = \frac{2\mu K}{(1 - 4\mu^2)\mu_{n+1}} \left\langle u_0, \Phi(x) P_{n+1}(-x) \left( P_m'(-x) - (H_{-1} P_m)(-x) \right) \right\rangle. \]

Writing \(P_m(x) = \theta_0 + \theta_1 x + \ldots + \theta_{m-1} x^{m-1} + x^m\), we can easily see that
\[ P_m'(-x) - (H_{-1} P_m)(-x) = xQ(x), \]

where \(Q\) is a polynomial of degree less than or equal to \(m - 1\) (with the convention that the degree of the zero polynomial is \(-\infty\)). Then,
\[ \left\langle v, P_m(x)P_n^{[1]} \right\rangle = \frac{2\mu K}{(1 - 4\mu^2)\mu_{n+1}} \left\langle u_0, x\Phi(x)Q(x)P_{n+1}(-x) \right\rangle. \]

Application of \((1.3)\) gives
\[ \left\langle v, P_m(x)P_n^{[1]} \right\rangle = \frac{2\mu K}{(1 - 4\mu^2)\mu_{n+1}} \left\langle u_0, B(x)Q(-x)P_{n+1}(x) \right\rangle. \]
Since \( B \) is a polynomial of degree less or equal to three then, from the orthogonality of \( \{P_n\}_{n \geq 0} \) with respect to \( u_0 \), we get
\[
\left\langle v, P_m(x)P_n^{[1]} \right\rangle = 0.
\]
For \( m = n \), a second use of (3.2) gives
\[
(3.30) \quad \left\langle v, P_n(x)P_n^{[1]} \right\rangle = \frac{1}{\mu_{n+1}} \left( P_nT_{\mu}v + P_n^{\prime}v + 2\mu H_{-1}P_n h_{-1}v, P_n^1 \right).
\]
Using (3.29) and the fact that \( \{P_n\}_{n \geq 0} \) is orthogonal with respect to the linear functional \( u_0 \), we get
\[
(3.31) \quad \left\langle P_nT_{\mu}v, P_{n+1} \right\rangle = -\Psi'(0)r_{n+1},
\]
where \( r_{n+1} \) is given in (2.3).
From (3.24), we obtain
\[
(3.32) \quad \left\langle P_n^{\prime}v, P_{n+1} \right\rangle = \frac{K}{1 - 4\mu^2} \left( \frac{1}{2} n\Phi''(0)r_{n+1} - 2\mu \left( u_0, \Phi(x)P_n^{\prime}(-x)P_{n+1}(-x) \right) \right)
\]
and
\[
(3.33) \quad \left\langle 2\mu H_{-1}P_n h_{-1}v, P_{n+1} \right\rangle = \frac{2\mu K}{1 - 4\mu^2} \left( u_0, \Phi(x)(H_{-1}P_n(-x)P_{n+1}(-x)) - \mu\Phi''(0)[n]r_{n+1} \right).
\]
Substitution of (3.31), (3.32) and (3.33) in (3.30) gives
\[
(3.34) \quad \left\langle v, P_n(x)P_n^{[1]} \right\rangle = \left( \Psi'(0) + \frac{1}{2} K\Phi''(0) \right) \frac{r_{n+1}}{\mu_{n+1}} - \frac{2\mu K}{\mu_{n+1}(1 - 4\mu^2)} \left( u_0, \Phi(x)(H_{-1}P_n(-x) - P_n^{\prime}(-x))P_{n+1}(-x) \right).
\]
Writing \( (H_{-1}P_n)(-x) - P_n^{\prime}(-x) = xQ(x) \), where \( Q \) is a polynomial of degree \( n - 2 \) with leading coefficient \( (-1)^{n-1}([n] - n) \) for \( n \geq 2 \) and \( Q = 0 \) for \( n \in \{0, 1\} \), and using (1.3), we get
\[
\left\langle u_0, \Phi(x)(H_{-1}P_n)(-x) - P_n^{\prime}(-x) \right\rangle P_{n+1}(-x) = -\frac{1}{6} B'''(0)([n] - n)r_{n+1}, \ n \geq 0.
\]
Therefore, (3.34) becomes
\[
\left\langle v, P_n(x)P_n^{[1]} \right\rangle = \left( \Psi'(0) + \frac{1}{2} K\Phi''(0) \right) \frac{r_{n+1}}{\mu_{n+1}} - \frac{2\mu K}{3(1 - 4\mu^2)} \left( [n] - n \right) r_{n+1},
\]
On account of condition (1.2), the last equation implies that
\[
\left\langle v, P_n(x)P_n^{[1]} \right\rangle \neq 0.
\]
So, the sequence \( \{P_n^{[1]}\}_{n \geq 0} \) is orthogonal with respect to the linear functional \( v \).
4. Examples

In order to illustrate Theorem 1.1, we present three Dunkl-classical linear functionals: the generalized Hermite, the generalized Gegenbauer and a non-symmetric Dunkl-classical linear functional.

4.1. Generalized Hermite linear functional

The generalized Hermite linear functional denoted by $\mathcal{H}(\mu)$ satisfies (see [4]):

\[ D(x\mathcal{H}(\mu)) + (2x^2 - (2\mu + 1))\mathcal{H}(\mu) = 0. \]  

(4.1)

The sequence of generalized Hermite polynomials $\{H_n^{(\mu)}\}_{n \geq 0}$ satisfies (2.4) with

\[ \beta_n = 0, \quad \gamma_{n+1} = \frac{1}{2} \mu_{n+1}, \quad n \geq 0, \]  

(4.2)

where the regularity condition is

\[ \mu \neq -n - \frac{1}{2}, \quad n \geq 0. \]  

(4.3)

The weight function for generalized Hermite polynomials in the positive definite case is given by

\[ w(x) = |x|^{2\mu}e^{-x^2}, \quad -\infty < x < \infty. \]

We will show that $\mathcal{H}(\mu)$ satisfies conditions (1.1) - (1.5).

Multiplying (4.1) by $(1 - 2\mu)x$ and using (2.1), we get (1.1)

\[ \Phi(x) = 1, \]  

(4.4)

\[ B(x) = -x, \]  

(4.5)

\[ \Psi(x) = 2x, \]  

(4.6)

\[ K = 1 + 2\mu. \]  

(4.7)

On the other hand, since $\mathcal{H}(\mu)$ is a symmetric linear functional, we have

\[ \mathcal{H}(\mu) = h_{-1}(\mathcal{H}(\mu)). \]

Multiplying the last equation by $x$, we get (1.3).

Finally, if we substitute (4.4)-(4.7) in the left hand side of (1.2), then we get

\[ \Psi'(0) + \frac{1}{2} K\Phi''(0) + \frac{1}{3} K B'''(0) - \alpha \mu([n] - n) = 2 \neq 0. \]

Therefore, Theorem 1.1 implies that $\mathcal{H}(\mu)$ is a Dunkl-classical linear functional. Furthermore, by virtue of (3.15) and (1.3), $\mathcal{H}(\mu)$ satisfies the following $T_\mu$-distributional equation:

\[ T_\mu(\mathcal{H}(\mu)) + 2x\mathcal{H}(\mu) = 0. \]  

(4.8)
Notice that Y. Ben Cheikh and M. Gaied [1] have proved differently that \( \mathcal{H}(\mu) \) is a Dunkl classical linear functional. But they did not give a \( T_\mu \)-distributional equation for \( \mathcal{H}(\mu) \).

### 4.2. Generalized Gegenbauer polynomials

The generalized Gegenbauer linear functional denoted by \( \mathcal{G}(\alpha, \beta) \) satisfies (see [2]):

\[
D(x^2 - 1)\mathcal{G}(\alpha, \beta) + \left(-2(\alpha + \beta + 1)x^2 + 2(\beta + 1)\right)\mathcal{G}(\alpha, \beta) = 0.
\]

This linear functional is regular for

\[
\alpha \neq -n, \beta \neq -n, \alpha + \beta \neq -n, n \geq 1.
\]

The weight function for generalized Gegenbauer polynomials in the positive definite case is given by

\[
w(x) = |x|^{2\alpha+1}(1 - x^2)^\alpha, \quad -1 < x < 1.
\]

Putting \( \beta = \mu - \frac{1}{2} \) in (4.9), we get

\[
D(x^2 - 1)\mathcal{G}(\alpha, \mu - \frac{1}{2}) + \left(-2(\alpha + \mu + \frac{3}{2})x^2 + 2\mu + 1\right)\mathcal{G}(\alpha, \mu - \frac{1}{2}) = 0.
\]

As in the previous example, multiplying (4.11) by \((1 - 2\mu)x\) and using the fact that \( \mathcal{G}(\alpha, \mu - \frac{1}{2}) \) is symmetric, we obtain (1.1) and (1.3), where

\[
\Phi(x) = x^2 - 1,
\]

\[
B(x) = -x(x^2 - 1),
\]

\[
\Psi(x) = (2\alpha + 2\mu + 3)x,
\]

\[
K = -\frac{(1 + 2\mu)(\alpha + \mu + \frac{3}{2})}{\alpha + 1}.
\]

The condition (1.2) follows, immediately, from (4.10). Indeed:

\[
\Psi''(0) + \frac{1}{2} K\Phi''(0) + \frac{1}{3} K\Phi'''(0) - n = \frac{2\alpha + 2\mu + 3}{2\alpha + 2}(2\alpha + 2 + 2\mu[n] + n) \neq 0, \quad n \geq 0.
\]

Hence, Theorem 1.1 follows that \( \mathcal{G}(\alpha, \mu - \frac{1}{2}) \) is a Dunkl-classical linear functional. Furthermore, by virtue of (3.15), the linear functional \( \mathcal{G}(\alpha, \mu - \frac{1}{2}) \) satisfies

\[
T_\mu\left(\left(x^2 - 1\right)\mathcal{G}(\alpha, \mu - \frac{1}{2})\right) - 2(\alpha + 1)x\mathcal{G}(\alpha, \mu - \frac{1}{2}) = 0.
\]
4.3. An example of non-symmetric Dunkl-classical linear functional

In this subsection, we will construct a non-symmetric Dunkl-classical linear functional by using the following result stated in [9]:

Let \( L \) be a regular linear functional and let \( c \) and \( \lambda \) be two complex numbers. The linear functional \( u \) defined by

\[
(4.17) \quad u = \lambda(x - c)^{-1}L + \delta_c
\]

is regular, for every complex \( \lambda \) such that the following condition:

\[
(4.18) \quad \lambda \neq 0, \quad P_n(c) + \lambda P_{n-1}(c) \neq 0, \quad n \geq 1,
\]

where \( \{P_n\}_{n \geq 0} \) is the MOPS corresponding to \( L \). Moreover, if \( L \) is a D-semiclassical linear functional satisfying (2.6), then \( u \) satisfies

\[
(4.19) \quad (\Phi u)' + \Psi u = 0,
\]

where,

\[
(4.20) \quad \Phi(x) = (x - c)\Phi(x), \quad \Psi(x) = (x - c)\Psi(x).
\]

Let us apply this result and take

\[
L = \mathcal{G}(\alpha, \mu - \frac{1}{2}), \quad c = 1, \quad \lambda = \frac{-2\alpha}{2\alpha + 2\mu + 1}, \quad \alpha \neq 0.
\]

We will show that the obtained linear functional \( u \) satisfies the conditions (1.1)-(1.5). But, first we will study the regularity of \( u \).

Let \( \{S_n\}_{n \geq 0} \) be the MOPS associated with \( \mathcal{G}(\alpha, \mu - \frac{1}{2}) \). It satisfies the recurrence relation (2.4) with [2]:

\[
(4.21) \quad \beta_n = 0, \quad \gamma_{n+1} = \frac{(n + 1 + \delta_n)(n + 1 + 2\alpha + \delta_n)}{4(n + \alpha + \mu + \frac{1}{2})(n + \alpha + \mu + \frac{3}{2})}, \quad \delta_n = \mu(1 + (-1)^n),
\]

where the regularity conditions are

\[
(4.22) \quad \alpha + n \neq 0, \quad 2\mu + 2n - 1 \neq 0, \quad 2\alpha + 2\mu + 2n - 1 \neq 0, \quad n \geq 1.
\]

Putting

\[
(4.23) \quad \lambda_{2n} = 2^n n! \frac{(1 + 2\mu)(3 + 2\mu)...(2n - 1 + 2\mu)}{(1 + 2\mu + 2\alpha)(3 + 2\mu + 2\alpha)...(4n - 1 + 2\mu + 2\alpha)}, \quad n \geq 0.
\]

\[
\lambda_{2n+1} = 2^n n! \frac{(1 + 2\mu)(3 + 2\mu)...(2n + 1 + 2\mu)}{(1 + 2\mu + 2\alpha)(3 + 2\mu + 2\alpha)...(4n + 1 + 2\mu + 2\alpha)}, \quad n \geq 0.
\]
From the regularity condition (4.22), we have

\( \lambda_n \neq 0. \) (4.24)

On the other hand, simple computations show that \((\lambda_n)_{n \geq 1}\) satisfies the recurrence:

\( \lambda_{n+2} = \lambda_{n+1} - \gamma_{n+1} \lambda_n, \quad n \geq 1. \) (4.25)

Let us prove, by recurrence on \( n \), that

\( \lambda_n = S_n(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_{n-1}^{(1)}(1), \quad n \geq 1. \) (4.26)

Using (2.4), (2.5), (4.21) and (4.23), we get

\[
S_1(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_0^{(1)}(1) = 1 - \frac{2\alpha}{2\alpha + 2\mu + 1}
\]
\[
= 1 + 2\mu
\]
\[
= \lambda_1.
\]

Hence, (4.26) is true for \( n = 1 \).

Using (2.4), (2.5), (4.21) and (4.23), we obtain

\[
S_2(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_1^{(1)}(1) = S_1(1) - \gamma_1 S_0(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_1^{(1)}(1)
\]
\[
= 1 - \frac{2\alpha}{2\alpha + 2\mu + 1}
\]
\[
= \frac{1 + 2\mu}{3 + 2\mu + 2\alpha}
\]
\[
= \lambda_2.
\]

Hence, (4.26) is true for \( n = 2 \).

Suppose that (4.26) is true until \( n + 1, \quad n \geq 1 \) and let us prove it for \( n + 2 \). From (4.25) and the recurrence hypothesis, we have

\[
\lambda_{n+2} = \lambda_{n+1} - \gamma_{n+1} \lambda_n
\]
\[
= \left( S_{n+1}(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_n^{(1)}(1) \right) - \gamma_{n+1} \left( S_n(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_{n-1}^{(1)}(1) \right)
\]
\[
= \left( S_{n+1}(1) - \gamma_{n+1} S_n(1) \right) - \frac{2\alpha}{2\alpha + 2\mu + 1} \left( S_n^{(1)}(1) - \gamma_{n+1} S_{n-1}^{(1)}(1) \right), \quad n \geq 1.
\]

Taking into account the recurrences (2.4) and (2.5), we get

\[
\lambda_{n+2} = S_{n+2}(1) - \frac{2\alpha}{2\alpha + 2\mu + 1} S_{n+1}^{(1)}(1), \quad n \geq 1.
\]
So, (4.26) is true for every nonnegative integer \( n \). Then, according to (4.24), condition (4.18) is fulfilled. Therefore, \( u \) is regular.

According to (4.11), (4.19) and (4.20), \( u \) satisfies the following D-Pearson equation:

\[
D \left( (x^2 - 1)(x - 1)u \right) + (x - 1) \left( -2(\alpha + \mu + \frac{3}{2})x^2 + 2\mu + 1 \right) u = 0. 
\]

(4.27)

Let us see when (4.27) can be simplified by \( x - 1 \). From (4.17), we have

\[
(u)_2 = (u)_1 = \frac{1 + 2\mu}{1 + 2\mu + 2\alpha}.
\]

Then,

\[
< u, (x^2 - 1)(x - 1) > + \theta_1((x - 1)(-2(\alpha + \mu + \frac{3}{2})x^2 + 2\mu + 1)) \geq 0.
\]

Therefore, we can divide both hand sides of (4.27) by \( x - 1 \) taking into account (2.7) does not hold. Thus, we obtain

\[
D \left( (x^2 - 1)u \right) + \left( -2(\alpha + \mu + 1)x^2 + x + 2\mu + 1 \right) u = 0.
\]

(4.28)

Multiplication of the last equation by \((1 + 2\mu)x\) gives (1.1), where

\[
\Phi(x) = (x - 1)(x + \frac{1 + 2\mu}{1 - 2\mu}), \\
B(x) = x(x - 1)(x - \frac{2\mu + 1}{1 - 2\mu}), \\
\Psi(x) = \frac{1 + 2\mu}{\gamma_1}(x - \beta_0), \\
K = \frac{2\mu - 1}{\gamma_1}\beta_0, \\
\beta_0 = \frac{1 + 2\mu}{1 + 2\mu + 2\alpha}.
\]

On the other hand, from (4.17), we have

\[
(x - 1)u = \frac{-2\alpha}{2\alpha + 2\mu + 1}G(\alpha, \mu - \frac{1}{2}).
\]

(4.34)

Taking into account the fact that \( G(\alpha, \mu - \frac{1}{2}) \) is symmetric, we get

\[
(x - 1)u = h_{-1}((x - 1)u).
\]

(4.35)

Multiplying (4.35) by \( x + \frac{1 + 2\mu}{1 - 2\mu} \), we obtain (1.3).

According to (4.29)- (4.33), (1.2) is equivalent to the following condition:

\[
2\mu + 2\alpha + 1 \neq 2\mu[n] - n, \ n \geq 0.
\]
Table 4.1: Coefficients of the $T_\mu$-distributional equation (4.36)

<table>
<thead>
<tr>
<th>Linear functional</th>
<th>$\Omega$</th>
<th>$\varphi$</th>
<th>Restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Hermite $\mathcal{H}(\mu)$</td>
<td>1</td>
<td>$2x$</td>
<td>$\mu \neq -n - \frac{1}{2}, \ n \geq 0$.</td>
</tr>
<tr>
<td>Generalized Gegenbauer $\mathcal{G}(\alpha, \mu - \frac{1}{2})$</td>
<td>$x^2 - 1$</td>
<td>$-2(\alpha + 1)x$</td>
<td>$\alpha \neq -n, \alpha + \mu - \frac{1}{2} \neq -n, \mu - \frac{1}{2} \neq -n, \ n \geq 1$.</td>
</tr>
<tr>
<td>Modified Generalized Gegenbauer $\ u = \lambda(x-1)^{-1}\mathcal{G}(\alpha, \mu - \frac{1}{2}) + \delta_1$, $\lambda = \frac{-2\alpha}{2\alpha + 2\mu + 1}$</td>
<td>$x^2 - 1$</td>
<td>$-(2\alpha + 2\mu + 1)x + 1 + 2\mu$</td>
<td>$\alpha \neq -n, \alpha + \mu - \frac{1}{2} \neq -n, \mu - \frac{1}{2} \neq -n, \ n \geq 1, \alpha \neq 0$.</td>
</tr>
</tbody>
</table>

This last condition is an immediate consequence of (4.22). So, according to Theorem 1.1, $u$ is a Dunkl-classical linear functional. Furthermore, by virtue of (3.15) and (4.35), the linear functional $u$ satisfies

$$T_\mu((x^2 - 1)u) + \left(- (2\alpha + 2\mu + 1)x + 1 + 2\mu\right)u = 0.$$  

Notice that, for all Dunkl-classical linear functionals discussed before, the $T_\mu$-distributional equation (3.15) is reduced to another one of type (4.36)

$$T_\mu(\Omega u) + \varphi u = 0.$$  

where $\Omega$ and $\varphi$ are two polynomials such that $\text{deg}\Omega \leq 2$, $\text{deg}\varphi \leq 1$. To conclude this paper, we will present a table where we give polynomials $\Omega$ and $\varphi$ in (4.36) for each example of D-semiclassical linear functional previously studied.

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