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FUZZY IMPLICATIVE IDEALS OF SHEFFER STROKE BG-ALGEBRAS

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Abstract. In this paper, an (implicative) ideal and a fuzzy ideal of Sheffer stroke BG-algebra are defined and some properties are presented. Then a fuzzy implicative and a sub-implicative ideals of a Sheffer stroke BG-algebra are described. Morever, an implicative Sheffer stroke BG-algebra and a medial Sheffer stroke BG-algebra are defined, and it is expressed that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra. Also, a fuzzy (completely) closed ideal and a fuzzy p-ideal are determined. Finally, the relationships between these structures are shown.
Keywords: Sheffer stroke BG-algebra, fuzzy ideal, fuzzy implicative ideal, fuzzy sub-implicative ideal.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([6], [7]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim introduced a new notion called a B-algebra [12]. C. B. Kim and H. S. Kim [8] introduced a BG-algebra as a generalization of B-algebra. Then a BG-algebra consists of a non-empty set X with a binary operation $*$ and a constant 0 satisfying some axioms.

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In 1965, Zadeh introduced the notion of a fuzzy set and fuzzy subset of a set [22]. As a generalization of this, intuitionistic fuzzy subset was defined by K. T. Atanassov ([2], [3], [4]) in 1986. In 1971, Rosenfield introduced the concept of fuzzy sub-group [20]. Ahn and Lee studied fuzzy subalgebra of BG-algebra in [1]. Muthuraj et al. presented fuzzy ideals in BG-algebras in [10]. Also, Muthuraj and Devi introduced a multi-fuzzy subalgebra of BG-algebras in [11].

Sheffer stroke (or Sheffer operation) was first introduced by H. M. Sheffer [21]. Because any Boolean function or axiom can be expressed by means of only this operation [9], the most important application is to have all diods on the chip forming processor in a computer, that is, it is enough to produce a single diod for Sheffer operation. Thus, it is simpler and cheaper than to produce different diods for other Boolean operations. In addition, it has many algebraic applications in algebraic structures such as Sheffer stroke BG-algebras [13], interval Sheffer stroke basic algebras [19], Sheffer stroke Hilbert algebras [14] and fuzzy filters [15], filters of strong Sheffer stroke non-associative MV-algebras [17], (fuzzy) filters of Sheffer stroke BL-algebras [18], Sheffer stroke UP-algebras [16] and Sheffer operation in ortholattices [5].

After giving basic definitions and notions about a Sheffer stroke BG-algebra, an (implicative) ideal of a Sheffer stroke BG-algebra is defined. It is proved that every implicative ideal of a Sheffer stroke BG-algebra is its ideal. By describing a fuzzy (implicative) ideal of this algebraic structure, the relationship between them is shown. After determining a fuzzy sub-implicative ideal of a Sheffer stroke BG-algebra, it is proved that every fuzzy sub-implicative ideal of a Sheffer stroke BG-algebra is the fuzzy ideal. An implicative Sheffer stroke BG-algebra is defined and it is indicated that every fuzzy ideal of a Sheffer stroke BG-algebra is its fuzzy sub-implicative ideal if the algebraic structure is implicative. Then a medial Sheffer stroke BG-algebra is described and it is expressed that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra. Morever, a fuzzy (completely) closed ideal and a fuzzy p-ideal of a Sheffer stroke BG-algebra are determined and the relationships between them are indicated. It is shown that every fuzzy completely closed ideal of an implicative Sheffer stroke BG-algebra is the fuzzy implicative ideal under one condition. Finally, it is stated that every fuzzy p-ideal of a Sheffer stroke BG-algebra is the fuzzy implicative ideal if this algebra equals to the BCA-part.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a BG-algebra.

Definition 2.1. [5] Let $\mathcal{A} = \langle A, |\rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the following conditions:
$(S1)$ $a_1|a_2 = a_2|a_1$,
$(S2)$ $(a_1|a_1)|(a_1|a_2) = a_1$,

$(S3)$ $a_1|((a_2|a_3)|(a_2|a_3)) = ((a_1|a_2)|(a_1|a_2))|a_3$,
$(S4)$ $(a_1|((a_1|a_1)|(a_2|a_2)))|(a_1|((a_1|a_1)|(a_2|a_2))) = a_1$.

Definition 2.2. [13] A Sheffer stroke BG-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:
$(sBG.1)$ $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0$,
$(sBG.2)$ $(0|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = a_1|a_1$,
for all $a_1, a_2 \in A$.

Let A be a Sheffer stroke BG-algebra, unless otherwise is indicated.

Lemma 2.1. [13] *Let A be a Sheffer stroke BG-algebra. Then the following features hold:*

1. $(0|0)|(a_1|a_1) = a_1$,
2. $(a_1|(0|0))|(a_1|(0|0)) = a_1$,
3. $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = (a_3|(a_2|a_2))|(a_3|(a_2|a_2))$ *implies* $a_1 = a_3$,
4. $(0|(0|(a_1|a_1))) = a_1|a_1$,
5. *If* $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$ *then* $a_1 = a_2$,
6. *If* $(0|(a_1|a_1)) = (0|(a_2|a_2))$ *then* $a_1 = a_2$,
7. $(((a_1|(0|(a_1|a_1)))|(a_1|(0|(a_1|a_1))))|(a_1|a_1)) = a_1|a_1$,
8. $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$,

for all $a_1, a_2, a_3 \in A$.

3. Some Types Of Fuzzy Ideals

Definition 3.1. Let I be a nonempty subset of a Sheffer stroke BG-algebra. Then I is called an ideal of A if it satisfies:
$(sI1)$ $0 \in I$,
$(sI2)$ $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$ and $a_2 \in I$ imply $a_1 \in I$.

Definition 3.2. A nonempty subset I of a Sheffer stroke BG-algebra A is called an implicative ideal of A if
(i) $0 \in I$,
(ii) $(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \in I$ and $a_3 \in I$ imply $a_1 \in I$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.1. *Every implicative ideal of a Sheffer stroke BG-algebra A is an ideal of A.*

Proof. Let I be an implicative ideal of A. Then $0 \in I$ from Definition 3.2 (i). Assume that $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$ and $a_2 \in I$. Since
$(((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2|a_2))|$
$(((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2|a_2))$
$= (((a_1|(0|0))|(a_1|(0|0)))|(a_2|a_2))|(((a_1|(0|0))|(a_1|(0|0)))|(a_2|a_2))$
$= (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I,$
from (S2), (sBG.1) and Lemma 2.1 (2), we obtain from Definition 3.2 (ii) that $a_1 \in I$. Therefore, I is an ideal of A. □

Definition 3.3. A fuzzy subset μ of a Sheffer stroke BG-algebra A is called a fuzzy ideal of A if it satisfies the following conditions:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu(a_1) \geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\}$,
for all $a_1, a_2 \in A$.

Lemma 3.1. *Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. If*

$$a_1 \leq a_2 \text{ if and only if } (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$$

holds for all $a_1, a_2 \in A$, then $\mu(a_1) \geq \mu(a_2)$ if $a_1 \leq a_2$.

Proof. Let $a_1 \leq a_2$. Then $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$. Thus,

$$\begin{aligned} \mu(a_1) &\geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\} \\ &= \min\{\mu(a_2), \mu(0)\} \\ &= \mu(a_2) \end{aligned}$$

from Definition 3.3 (ii) and (i), respectively. □

Lemma 3.2. *Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. If $\mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) = \mu(0)$, then $\mu(a_1) \geq \mu(a_2)$, for any $a_1, a_2 \in A$.*

Proof. It is obvious from Definition 3.3. □

Definition 3.4. A fuzzy subset μ of a Sheffer stroke BG-algebra A is called a fuzzy implicative ideal of A if it satisfies:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu(a_1) \geq \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))$
$|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\}$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.2. *Every fuzzy implicative ideal of a Sheffer stroke BG-algebra A is a fuzzy ideal of A.*

Proof. Let μ be a fuzzy implicative ideal of a Sheffer stroke BG-algebra A. Then $\mu(0) \geq \mu(a_1)$ from Definition 3.4 (i). Also,

$$\begin{aligned}\mu(a_1) &\geq \min\{\mu(a_2), \mu((((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2 \\ &\quad |a_2))|(((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2|a_2)))\} \\ &= \min\{\mu(a_2), \mu((((a_1|(0|0))|(a_1|(0|0)))|(a_2 \\ &\quad |a_2))|(((a_1|(0|0))|(a_1|(0|0)))|(a_2|a_2)))\} \\ &= \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\}\end{aligned}$$

by Definition 3.4 (ii), (S2), (sBG.1) and Lemma 2.1 (2). Therefore, μ is a fuzzy ideal of A. □

Theorem 3.1. *Let μ be a fuzzy subset of a Sheffer stroke BG-algebra A. Then μ is a fuzzy (implicative) ideal of A if and only if a level subset $\mu_x = \{a \in A : \mu(a) \geq x\} \neq \emptyset$ of A is an (implicative) ideal of A.*

Proof. Let μ be a fuzzy ideal of A and $\mu_x \neq \emptyset$. Since it follows from Definition 3.3 (i) that $\mu(0) \geq \mu(a) \geq x$, for $a \in \mu_x$, we get that $0 \in \mu_x$. Assume that $a_2, (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in \mu_x$. Since $\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \geq x$, it is obtained from Definition 3.3 (ii) that

$$\mu(a_1) \geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\} \geq x,$$

which implies that $a_1 \in \mu_x$. Thus, μ_x is an ideal of A. Also, let μ be a fuzzy implicative ideal of A and $\mu_x \neq \emptyset$. Suppose that a_3, $(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \in \mu_x$. Since

$$\begin{aligned}&\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \\ &\quad |(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))) \geq x\end{aligned}$$

we have from Definition 3.4 (ii) that

$$\begin{aligned}&\mu(a_1) \geq \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))| \\ &(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\} \geq x,\end{aligned}$$

which means that $a_1 \in \mu_x$. Hence, μ_x is an implicative ideal of A.

Conversely, let $\mu_x \neq \emptyset$ be an ideal of A. Assume that $\mu(0) < \mu(a)$, for some $a \in A$. If $x = (\mu(0)+\mu(a))/2 \in (0, 1]$, then $\mu(0) < x < \mu(a)$. So, $0 \notin \mu_x$, which is contradiction with (sI1). Thereby, $\mu(0) \geq \mu(a)$, for all $a \in A$. Suppose that $x_1 = \mu(a_1) < \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\} = x_2$. If If $x_0 = (x_1 + x_2)/2 \in (0, 1]$, then $x_1 < x_0 < x_2$. Thus, $a_2, (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in \mu_{x_0}$ but $a_1 \notin \mu_{x_0}$, which contradicts with (sI2). Then $\mu(a_1) \geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\}$, for all $a_1, a_2 \in A$. Hence, μ is a fuzzy ideal of A. Moreover, let $\mu_x \neq \emptyset$ be an implicative ideal of A. Assume that $y_1 = \mu(a_1) < \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\} = y_2$. If $x^* = (y_1 +$

$y_2)/2 \in (0,1]$, then $y_1 < x^* < y_2$. So, a_3, $(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \in \mu_{x^*}$ but $a_1 \notin \mu_{x^*}$, which contradicts with Definition 3.2 (ii). Hence,

$$\mu(a_1) \geq \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))) |(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\},$$

for all $a_1, a_2, a_3 \in A$. Therefore, μ is a fuzzy implicative ideal of A. □

Definition 3.5. A fuzzy subset μ of a Sheffer stroke BG-algebra A is called a fuzzy sub-implicative ideal of A if it satisfies:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) \geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1 |(a_2|a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.3. *Let A be a Sheffer stroke BG-algebra. Then every fuzzy sub-implicative ideal of A is a fuzzy ideal of A.*

Proof. Let μ be a fuzzy sub-implicative ideal of A. Then $\mu(0) \geq \mu(a_1)$ from Definition 3.5 (i). We get from (sBG.1), Lemma 2.1 (2), Definition 3.5 (ii) that

$$\begin{aligned}
\mu(a_1) &= \mu((a_1|(0|0))|(a_1|(0|0))) \\
&= \mu((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1)))) \\
&\geq \min\{\mu((((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_3|a_3))| \\
&\quad (((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\} \\
&= \min\{\mu((((a_1|(0|0))|(a_1|(0|0)))|(a_3|a_3))| \\
&\quad (((a_1|(0|0))|(a_1|(0|0)))|(a_3|a_3))), \mu(a_3)\} \\
&= \min\{\mu((a_1|(a_3|a_3))|(a_1|(a_3|a_3))), \mu(a_3)\}.
\end{aligned}$$

Therefore, μ is a fuzzy ideal of A. □

Theorem 3.2. *Let A be a Sheffer stroke BG-algebra and μ be a fuzzy ideal of A. Then μ is a fuzzy sub-implicative ideal of A if and only if*

$$\begin{aligned}
&\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) \\
&\geq \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))).
\end{aligned} \tag{3.1}$$

Proof. Let μ be a fuzzy sub-implicative ideal of A. We have from Lemma 2.1 (2) and Definition 3.5 (ii) that

$$\begin{aligned}
\mu(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) &\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1| \\
&\quad (a_2|a_2))))|(0|0))|(((a_1|(a_1|(a_2|a_2))) \\
&\quad |(a_1|(a_1|(a_2|a_2))))|(0|0))), \mu(0)\} \\
&= \min\{\mu(a_1|(a_1|(a_2|a_2)))|, \\
&\quad (a_1|(a_1|(a_2|a_2)))\mu(0)\} \\
&= \mu(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))).
\end{aligned}$$

Conversely, since μ is a fuzzy ideal, it follows that
(i) $\mu(0) \geq \mu(a_1)$,
(ii)

$$\begin{aligned}\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) &\geq \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))\\ &\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2\\ &\quad |a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2)))|\\ &\quad (a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}.\end{aligned}$$

Therefore, μ is a fuzzy sub-implicative ideal of A. □

Definition 3.6. A Sheffer stroke BG-algebra is said to be implicative if it satisfies the condition

$$a_1|(a_1|(a_2|a_2)) = a_2|(a_2|(a_1|a_1)), \tag{3.2}$$

for all $a_1, a_2 \in A$.

Theorem 3.3. *Let A be an implicative Sheffer stroke BG-algebra. Then every fuzzy ideal of A is a fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy ideal of A. Then
(i) $\mu(0) \geq \mu(a_1)$,
(ii)

$$\begin{aligned}&\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))\\ &\geq \min\{\mu((((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))|\\ &\quad (((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\}\\ &= \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))|\\ &\quad (((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}.\end{aligned}$$

Thereby, μ is a fuzzy sub-implicative ideal of A. □

Definition 3.7. A Sheffer stroke BG-algebra A is called medial if

$$a_1|(a_1|(a_2|a_2)) = a_2|a_2, \tag{3.3}$$

for all $a_1, a_2 \in A$.

Lemma 3.3. *In a Sheffer stroke BG-algebra A, the following property holds:*

$$((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)) = a_1|(a_1|(a_2|a_2)),$$

for all $a_1, a_2 \in A$.

Proof. It follows from (S1), (S2) and (S3) that

$$((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1))$$
$$= (((a_1|(a_2|a_2))|a_1)|((a_1|(a_2|a_2))|a_1))|(a_2|(a_1|a_1))$$
$$= (a_1|(a_2|a_2))|((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))$$
$$=(a_1|(a_2|a_2))|((((a_1|a_1)|(a_1|a_1))|(a_2|(a_1|a_1)))|(((a_1|a_1)|(a_1|a_1))|(a_2|(a_1|a_1))))$$
$$= (a_1|(a_2|a_2))|((((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2))|(((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2)))$$
$$= (a_1|(a_2|a_2))|((a_1|a_1)|(a_1|a_1))$$
$$= a_1|(a_1|(a_2|a_2)).$$

□

Theorem 3.4. *Every fuzzy ideal of a medial Sheffer stroke BG-algebra A is a fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy ideal of a medial Sheffer stroke BG-algebra A. It is obtained from (S2), Definition 3.3, Definition 3.7 and Lemma 3.3 that
(*i*) $\mu(0) \geq \mu(a_1)$,
(*ii*)

$$\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))$$
$$= \mu(a_1)$$
$$\geq \min\{\mu((a_1|(a_3|a_3))|(a_1|(a_3|a_3))), \mu(a_3)\}$$
$$= \min\{\mu((((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))|$$
$$(((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\}$$
$$= \min\{\mu(((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))|$$
$$(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))|(a_3|a_3))|$$
$$((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))|(((a_1|(a_1$$
$$|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))|(a_3|a_3))), \mu(a_3)\}$$
$$= \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))|$$
$$(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}.$$

Hence, μ is a fuzzy sub-implicative ideal of A. □

Theorem 3.5. *Let A be a Sheffer stroke BG-algebra satisfying*

$$\mu((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) \geq \mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \tag{3.4}$$

for all $a_1, a_2, a_3 \in A$. Then every fuzzy ideal of A is a fuzzy sub-implicative ideal of A.

Proof. It is obtained from the inequality (3.3), (S2) and Lemma 3.3 that

$$\begin{aligned}\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) &\geq \mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \\ &\quad |(a_2|(a_1|a_1)))|(((a_1|(a_1|(a_2|a_2)))| \\ &\quad (a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))) \\ &= \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))).\end{aligned}$$

Thus, μ is a fuzzy sub-implicative ideal of A by Theorem 3.2. □

Theorem 3.6. *Every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra.*

Proof. Let A be a medial Sheffer stroke BG-algebra. Then it follows from Lemma 3.3, Definition 3.7 and (S2) that

$$\begin{aligned}a_1|(a_1|(a_2|a_2)) &= ((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)) \\ &= ((a_2|a_2)|(a_2|a_2))|(a_2|(a_1|a_1)) \\ &= a_2|(a_2|(a_1|a_1)).\end{aligned}$$

Therefore, A is an implicative Sheffer stroke BG-algebra. □

Theorem 3.7. *Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. Then μ is a fuzzy implicative ideal of A if and only if μ satisfies the following condition:*

$$\mu(a_1) \geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))), \tag{3.5}$$

for all $a_1, a_2 \in A$.

Proof. ($\Rightarrow$) Let μ be a fuzzy implicative ideal of A. Then we get from Lemma 2.1 (2) that

$$\begin{aligned}\mu(a_1) &\geq \min\{\mu(0), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0 \\ &\quad |0))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|0)))\} \\ &= \min\{\mu(0), \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))\} \\ &= \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))),\end{aligned}$$

for all $a_1, a_2 \in A$.

($\Leftarrow$) Let μ be a fuzzy ideal of A satisfying the inequality (3.4). Then it is clear that $\mu(0) \geq \mu(a_1)$, for all $a_1 \in A$. Since

$$\begin{aligned}\mu(a_1) &\geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))) \\ &\geq \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))| \\ &\quad (((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\},\end{aligned}$$

for all $a_1, a_2, a_3 \in A$, we have that μ is a fuzzy implicative ideal of A. □

Theorem 3.8. *Let A be a medial Sheffer stroke BG-algebra satisfying*

$$\mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))), \tag{3.6}$$

for all $a_1, a_2 \in A$. Then every fuzzy sub-implicative ideal of A is a fuzzy implicative ideal of A.

Proof. Let μ be a fuzzy sub-implicative ideal of a medial Sheffer stroke BG-algebra A satisfying the inequality (3.5). Then we obtain from Definition 3.7, (S2), Definition 3.5, Lemma 2.1 (2) and the inequality (3.5) that

$$\begin{aligned}
\mu(a_1) &= \mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) \\
&\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(0|0)) \\
&\quad |(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(0|0))), \mu(0)\} \\
&= \min\{\mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))), \mu(0)\} \\
&= \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \\
&\geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))).
\end{aligned}$$

Thus, μ is a fuzzy implicative ideal of A by Theorem 3.7. □

Theorem 3.9. *Let A be an implicative Sheffer stroke BG-algebra. Then every fuzzy implicative ideal of A is a fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy implicative ideal of an implicative Sheffer stroke BG-algebra A. Then μ is a fuzzy ideal of A by Proposition 3.2. So, it is obvious that $\mu(0) \geq \mu(a_1)$, for all $a_1 \in A$. Thus, it follows from Definition 3.6 and Definition 3.3 (ii) that

$$\begin{aligned}
\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) &= \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \\
&\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2 \\
&\quad |a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2))) \\
&\quad |(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\},
\end{aligned}$$

for all $a_1, a_2, a_3 \in A$. Hence, μ is a fuzzy sub-implicative ideal of A. □

Corollary 3.1. *Let A be a medial Sheffer stroke BG-algebra. Then every fuzzy implicative ideal of A is a fuzzy sub-implicative ideal of A.*

Definition 3.8. A fuzzy ideal μ of a Sheffer stroke BG-algebra A is said to be fuzzy closed if

$$\mu((0|(a_1|a_1))|(0|(a_1|a_1))) \geq \mu(a_1), \tag{3.7}$$

for all $a_1 \in A$.

Definition 3.9. Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. Then μ is called a fuzzy completely closed ideal of A if

$$\mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \geq \min\{\mu(a_1), \mu(a_2)\},$$

for all $a_1, a_2 \in A$.

Theorem 3.10. *Let A be a Sheffer stroke BG-algebra satisfying*

$$((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3))))|(a_3|(a_2|a_2)) = 0|0, \tag{3.8}$$

for all $a_1, a_2, a_3 \in A$. Then A is implicative if and only if every fuzzy closed ideal of A is a fuzzy implicative ideal of A.

Proof. Let A be a Sheffer stroke BG-algebra satisfying the equation (3.8).

($\Rightarrow$) Assume that A is implicative and μ is a fuzzy closed ideal of A. Then μ is a fuzzy ideal of A. Thus,
(i) $\mu(0) \geq \mu(a_1)$.
(ii)

$$\begin{aligned}\mu(a_1) &\geq \min\{\mu(a_3), \mu((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))\} \\ &= \min\{\mu(a_3), \mu(((((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2))|(a_3 \\ &\quad |a_3))|((((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2))|(a_3|a_3)))\} \\ &= \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3 \\ &\quad |a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\},\end{aligned}$$

which means that μ is a fuzzy implicative ideal of A.

($\Leftarrow$) Suppose that every fuzzy closed ideal of A is a fuzzy implicative ideal of A. So, it follows from the equation (3.8), (S1)-(S2) and Lemma 2.1 (5) that $a_3|(a_2|a_2) = (a_1|(a_3|a_3))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))$. Since $a_3|(a_2|a_2) = (a_1|(a_3|a_3))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) = ((a_1|(a_1|(a_3|a_3)))|(a_1|(a_1|(a_3|a_3))))|(a_2|a_2)$ from (S1) and (S3), it is obtained from (S2) and Lemma 2.1 (3) that $a_3 = (a_1|(a_1|(a_3|a_3)))|(a_1|(a_1|(a_3|a_3)))$. Thus, we get from (S1)-(S3) and Lemma 2.1 (8) that

$$\begin{aligned}a_1|(a_1|(a_2|a_2)) &= ((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(((a_2 \\ &\quad |(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_2|a_2)) \\ &= ((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_2|(((a_2 \\ &\quad |a_2)|(a_2|(a_1|a_1)))|((a_2|a_2)|(a_2|(a_1|a_1))))) \\ &= ((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_2|(a_2|a_2)) \\ &= (((a_2|(a_2|a_2))|((a_2|a_2)|(a_2|a_2)))|((a_2| \\ &\quad (a_2|a_2))|((a_2|a_2)|(a_2|a_2))))|(a_2|(a_1|a_1)) \\ &= a_2|(a_2|(a_1|a_1)),\end{aligned}$$

for all $a_1, a_2 \in A$, which means that A is implicative. □

Proposition 3.4. *Let A be an implicative Sheffer stroke BG-algebra satisfying the equation (3.8). Then every fuzzy completely closed ideal of A is a fuzzy implicative ideal of A.*

Proof. Let μ be a fuzzy completely closed ideal of an implicative Sheffer stroke BG-algebra A. Then μ is a fuzzy ideal of A. Since $\mu((0|(a_2|a_2))|(0|(a_2|a_2))) \geq \min\{\mu(0), \mu(a_2)\} = \mu(a_2)$, it is obtained that μ is a fuzzy closed ideal of A. Therefore, μ is a fuzzy implicative ideal of A from Theorem 3.10. □

Corollary 3.2. *Let A be a medial Sheffer stroke BG-algebra satisfying the equation (3.8). Then every fuzzy completely closed ideal of A is a fuzzy implicative ideal of A.*

Definition 3.10. A fuzzy set μ of a Sheffer stroke BG-algebra A is called a fuzzy p-ideal of A if it satisfies:
(*i*) $\mu(0) \geq \mu(a_1)$,
(*ii*) $\mu(a_1) \geq \min\{\mu((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_3|$
$a_3)))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_3|a_3)))), \mu(a_2)\}$,
for all $a_1, a_2, a_3 \in A$.

Definition 3.11. Let A be a Sheffer stroke BG-algebra. Then the set $A_+ = \{a_1 \in A : (0|(a_1|a_1))|(0|(a_1|a_1)) = 0\}$ is called the BCA-part of A.

Theorem 3.11. *Let $A = A_+$ be a Sheffer stroke BG-algebra. Then every fuzzy p-ideal of A is a fuzzy implicative ideal of A.*

Proof. Let μ be a fuzzy p-ideal of A. Since

$$\begin{aligned}
\mu(a_1) &\geq \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|(a_2|(a_1|a_1))))| \\
&\quad (((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|(a_2|(a_1|a_1))))), \mu(0)\} \\
&= \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|0))| \\
&\quad (((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|0))), \mu(0)\} \\
&= \min\{\mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))), \mu(0)\} \\
&= \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))),
\end{aligned}$$

from Definition 3.10 (i)-(ii), (S2) and Lemma 2.1 (2), it follows from Theorem 3.7 that μ is a fuzzy implicative ideal of A. □

4. Conclusion

In this study, we introduce a fuzzy ideal, a fuzzy implicative ideal, a fuzzy sub-implicative ideal, a fuzzy (completely) closed ideal and a fuzzy p-ideal of a Sheffer

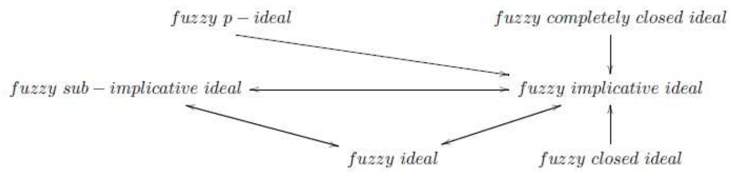

Fig. 3.1: Diagram of some types of fuzzy ideals

stroke BG-algebra and investigate some properties. After giving basic definitions and notions about a Sheffer stroke BG-algebra, we define an (implicative) ideal of a Sheffer stroke BG-algebra and prove that every implicative ideal of a Sheffer stroke BG-algebra is the ideal. Also, we determine a fuzzy ideal, a fuzzy implicative ideal and a fuzzy sub-implicative ideal on this algebraic structure. Besides, we construct an (implicative) ideal of a Sheffer stroke BG-algebra by means of its fuzzy (implicative) ideal and vice versa. It is shown that every fuzzy ((sub-)implicative) ideal of a Sheffer stroke BG-algebra is its fuzzy ideal. Besides, we examine the cases which the inverses hold. Morever, we describe an implicative Sheffer stroke BG-algebra and a medial Sheffer stroke BG-algebra and indicate that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra. It is demonstrated that every fuzzy ideal of an implicative (or medial) Sheffer stroke BG-algebra is the fuzzy sub-implicative ideal. It is indicated that every fuzzy sub-implicative ideal of a Sheffer stroke BG-algebra is the fuzzy implicative ideal when the algebra is a medial Sheffer stroke BG-algebra with a special condition, and every fuzzy implicative ideal of an implicative (or medial) Sheffer stroke BG-algebra is its fuzzy sub-implicative ideal. Finally, a fuzzy (completely) closed ideal and a fuzzy p-ideal of this algebraic structure are determined and the relationship between them are examined. By BCA-part of a Sheffer stroke BG-algebra, we prove that every fuzzy p-ideal of a Sheffer stroke BG-algebra is its fuzzy implicative ideal when the algebraic structure equals to the BCA-part.

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REFERENCES

1. S. S. Ahn and H. D. Lee: *Fuzzy subalgebras of BG-algebras*. Commun. Korean Math. Soc. **19** (2004), 243–251.
2. K. T. Atanassov: *More on intuitionistic fuzzy sets*. Fuzzy Sets and Systems **33(1)** (1989), 37–45.
3. K. T. Atanassov: *On intuitionistic fuzzy sets theory*. SpringerVerlag Berlin Heidelberg, 2002.

4. K. T. Atanassov: *Intuitionistic fuzzy sets*. In: VII ITKR's Session, Sofia, Deposed in Central Sci. -Techn. Library of Bulg. Acad. of Sci., 1983.
5. I. Chajda: *Sheffer operation in ortholattices*. Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **44(1)** (2005), 19–23.
6. K. Iséki and S. Tanaka: *An introduction to the theory of BCK-algebras*. Mathematica Japonica **23** (1978), 1–26.
7. K. Iséki: *On BCI-algebras*. Mathematics Seminar Notes **8** (1980), 125–130.
8. C. B. Kim and H. S. Kim: *On BG-algebras*. Demonstratio Mathematica **41** (2008) 497–505.
9. W. McCune and R. Veroff and B. Fitelson and K. Harris and A. Feist and L. Wos: *Short single axioms for Boolean algebra*. Journal of Automated Reasoning **29(1)** (2002), 1–16.
10. R. Muthuraj and M. Sridharan and P. M. S. Selvam *Fuzzy BG-ideals in BG-algebras*. International Journal of Computer Applications **2(1)** (2010), 26–30.
11. R. Muthuraj and S. Devi: *Multi-fuzzy subalgebras of BG-Algebra and its level subalgebras*. International Journal of Applied Mathematical Sciences **9(1)** (2016), 113–120.
12. J. Neggers and H. S. Kim: *On B-algebras*. Mat. Vesnik **54** (2002), 21–29.
13. T. Oner and T. Kalkan and N. Kircali Gursoy: *Sheffer stroke BG-algebras*. International Journal of Maps in Mathematics **4(1)** (2021), 27–30.
14. T. Oner and T. Katican and A. Borumand Saeid: *Relation between Sheffer stroke operation and Hilbert algebras*. Categories and General Algebraic Structures with Applications **14(1)** (2021), 245–268.
15. T. Oner and T. Katican and A. Borumand Saeid: *Fuzzy filters of Sheffer stroke Hilbert algebras*. Journal of Intelligent and Fuzzy Systems **40(1)** (2021), 759–772.
16. T. Oner and T. Katican and A. Borumand Saeid: *On Sheffer stroke UP-algebras*. Discussiones Mathematicae General Algebra and Applications (2021).
17. T. Oner and T. Katican and A. Borumand Saeid and M. Terziler: *Filters of strong Sheffer stroke non-associative MV-algebras*. Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica **29(1)** (2021), 143–164.
18. T. Oner and T. Katican and A. Borumand Saeid: *(Fuzzy) filters of Sheffer stroke BL-algebras*. Kragujevac Journal of Mathematics **47(1)** (2023), 39–55.
19. T. Oner and T. Katican and A. Ulker: *Interval Sheffer stroke basic algebras*. TWMS Journal of Applied and Engineering Mathematics **9(1)** (2019), 134–141.
20. A. Rosenfield: *Fuzzy groups*. J. Math. Anal. Appl. **35** (1971), 512–517.
21. H. M. Sheffer: *A set of five independent postulates for Boolean algebras, with application to logical constants*. Transactions of the American Mathematical Society **14(4)** (1913), 481–488.
22. L. A. Zadeh: *Fuzzy sets*. Inform. Control. **8** (1965), 338–353.