FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 1 (2022), 121–135
https://doi.org/10.22190/FUMI210301011M
Original Scientific Paper

ON THE NUMBER OF CYCLES OF GRAPHS AND VC-DIMENSION *

Alireza Mofidi[1,2]

[1] **Department of Mathematics and Computer Science**
Amirkabir University of Technology (Tehran Polytechnic), Iran
[2] **School of Mathematics, Institute for Research in Fundamental Sciences (IPM)**
P.O. Box 19395-5746, Tehran, Iran

Abstract. The number of cycles in a graph is an important well-known parameter in graph theory and there are a lot of investigations carried out in the literature for finding suitable bounds for it. In this paper, we delve into studying this parameter and the cycle structure of graphs through the lens of the cycle hypergraphs and VC-dimension and find some new bounds for it, where the cycle hypergraph of a graph is a hypergraph with the edges of the graph as its vertices and the edge sets of the cycles as its hyperedges respectively. Note that VC-dimension is an important notion in extremal combinatorics, graph theory, statistics, machine learning and logic. We investigate cycle hypergraph from the perspective of VC-theory, specially the celebrated Sauer-Shelah lemma, in order to give our upper and lower bounds for the number of cycles in terms of the (dual) VC-dimension of the cycle hypergraph and nullity of graph. We compute the VC-dimension and the mentioned bounds in some graph classes and also show that in certain classes, our bounds are sharper than many previous ones in the literature.
Key words: VC-dimension, Number of cycles, cycle hypergraph, cycle structure of graphs

Received March 1, 2021, accepted: December 10, 2021
Communicated by Bojan Bašić
Corresponding Author: Alireza Mofidi, Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran | School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran | E-mail: mofidi@aut.ac.ir
2010 *Mathematics Subject Classification.* 05C38, 05C65, 03C13.

*The author is indebted to Institute for Research in Fundamental Sciences, IPM, for support. This research was in part supported by a grant from IPM (No.1400030117).

1. Introduction

The number of cycles in a graph is an important parameter in graph theory. Investigation on this number and finding suitable bounds for that has a long history in the literature of graph theory. For instance, this problem was studied for many classes of graphs such as the ones with a given cyclomatic number, planar graphs, 3-regular graphs and 4-regular graphs. Also, many efforts have been made to bound this parameter in terms of various other parameters of the graph such as the number of the vertices and edges (see for example [1], [2], [4], [11] and [14]). It is worth to mention that counting cycles in graphs has great importance in the study of networks, in particular social networks. It is also notable to say that from the algorithmic and computational point of view, it is known that counting cycles in graphs is a NP-complete problem.

Beside graph theory, the theory of hypergraphs (or set systems) is also a major line of research in combinatorics (see [6], [7], [9], [13], [17] as some classical references). These two subjects have intimate interactions to each other. Historically, considering hypergraphs associated to a graph and using VC-theory and many other hypergraph theoretic aspects of those hypergraphs for studying that graph has been among important examples of such interactions. In hypergraph theory, there are various invariants and parameters associated to a given hypergraph that encode many information about it. Amongst them, VC-dimension is a very important one. In numerous papers such as [5], [8], [10], [15], [16], [18], [19], [20], this notion was considered from different viewpoints. For example, VC-dimensions of many hypergraphs associated to a graph (such as the hypergraphs of connected sets of vertices (edges), paths, neighbourhoods, etc) were related to the study of various features of the graph. VC-theory was first initiated in the works of Vapnic and Chervonenkis on the base of the notion of VC-dimension. Then, the theory was developed by discovering important results such as the celebrated Sauer-Shelah lemma. Nowadays, VC-theory is an important part of many areas of mathematics, computer science and statistics, in particular, extremal combinatorics, the theory of machine learning and logic. An interested reader can see [3], [13] and [17] for more information on combinatorial aspects of the VC-dimension and VC-theory.

During the course of our investigation in this paper, we pursue two main goals. The first one is to relate (in Subsection 3.1.) the problem of finding suitable bounds for the number of cycles of a graph to the VC-dimension of its cycle hypergraph and a few other parameters of the graph and finding new bounds. Note that by the cycle hypergraph of a graph we mean a hypergraph with the edge set of the graph as its vertices and the edge sets of the cycles of the graph as its hyperedges. We will have a hypergraph and VC-theoretic viewpoints in the study of graphs and their cycle structures and find some upper and lower bounds for the number of cycles of a graph in terms of the VC-dimension and dual VC-dimension of the cycle hypergraph as well as some other important graph parameters such as the nullity of the graph. Through the paper, the machinery of VC-theory, in particular the celebrated Sauer-Shelah lemma, provides us with some important conceptual and technical tools. Indeed, our bounds rely on the analysis of the VC and dual

VC dimensions, the nullity of the graph, transversally witness number of the cycle hypergraph, and application of this lemma. Also, the notion of twin-freeness of the cycle hypergraph and our characterization of it come to the picture naturally. Then, in subsections 3.2. and 3.3., we compute VC-dimension and the mentioned bounds in some graph classes such as the friendship graphs and wheel graphs and furthermore, show that in certain graph classes, including these two as well as some classes of graphs with large edge-density, the given bounds are sharper than many of the previous ones in the literature such as the ones mentioned in [2] or in a conjecture by Kiraly in [14]. As the second goal of the paper, on the way to the first one, we try to elaborate the idea of studying graphs by means of the hypergraphs associated to them (in here mostly cycle hypergraph) and their parameters.

Organization of the rest of the paper is as follows. We review necessary preliminaries in Section 2. Then, we prove the main results as described above in Section 3.

2. Preliminaries

We start to review some notions from graph theory. Through the paper, we use the notation $G = (V(G), E(G))$ for representing a graph with vertex set $V(G)$ and edge set $E(G)$. Also if G' is a subgraph of G, then by $V(G')$ and $E(G')$ we mean the vertex set and edge set of G' respectively. For a subset U of the edges of G, by $V(U)$ we mean the set of the vertices appeared in the edges in U. Also by $[U]$ we mean the graph $(V(U), U)$. When clear from the context, we may use U instead of $[U]$. If each of two arbitrary sets A and B is a subgraph or subset of the edges of G, then by the notation $A[B]$ we mean the subgraph $[C]$ where C is the set of the edges of A with both ending vertices belonging to $V(A) \cap V(B)$. The maximum and minimum degrees of the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. By the circumference of G, denoted by $circ(G)$, we mean the size of the longest cycle in G. For any $A \subseteq E(G)$, by $G \setminus A$ we mean the graph $(V(G), E(G) \setminus A)$. Also if A is a subgraph of G, then by $G \setminus A$ we mean the graph $(V(G), E(G) \setminus E(A))$. For any subgraph A and edge e, by $A \setminus e$ and $A - e$ we mean $(V(A), E(A) \setminus \{e\})$. Similarly, by $A + e$ and $A \cup \{e\}$ we mean the subgraph $(V(A) \cup V(\{e\}), E(A) \cup \{e\})$. By a cut-set in a connected graph we mean a subset of the edges whose removal makes the graph disconnected. We call an edge of a connected graph a *bridge* if the set containing that single edge is a cut-set. We call a graph *bridgeless* if it has no bridge. By a minimal cut-set we mean a cut-set whose strict subsets are not cut-sets anymore. By a cut-vertex in a graph we mean a vertex whose removal increases the number of the connected components of the graph.

Let T be a spanning tree of a connected graph G. Note that for every edge $e \in E(G) \setminus E(T)$, the graph $T + e$ contains a unique cycle denoted by C_e. Each such cycle C_e is usually called a *fundamental cycle* of graph G with respect to the spanning tree T and edge e. On the other hand, for every $e \in E(T)$, the graph $T - e$ has exactly two connected components. The set of the edges of G with one end in each of those components is usually denoted by D_e and is called the *fundamental cut* of G with respect to spanning tree T and edge e. The cospanning tree corresponding to a spanning tree T of G is the graph $(V(G), E(G) \setminus E(T))$.

Definition 2.1. In a connected graph G with n vertices and m edges, by the nullity of G we mean the parameter $null(G) := m - n + 1$.

It is known that for any connected graph G, the parameter $null(G)$ is equal to the dimension of the null space of the oriented incidence matrix of G.

In the following, we recall the definition of two well-known classes of graphs, namely, "wheel graphs" and "friendship graphs". We will use them later in the paper.

Definition 2.2. By the *wheel graph* W_n of order $n \geqslant 4$, we mean a graph that consists of a cycle of length $n-1$ plus one other vertex which is connected to all vertices of that cycle and is called the center of the wheel graph.

Definition 2.3. For every $t \in \mathbb{N}$, the *friendship graph* F_t is the graph constructed by union of t copies of the cycle C_3 with a common vertex.

Now we review some notions from the theory of hypergraphs. By a hypergraph (or set system) $(X, \mathcal{F})$ in this paper we mean a finite set X, which is called the vertex set (or domain) of the hypergraph, equipped with a family $\mathcal{F}$ of subsets of X called hyperedges. We call $|X|$ and $|\mathcal{F}|$ the order and the size of the hypergraph respectively. For a hypergraph $(X, \mathcal{F})$ and $Y \subseteq X$ define $\mathcal{F} \cap Y := \{A \cap Y : A \in \mathcal{F}\}$. We call the new hypergraph $(Y, \mathcal{F} \cap Y)$ the *trace* of the hypergraph $(X, \mathcal{F})$ on Y. Also for every $A \in \mathcal{F}$, we call $A \cap Y$ the trace of A on Y. In a hypergraph $(X, \mathcal{F})$, a subset $Y \subseteq X$ is called *shattered* by $(X, \mathcal{F})$ if $\mathcal{F} \cap Y = \mathcal{P}(Y)$ where $\mathcal{P}(Y)$ is the power set of Y. The VC-dimension of $(X, \mathcal{F})$, denoted by $VC(X, \mathcal{F})$ is the largest integer n such that there exists a subset of X of size n which is shattered by $(X, \mathcal{F})$.

Definition 2.4. For every $d \in \mathbb{N} \cup \{0\}$, define the function $\phi_d : \mathbb{N} \to \mathbb{N}$ by $\phi_d(n) := \sum_{i=0}^{d} \binom{n}{i}$ for every $n \geqslant d$ and $\phi_d(n) := 2^n$ for every $n < d$. Moreover, for every $m \in \mathbb{N}$ and $d \geqslant 1$ define $\phi_d^{-1}(m) := \min\{n : \phi_d(n) \geqslant m\}$.

One can find the following important classical result in [21] or [22].

Theorem 2.1. Sauer-Shelah lemma *Let $(X, \mathcal{F})$ be a hypergraph with $d := VC(X, \mathcal{F})$. Then, for every $Y \subseteq X$ we have $|\mathcal{F} \cap Y| \leqslant \phi_d(|Y|)$.*

Corollary 2.1. *For every $n \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{0\}$, we have $\phi_d(n) \leqslant (n+1)^d$.*

Proof.

$$\begin{aligned}\phi_d(n) &= \sum_{i=0}^{d} \binom{n}{i} = \sum_{i=0}^{d} \frac{n!}{i!(n-i)!} \leqslant \sum_{i=0}^{d} \frac{n^i}{i!} \\ &\leqslant \sum_{i=0}^{d} n^i \frac{d!}{i!(d-i)!} = \sum_{i=0}^{d} n^i \binom{d}{i} = (n+1)^d.\end{aligned}$$

□

In the following, we define some important notions associated to hypergraphs.

Definition 2.5. By a *witness set* (or separating set) of a hypergraph $(X, \mathcal{F})$ we mean a subset $W \subseteq X$ such that for every $A, B \in \mathcal{F}$ we have $A \cap W \neq B \cap W$. By a *transversally witness set* of the hypergraph, we mean a witness set W such that every hyperedge in $\mathcal{F}$ (except $\varnothing$ in the case that $\varnothing \in \mathcal{F}$) has nonempty intersection with W. By a *minimal transversally witness set* we mean a transversally witness set whose strict subsets are not transversally witness sets anymore. We denote the smallest size among all minimal transversally witness sets of the hypergraph $(X, \mathcal{F})$ by $twt(X, \mathcal{F})$ and call it the *transversally witness number* of $(X, \mathcal{F})$. We denote the family of all minimal transversally witness sets of $(X, \mathcal{F})$ by $minTWT(X, \mathcal{F})$. Also we call the hypergraph $(X, minTWT(X, \mathcal{F}))$, the *minimal transversally witness hypergraph* of $(X, \mathcal{F})$.

Definition 2.6. Let $(X, \mathcal{F})$ be a hypergraph. Two distinct elements $x, y \in X$ are called *twins* if every $A \in \mathcal{F}$ either contains both of x and y or none of them. We call $(X, \mathcal{F})$ *twin-free* if there are no such twins x and y in X.

In the following definition, we recall a notion of duality in the hypergraph theory which we call the hypergraph duality. This notion has been an important tool in studying hypergraphs in many works in the literature. An interested reader can refer to for instance, the book [6](Chapter 17) for some more details about that.

Definition 2.7. The *hypergraph dual* (or simply, dual) of a hypergraph $(X, \mathcal{F})$ is a hypergraph $(X^*, \mathcal{F}^*)$ where $X^* := \mathcal{F}$ and $\mathcal{F}^* := \{\mathcal{A}_x : x \in X\}$ where for each $x \in X$, we define $\mathcal{A}_x := \{A \in \mathcal{F} : x \in A\}$.

Note that in Definition 2.7, we consider $\mathcal{F}^*$ as a set and not a multiset. In other words, if $\mathcal{A}_x = \mathcal{A}_y$ for two $x, y \in X$, then we identify $\mathcal{A}_x$ and $\mathcal{A}_y$ and they are counted as one member of $\mathcal{F}^*$ and not two members.

Definition 2.8. Let $(X, \mathcal{F})$ be a hypergraph. By the dual VC-dimension of $(X, \mathcal{F})$, denoted by $VC^*(X, \mathcal{F})$, we mean the VC-dimension of its dual. In other words, $VC^*(X, \mathcal{F}) := VC(X^*, \mathcal{F}^*)$.

3. VC-dimension and bounding the number of cycles in graphs

Investigation of the cycle structure of graphs has been always an important aspect of the study of graphs and networks. In particular, the problem of counting the number of cycles, usually denoted by $c(G)$ in a graph G, and bounding $c(G)$ for different classes of graph has a long history. Finding suitable bounds for $c(G)$ in terms of the various parameters of the graph was worked out in many papers such [1], [2], [4], [11] and [14]. It is worth to mention that counting cycles in graphs has many applications in network theory, in particular social networks.

We first review some classical results concerning bounds on $c(G)$. It was shown in [1] that $m-n+k \leqslant c(G) \leqslant 2^{m-n+k}-1$ for any graph G with n vertices, m edges and k connected components. The upper bound was improved in [2] for connected graphs by showing the following statement.

Theorem 3.1. (Aldred-Thomassen) *For any connected graph G we have $c(G) \leqslant \frac{15}{16}2^{m-n+1}$.*

The following statement was conjectured by Kiraly in [14].

Kiraly's conjecture: There is a constant α such that for any graph G with m edges $c(G) \leqslant \alpha(1.4)^m$.

An interested reader can refer to papers such as [4] and [11] to see the results about the above statement.

In this section, we consider the problem of finding suitable bounds for $c(G)$ from the perspective of the hypergraph and VC-theoretical viewpoints and give some bounds for $c(G)$ in terms of the new graph parameters defined in this paper, namely, the VC-dimension and dual VC-dimension of the cycle hypergraph (as defined below) and also some other important parameters of the graph such as its nullity.

Definition 3.1. Let G be a graph. By the *cycle hypergraph* of G, we mean the hypergraph $(E(G), \mathcal{CYC}(G))$ where $\mathcal{CYC}(G) := \{E(C) : \ C \text{ is a cycle in } G\}$.

The hypergraph theory (or the theory of set systems) is one of the central areas of research in combinatorics. In this theory, there are various features and notions associated to a given hypergraph and the hypergraph is usually studied from the viewpoints of those notions. VC-dimension and more generally, VC-theoretic features are among the most important aspects of a hypergraph. We have mentioned a brief history of VC-theory and the notion of VC-dimension and its connection to graphs in the introduction of the paper. In Subsection 3.1. below, we give some upper and lower bounds for the number of cycles of graphs by applying Sauer-Shelah lemma and utilizing VC-dimension and dual VC-dimension of the cycle hypergraph. Also in subsections 3.2. and 3.3., we will see that for certain classes of graphs, our bounds are sharper than some of the previous ones in the literature such as the ones in [2] and [14].

3.1. Some new upper and lower bounds for the number of cycles of graphs

We want to start analysing the number of cycles of graphs by using the notion of VC-dimension. We first prove a general statement for hypergraphs and use it later. We remind that the notion of dual hypergraph and notations such as $\mathcal{A}_x$ was defined in Definition 2.7. In the remaining of the paper, unless specifically mentioned otherwise, we consider 2 as the base of the logarithm in the notation log.

Proposition 3.1. *Let $(X, \mathcal{F})$ be a hypergraph with $\mathcal{F} \neq \varnothing$. Then, the following hold.*

1. *Let S be such that $\varnothing \neq S \subseteq X$. Assume that S is shattered by the hypergraph $(X, \mathcal{F})$.*

   Then, for every $x \in S$, we have $|S| \leqslant 1 + \log |\mathcal{A}_x|$.

2. *We have $VC(X, \mathcal{F}) \leqslant 1 + \log r$ where $r := \max\{|\mathcal{A}_x| : x \in X\}$.*

3. *We have $|\mathcal{F}| \leqslant 2^{twt(X,\mathcal{F})}$. Moreover, if $\varnothing \notin \mathcal{F}$, then $|\mathcal{F}| \leqslant 2^{twt(X,\mathcal{F})} - 1$.*

Proof. 1) Define $d := |S|$. Fix some $x \in S$ and let $\mathcal{H}$ be the family of subsets of S containing x. It is clear that $|\mathcal{H}| = 2^{d-1}$. Since S is a shattered set, there is a subfamily $\mathcal{U} \subseteq \mathcal{F}$ with $|\mathcal{U}| = 2^{d-1}$ such that $\mathcal{U} \cap S = \mathcal{H}$. Thus, every $A \in \mathcal{U}$ contains x. It follows that $\mathcal{U} \subseteq \mathcal{A}_x$. Therefore, $|\mathcal{A}_x| \geqslant |\mathcal{U}| = 2^{d-1}$ which follows that $d \leqslant 1 + \log |\mathcal{A}_x|$.

2) We may assume that $VC(X, \mathcal{F}) > 0$ since otherwise the result would be clear. Let $S \subseteq X$ be a shattered set with size $VC(X, \mathcal{F})$. Then, by using Part 1 of the present proposition, we have $VC(X, \mathcal{F}) = |S| \leqslant 1 + \log r$.

3) Let W be a transversally witness set of size $twt(X, \mathcal{F})$. Then, by the definition of transversally witness sets, for every $A, B \in \mathcal{F}$ we have $A \cap W \neq B \cap W$. It follows that $|\mathcal{F}| = |\mathcal{F} \cap W| \leqslant |\mathcal{P}(W)| = 2^{twt(X,\mathcal{F})}$. Moreover, if $\varnothing \notin \mathcal{F}$, then since W is a transversally witness set, we have $\varnothing \notin \mathcal{F} \cap W$. It implies that $|\mathcal{F}| = |\mathcal{F} \cap W| \leqslant |\mathcal{P}(W)| - 1 = 2^{twt(X,\mathcal{F})} - 1$. $\square$

The following statement provides a relationship between the dual VC-dimension of the cycle hypergraph of any graph G and the parameter $circ(G)$, the size of the longest cycle of G.

Corollary 3.1. *Let G be a graph containing at least one cycle and $d^* := VC^*(E(G), \mathcal{CYC}(G))$. Then, $2^{d^*-1} \leqslant circ(G)$.*

Proof. It is not very hard to see that by applying Proposition 3.1(2) for the hypergraph $(E(G)^*, \mathcal{CYC}(G)^*)$, which is the dual hypergraph of the cycle hypergraph, we have $d^* \leqslant 1 + \log (circ(G))$. The result follows. $\square$

The following two theorems, which are among the main results of this paper, give upper and lower bounds for the number of cycles in a graph in terms of the VC-dimension and the dual VC-dimension of the cycle hypergraph and some other parameters. In particular, the following theorem gives an upper bound for the number of cycles in terms of two important notions of VC-dimension and nullity.

Theorem 3.2. *Let G be a connected graph with n vertices and m edges and at least one cycle. Also let $d := VC(E(G), \mathcal{CYC}(G))$. Then, the following upper bounds hold for the number of cycles.*

$$c(G) \leqslant \phi_d(null(G)) \leqslant (null(G) + 1)^d \leqslant (null(G) + 1)^{1+\log s}, \qquad (*)$$

where s is the largest number of appearances of an edge in the cycles of G.

Proof. First, for every transversally witness set $W \subseteq E(G)$ of the hypergraph $(E(G), \mathcal{CYC}(G))$, define $d_W := VC(W, \mathcal{F}_W)$ where $\mathcal{F}_W := \mathcal{CYC}(G) \cap W$. For every minimal transversally witness set W of $(E(G), \mathcal{CYC}(G))$ we have $|\mathcal{F}_W| = |\mathcal{CYC}(G)| = c(G)$. So by using Sauer-Shelah lemma (Theorem 2.1), we have $c(G) = |\mathcal{F}_W| \leqslant \phi_{d_W}(|W|)$. It follows that

$$\begin{aligned} c(G) \quad \leqslant \quad & \min(\{\phi_{d_W}(|W|) \\ & \quad : W \text{ is a minimal transversally witness set of } (E(G), \mathcal{CYC}(G))\}). \end{aligned}$$

Now we consider the structure of the minimal transversally witness sets in the following claim.

Claim. A subset $A \subseteq E(G)$ is a minimal transversally witness set of $(E(G), \mathcal{CYC}(G))$ if and only if A is the edge set of a cospanning tree of G.

Proof of Claim: First assume that $A \subseteq E(G)$ is the edge set of a cospanning tree T_0 of G. So, since $E(G) \setminus A$ is the edge set of a tree, the edge set of every cycle of G has a nonempty intersection with A. We recall that for any spanning tree T of a graph, the edges of the cospanning tree of T correspond to the fundamental cycles with respect to T which in turn correspond to the elements of a basis for the cycle space of the graph as a vector space over the field $\mathbb{Z}_2$. We call the vector in the cycle space of G corresponding to an edge $e \in A$ by an A-fundamental vector of the cycle space of G and denote it by u_e. Hence, every cycle C in G, being viewed as a vector in the cycle space, is a unique linear combination (with coefficients from $\mathbb{Z}_2$) of the A-fundamental vectors of the cycle space of G. In such linear combination, call those A-fundamental vectors having coefficient 1 in the combination by the generating A-fundamental vectors of the cycle C. It is easily seen that any two different cycles of G have different sets of generating A-fundamental vectors. It is also not very difficult to observe that, for every cycle C of G and edge $e \in A$, we have $e \in E(C) \cap A$ if and only if u_e belongs to the set of generating A-fundamental vectors of C. Combining these facts, for any two cycles C_1 and C_2 we have $E(C_1) \cap A \neq E(C_2) \cap A$. It follows that A is a witness set for the hypergraph $(E(G), \mathcal{CYC}(G))$. So, since as mentioned above $E(C) \cap A \neq \varnothing$ for every cycle C, the set A is a transversally witness set for the cycle hypergraph. Moreover, for every $e \in A$, $A \setminus \{e\}$ has empty intersection with the edge set of some cycle (indeed the cycle C_e, the fundamental cycle with respect to the spanning tree on $E(G) \setminus T_0$ and edge e). So, $A \setminus \{e\}$ is not a transversally witness set. It follows that A is a minimal transversally witness set for the hypergraph $(E(G), \mathcal{CYC}(G))$.

For the other direction, assume that $A \subseteq E(G)$ is a minimal transversally witness set for the cycle hypergraph of G. So A has at least one common edge with every cycle of G. It follows that $E(G) \setminus A$ does not contain any cycle, or in other words is the edge set of a forest. Therefore, it is not hard to verify that A contains the edge set of some cospanning tree S. Now by the proof of the other direction mentioned above, the edge set of S is a transversally witness set for the cycle hypergraph. So, since A is a minimal transversally witness set, A must be the same as the edge set of S. It follows that A is the edge set of a cospanning tree of G. **Claim** $\square$

By using the above claim, for any minimal transversally witness set W of the hypergraph $(E(G), \mathcal{CYC}(G))$, we have $|W| = null(G)$. So, by combining the above facts, we have $c(G) \leqslant \phi_{d_W}(null(G))$ for any minimal transversally witness set W. Also it is easy to see that $d_W \leqslant d$. It implies that $\phi_{d_W}(null(G)) \leqslant \phi_d(null(G))$. Therefore, first two inequalities of $(*)$ are followed by combining these with Corollary 2.1.

Furthermore, one can observe that $s = \max\{|\mathcal{A}_e| : e \in E(G)\}$ (see Definition 2.7 for the notation $\mathcal{A}_e$ in the context of the dual systems). So, by Proposition 3.1(2), we have $d \leqslant 1 + \log s$. Hence, $(null(G)+1)^d \leqslant (null(G)+1)^{1+\log s}$. This completes the proof. $\square$

The following statement gives a lower bound for the number of cycles in terms of the (dual) VC-dimension. We remind that the notation $\phi_{d^*}^{-1}$ in the following theorem was defined in Definition 2.4.

Theorem 3.3. *Let G be a connected graph and let $d^* := VC^*(E(G), \mathcal{CYC}(G))$.*

1. *If the hypergraph $(E(G), \mathcal{CYC}(G))$ is twin-free (see Part 2 for a characterization of this property for the cycle hypergraphs), then we have*

   (i) $$c(G) \geqslant \phi_{d^*}^{-1}(m),$$

   Also in terms of the parameters m and ℓ we have

   (ii) $$c(G) \geqslant 2^{\frac{\log m}{1+\log \ell}} - 1,$$

   where $\ell := circ(G)$ is the size of the longest cycle of G.

2. *The hypergraph $(E(G), \mathcal{CYC}(G))$ is twin-free if and only if either G is 3-edge-connected or it has a bridge edge e such that two connected components of $G \setminus \{e\}$ are 3-edge-connected.*

Proof. 1) We consider the dual hypergraph of the cycle hypergraph, namely, $(E(G)^*, \mathcal{CYC}(G)^*)$ and use the notations of Definition 2.7. Since $(E(G), \mathcal{CYC}(G))$ is assumed to be twin-free, for any two $e_1, e_2 \in E(G)$, their corresponding members of the dual hypergraph, namely $\mathcal{A}_{e_1}$ and $\mathcal{A}_{e_2}$ are distinct. It implies that $m = |\mathcal{CYC}(G)^*|$. Now by applying Sauer-Shelah lemma (Theorem 2.1) on the dual hypergraph we get $m = |\mathcal{CYC}(G)^*| \leqslant \phi_{d^*}(|E(G)^*|) = \phi_{d^*}(c(G))$. It follows that $\phi_{d^*}^{-1}(m) \leqslant c(G)$, which establishes (i).

Moreover, by using Corollary 3.1, we have $d^* \leqslant 1 + \log \ell$. So, combining this fact with previous ones we get

$$m \leqslant \phi_{d^*}(c(G)) \leqslant \phi_{1+\log \ell}(c(G)) \leqslant (1 + c(G))^{1+\log \ell},$$

where we have used Corollary 2.1 in the last inequality. It follows that $2^{\frac{\log m}{1+\log \ell}} - 1 \leqslant c(G)$ which establishes (ii).

2) We first show the direction from left to right. If $(E(G), \mathcal{CYC}(G))$ is twin-free, then G has at most one bridge since any two bridges are twins in the cycle hypergraph since they both belong to no cycle. Now, as in the first case, assume that G has no bridge. We again use twin-freeness to show that in this case G is 3-edge-connected. Assume the opposite. Since G has no bridge, it is 2-edge-connected. So, since G is assumed to not be 3-edge-connected, it must contain a cut-set of size exactly 2, say $\{e_1, e_2\}$. Now it is easy to see that every cycle of G either contains both e_1 and e_2 or contains none of them. Hence, e_1 and e_2 are twins. This is a contradiction. Therefore, in this case G is 3-edge-connected. As in the second case, we assume that G has exactly one bridge e. So, each of the two connected components of $G \setminus \{e\}$ are 2-edge-connected. Now, one can repeat the above argument for each of those components of $G \setminus \{e\}$, showing that they are both 3-edge-connected.

Now we prove right to left side. Fix two arbitrary edges e_1 and e_2 of G. It is enough to show that they are not twins in the cycle hypergraph of G. If one of e_1 or e_2, say e_1, is a bridge, then by using the assumptions, e_1 would be the only bridge of G. So, in this case we are done since e_1 is not twin with any other edge of G, in particular e_2, since e_1 is the only edge not belonging to any cycle of G. Hence, we may assume that none of e_1 and e_2 are bridge edges. Let T be any spanning tree containing e_1 but not e_2. By using the assumptions, the fundamental cut of G with respect to the spanning tree T and edge e_1 contains at least 3 edges. Thus, at least one of those edges is different from e_1 and e_2. We denote that edge by e_3. Obviously, $e_3 \notin E(T)$. Now $T + e_3$ contains a unique cycle (which is indeed the fundamental cycle C_{e_1}) and that cycle contains e_1 but not e_2. Therefore, e_1 and e_2 are not twins in the cycle hypergraph. Now we can conclude that the hypergraph $(E(G), \mathcal{CYC}(G))$ is twin-free. $\square$

3.2. Examples of some classes with improved bounds

In this part, we will give some examples of graph classes in which the bounds given in this paper for the number of cycles are sharper than several earlier bounds in the literature. Indeed, comparing the upper bound $c(G) \leqslant (null(G) + 1)^d$ given in Theorem 3.2 with the Aldred-Thomassen bound $c(G) \leqslant \frac{15}{16} 2^{null(G)}$ (Theorem 3.1) or the bound in Kiraly's conjecture easily shows that in certain classes of graphs (mostly those with small VC-dimensions of the cycle hypergraphs comparing to their nullities), our bound given in Theorem 3.2 is sharper (as we will see in the following examples). For example, for a given d and for every graph with VC-dimension of the cycle hypergraph equal to d and large enough nullity, our bound is sharper than the other mentioned bounds. Note that for every given d, there are graphs with arbitrary large nullities but with the VC-dimension of their cycle hypergraphs equal to d. For instance, in each of the classes of graphs we consider in the following two examples, the VC-dimension of the cycle hypergraph is fixed while the nullity can be arbitrary large in the class.

The class of the friendship graphs

The friendship graph F_t (see Definition 2.3) has $2t+1$ vertices and $3t$ edges. So we have $null(F_t) = t$. Also it has exactly t cycles. Moreover, it is easy to see that $d := VC(E(F_t), \mathcal{CYC}(F_t)) = 1$ for every $t \geqslant 2$. In this case, the upper bound given in Aldred-Thomassen theorem (Theorem 3.1) for the number of cycles would be $\frac{15}{16}2^t$. On the other hand, our bound using the VC-dimension given in Theorem 3.2 would be $(null(F_t)+1)^d = null(F_t)+1$, which is equal to $t+1$. Therefore, our upper bound is almost sharp in the class of friendship graphs and is much sharper than the bound given by Aldred-Thomassen theorem. Also, in this case it is a sharper bound than the one in Kiraly's conjecture.

In the following statement, we generalize the situation of friendship graphs discussed above and show that the bound given in Theorem 3.2 for the number of cycles is almost sharp for every connected graph with the VC-dimension of the cycle hypergraph equal to 1.

Proposition 3.2. *The bound $(null(G)+1)^d$ given in Theorem 3.2 for $c(G)$ is either equal to $c(G)$ or to $c(G)+1$ for every connected graph G with $VC(E(G), \mathcal{CYC}(G)) = 1$.*

Proof. Let G be a connected graph with n vertices, m edges and assume that $VC(E(G), \mathcal{CYC}(G)) = 1$. It is easily seen that G has at least $m-n+1 (= null(G))$ cycles, since if we consider any spanning tree of G, then G has $m-n+1$ many fundamental cycles with respect to that spanning tree. On the other hand, in this case the bound given in Theorem 3.2 would be $(null(G)+1)^1 = null(G)+1$. Therefore, we have $null(G) \leqslant c(G) \leqslant null(G)+1$. The result follows. □

The class of the wheel graphs

In the following, we will show that in the class of the wheel graphs, the bounds given in this paper for the number of cycles are sharper than bounds from several earlier results in the literature. It is easy to see that in the wheel graph W_n with n vertices, the number of edges is $m = 2n-2$. It follows that $null(W_n) = m-n+1 = n-1$. In the following statement, we find the VC-dimension of the cycle hypergraph of the wheel graphs. Then, using that we can compute our bound given in Theorem 3.2 for the number of cycles and compare that bound with the earlier bounds in the literature as well as the exact number of cycles.

Proposition 3.3. *For every $n \geqslant 5$, the VC-dimension of the cycle hypergraph of W_n is 3.*

Proof. Fix some $n \geqslant 5$. By the star edges of the wheel graph W_n we mean those edges connected to the center of W_n. Also we call the rest of the edges the outer edges. Denote the set of the star edges and outer edges in W_n by E_1 and E_2 respectively. Denote the cycle of W_n with the edge set E_2 by C_0. Note that any cycle of G is either the cycle C_0 or has exactly two edges from E_1 and the rest of the edges from E_2. Assume that $S \subseteq E(W_n)$ is shattered by the cycle hypergraph of W_n. So, there exists some cycle C such that $E(C) \cap S = S$. Thus, $S \subseteq E(C)$. It follows that $[S]$ is either a cycle or a vertex disjoint union of some paths. So S contains at most two edges of E_1. We show that $|S| \leqslant 3$. We distinguish three cases.

Case (i): In this case we assume that S has exactly two edges of E_1, say edges e_1 and e_2. It is easily seen that there are exactly two cycles in W_n containing both e_1 and e_2. Since S is shattered by the cycle hypergraph, for each subset $X \subseteq S \cap E_2$ there exists a cycle C such that $E(C) \cap S = X \cup \{e_1, e_2\}$. Hence, every such C contains $\{e_1, e_2\}$. So, there are at most two such C's. It follows that there are at most two subsets $X \subseteq S \cap E_2$. Therefore, $|S \cap E_2| \leqslant 1$. It follows that $|S| \leqslant 3$.

Case (ii): In this case we assume that S contains exactly one edge of E_1, say e_1. It is sufficient to show that $|S \cap E_2| \leqslant 2$. Assume for contradiction that $|S \cap E_2| > 2$. Let v_0 be the common vertex of the edge e_1 and the cycle C_0. Also let e_2 and e_3 be the first and the last edges of $S \cap E_2$ appearing when we start moving on the cycle C_0 from v_0 in one direction (clockwise or counter-clockwise) until again we get back to v_0. Since we assumed that $|S \cap E_2| > 2$, $S \cap E_2$ has at least one other edge, say e_4, distinct from e_2 and e_3. Now it is not difficult to observe that each cycle of G containing e_1 and e_4 must contain e_2 or e_3 too. It follows that the set $\{e_1, e_2, e_3, e_4\}$ is not shattered by the cycle hypergraph. Hence, since $\{e_1, e_2, e_3, e_4\} \subseteq S$, the set S is not shattered by the cycle hypergraph, which is a contradiction. So, we have $|S \cap E_2| \leqslant 2$. Therefore, in this case we conclude that $|S| \leqslant 3$.

Case (iii): In this case we assume that S does not have any edge of E_1. It is sufficient to show that no subset of E_2 consisting of four edges is shattered by the cycle hypergraph. Let e_1, e_2, e_3 and e_4 be four arbitrary edges of E_2 in the clockwise ordering of the cycle C_0. It is not hard to see that there is not any cycle C of G with $E(C) \cap \{e_1, e_2, e_3, e_4\} = \{e_2, e_4\}$. So the set $\{e_1, e_2, e_3, e_4\}$ is not shattered by the cycle hypergraph. It follows that in this case we have $|S| \leqslant 3$.

So far, by the above analysis we have $VC(E(W_n), \mathcal{CYC}(W_n)) \leqslant 3$. It is not hard to see that every subset of E_2 of size 3 is shattered by $(E(W_n), \mathcal{CYC}(W_n))$. It follows that for every $n \geqslant 5$, $VC(E(W_n), \mathcal{CYC}(W_n)) \geqslant 3$. Combining these facts we have that $VC(E(W_n), \mathcal{CYC}(W_n)) = 3$ for every $n \geqslant 5$. $\square$

Corollary 3.2. *For the class of wheel graphs, the upper bound given in Theorem 3.2 is sharper than the bounds given in the Aldred-Thomassen Theorem and Kiraly's conjecture.*

Proof. By Proposition 3.3 and our upper bound given in Theorem 3.2 for the number of cycles, we have $c(W_n) \leqslant (null(W_n) + 1)^3 = n^3$ (and even stronger, $c(W_n) \leqslant \phi_3(null(W_n)) = \frac{1}{6}(n^3 - 3n^2 + 8n)$), while the upper bound given by Aldred-Thomassen (Theorem 3.1) is $\frac{15}{16}2^{n-1}$. Obviously, for large enough n, the bound given in Theorem 3.2 is sharper. Also in this case our bound is asymptotically sharper than the bound in Kiraly's conjecture. □

Note that it is known that the number of cycles of the wheel graph W_n is $n^2 - 3n + 3$ (see the sequence A002061 in OEIS [12]). It is obvious that our bound in this case is much closer than earlier bounds to the exact number.

3.3. Improving Kiraly's bound for some classes of dense graphs

In the previous subsection, we gave some examples of graph classes in which the bounds given in Subsection 3.1. for the number of cycles are sharper than many earlier bounds. In this subsection, we aim to improve the bound mentioned in Kiraly's conjecture in some more graph classes. These classes possess a density condition defined below. Our methods rely on using the upper bounds given in our result in Theorem 3.2.

In the following definition, we introduce some classes of graphs which are dense in the sense that, roughly speaking, the number of their edges are large (in the following sense) comparing to the number of their vertices.

Definition 3.2. Let $r > 1$ be a real number. We define the class of graphs $\mathcal{U}_r$ as follows. A graph G with n vertices and m edges belongs to $\mathcal{U}_r$ if and only if it is connected and $m \geqslant 2n \log_r n$.

Lemma 3.1. *For every $r > 1$ and each graph $G \in \mathcal{U}_r$ with n vertices and m edges we have $null(G) \leqslant r^{\frac{m}{n}} - 1$. Moreover, $c(G) \leqslant r^m$.*

Proof. Fix some $r > 1$ and let $G \in \mathcal{U}_r$ with n vertices and m edges. So, by definition, we have $m \geqslant 2n \log_r n$. Also since $m \leqslant \binom{n}{2}$, we have $m - n + 2 \leqslant n^2$. Combining these, we have $n \leqslant \frac{m}{\log_r n^2} \leqslant \frac{m}{\log_r (m-n+2)}$. Therefore, $n \log_r (m - n + 2) \leqslant m$. Thus, $(m - n + 2)^n \leqslant r^m$, which implies that $null(G) \leqslant r^{\frac{m}{n}} - 1$. This establishes the first part of the lemma. Since the length of the longest cycle is at most n, it is easily seen that $d \leqslant n$ where $d := VC(E(G), \mathcal{CYC}(G))$. So, by using Theorem 3.2 and the result of the first part of the this lemma, we have

$$c(G) \leqslant (null(G) + 1)^d \leqslant (null(G) + 1)^n \leqslant r^m.$$

This completes the proof. □

The following statement improves the upper bound for the number of cycles mentioned in Kiraly's conjecture for the graphs in the class of graphs $\mathcal{U}_r$ for each r, $1 < r < 1.4$. Also it gives a short proof for the Kiraly's conjecture restricted to the class of graphs $\mathcal{U}_r$ for $r = 1.4$.

Proposition 3.4. *1. For every r, $1 < r < 1.4$ and $G \in \mathcal{U}_r$ with m edges, we have $c(G) \leqslant r^m$ (this is a sharpening of the bound mentioned in Kiraly's conjecture for the particular class of graphs $\mathcal{U}_r$).*

2. Kiraly's conjecture holds for graphs in the class of graphs $\mathcal{U}_{1.4}$ with $\alpha = 1$ in the statement.

Proof. 1) The result is clear by Lemma 3.1.

2) Letting $r = 1.4$ in Lemma 3.1 implies that the bound in Kiraly's conjecture holds (with letting $\alpha = 1$) for every graph in $\mathcal{U}_{1.4}$. □

REFERENCES

1. W. Ahrens: *Ueber das gleichungssystem einer kirchhoff'schen galvanischen stromverzweigung.* Math. Ann. 49 (1897) 311–324 (German).
2. R. E. L. Aldred and C. Thomassen: *On the maximum number of cycles in a planar graph.* J. Graph Theory 57 (2008) 255–264.
3. R. Anstee, L. Ronyai and A. Sali: *Shattering news.* Graphs Combin 18 (1) (2002) 59–73.
4. A. Arman and S. Tsaturian: *The maximum number of cycles in a graph with fixed number of edges.* Electron. J. Comb. 26 (2019) P4.42.
5. L. Beaudou, P. Dankelmann, F. Foucaud, M. A. Henning, A. Mary and A. Parreau: *Bounding the order of a graph using its diameter and metric dimension: a study through tree decomposition and VC dimension.* SIAM Journal on Discrete Mathematics 32 (2018) 902–918.
6. C. Berge: *Graphs and Hypergraphs.* North-Holland (translation and revision of Graphes et Hypergraphes, Dunod, 1970), 1973.
7. B. Bollobas: *Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability.* Cambridge University Press, 1986.
8. N. Bousquet and S. Thomasse: *VC-dimension and Erdos-Posa property.* Discrete Math. 338 (2015) 2302–2317.
9. A. Bretto: *Hypergraph theory, an introduction.* Springer, New york, 2013.
10. V. D. Chepoi, B. Estellon and Y. Vaxes: *Covering planar graphs with a fixed number of balls.* Discrete Comput Geom 37 (2007) 237–244.
11. Z. Dvorak, N. Morrison, J. A. Noel, S. Norin and L. Postle: *Bounding the number of cycles in a graph in terms of its degree sequence.* European Journal of Combinatorics, 91(5) (2021), 103206.
12. https://oeis.org, The on-line encyclopedia of integer sequences.
13. S. Jukna: *Extremal Combinatorics. With Applications in Computer Science, 2nd ed.* Springer, Heidelberg, 2011.
14. Z. Kiraly: *Maximum number of cycles and Hamiltonian cycles in sparse graphs.* Tech. rep., Egervary Research Group (2009).
15. E. Kranakis, D. Krizanc, B. Ruf, J. Urrutia and G. Woeginger: *The VC-dimension of set systems defined by graphs.* Discrete Appl. Math. 77 (3) (1997) 237–257.
16. E. Kranakis, D. Krizanc and G. Woeginger: *VC-dimensions for graphs.* in: M. Nagl, editor, Graph-theoretic concepts in computer science, LNCS 1017, 1995, pp. 1–13.

17. J. Matousek: *Lectures on Discrete Geometry.* Springer-Verlag New York, 2002.

18. A. Mofidi: *On partial cubes, well-graded families and their duals with some applications in graphs.* Discrete Appl. Math, 283 (2020) 207–230.

19. A. Mofidi: *On the VC-dimension, covering and separating properties of the cycle and spanning tree hypergraphs of graphs.* Trans. Comb. 11(1) (2022) 29–43, Doi:10.22108/TOC.2021.127880.1832.

20. A. Mofidi: *On some dynamical aspects of NIP theories.* Arch. Math. Logic, 57(1), (2018) 37–71.

21. N. Sauer: *On the density of families of sets.* J. Comb. Theory, Ser. A 13 (1972) 145–147.

22. S. Shelah: *A combinatorial problem; stability and order for models and theories in infinitary languages.* Pac. J. Math. 41 (1972) 247–261.