FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 5 (2021), 1089-1097
https://doi.org/10.22190/FUMI210507079F
Original Scientific Paper

A FIXED POINT THEOREM FOR F-CONTRACTION MAPPINGS IN PARTIALLY ORDERED BANACH SPACES

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Abstract. In this paper, we first introduce a new notion of an F-contraction mapping, also we establish a fixed point theorem for such mappings in partially ordered Banach spaces. Moreover, two examples are represented to show the compatibility of our results.
Keywords: F-Contraction; Fixed point; Partially ordered.

1. Introduction and Preliminaries

F-contractions were introduced initially by Wardowski [24]. Indeed, Wardowski [24] extended the Banach Contraction Principle and proved some fixed-point results for F-contraction mappings. Since then, several authors proved many fixed point results for F-contraction mappings (refer to [1, 4, 5, 8, 11, 12, 13, 15, 19, 21, 22, 25]).

Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:
(F_1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, +\infty)$ with $\alpha < \beta$ we have $F(\alpha) < F(\beta)$,
(F_2) for each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to+\infty} \alpha_n = 0 \;\; if\, and\, only\, if \;\; \lim_{n\to+\infty} F(\alpha_n) = -\infty;$$

Received May 07, 2021. accepted August 30, 2021.
Communicated by Dijana Mosić
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2010 *Mathematics Subject Classification.* 47H10; 54H25

(F_3) there exists $k \in (0, +\infty)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Let $F_1(\alpha) = ln(\alpha), F_2(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_3(\alpha) = \alpha + ln(\alpha)$ for $\alpha > 0$, then $F_1, F_2, F_3 \in \mathcal{F}$. A mapping $T : X \to X$ is called an F-contraction if there exists $\tau > 0$ and $F \in \mathcal{F}$ shch that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

holds for all $x, y \in X$ with $d(Tx, Ty) > 0$. From (F_1) and (1.1), we can easily see that any F-contraction is a contractive mapping. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be givin by $F(\alpha) = ln\,\alpha$. By (1.1), we obtain

$$d(Tx, Ty) \leq e^{-\tau} d(x, y),$$

for all $x, y \in X$ and $d(Tx, Ty) > 0$. Let $F(\alpha) = \alpha + ln\,\alpha$ for $\alpha > 0$. From (1.1), we get

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau},$$

for any $x, y \in X$ and $d(Tx, Ty) > 0$. Wardowski [24] proved the following fixed point theorem.

Theorem 1.1. *[24] Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a fixed point x^* and for an arbitrary point $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.*

Let X be an ordered normed space, i.e., a vector space over the real equipped with a partial order $\preccurlyeq$ and a norm $||.||$. For every $\alpha \geq 0$ and $x, y, z \in X$ with $x \preccurlyeq y$ one has that $x + z \preccurlyeq y + z$ and $\alpha x \preccurlyeq \alpha y$. Two elements $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A self-mapping T on X is called non-decreasing if $Tx \preccurlyeq Ty$ whenever $x \preccurlyeq y$ for all $x, y \in X$.

Ran and Reurings [18] initiate the fixed point theory in the metric spaces equipped with a partial order relation. Thereafter, several authors obtained many fixed point results in ordered metric space (see [2, 3, 6, 7, 10, 16, 17, 23] and references therein).

Definition 1.1. [9] Let E be a Banach space. A subset P of E is called cone if the following conditions are satisfied:
1) P is nonempty closed set and $P \neq \{\theta\}$, where θ denotes the zero element in E;
2) if $x, y \in P$ and $a, b \in \mathbb{R}, a, b \geq 0$, then $ax + by \in P$;
3) if $x \in P$ and $-x \in P$, then $x = \theta$.

Let $P \subseteq E$ be a cone. We define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $L > 0$ such that

$$\theta \preccurlyeq x \preccurlyeq y \textit{ implies } ||x|| \leq L||y||,$$

for all $x, y \in E$. The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.2. [14, 20] A set $P \subseteq E$ is said to be a lattice under the partial ordering $\preccurlyeq$, if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in P$.

Lemma 1.1. *[9] A cone P in a normed space $(E, ||.||)$ is normal if and only if there exists a norm $||.||_1$ on E, equivalent to the given norm $||.||$, such that the cone P is monotone w.r.t. $||.||_1$.*

Lemma 1.2. *[9] Let E be a real Banach space, P be a normal cone and $\{x_{n_k}\}$ be a subsequence converging to p of monotone sequence $\{x_n\}$. Then $\{x_n\}$ converges to p. Also if $\{x_n\}_{n\in\mathbb{N}}$ is an increasing(decreasing) sequence, then $x_n \preccurlyeq p(p \preccurlyeq x_n)$ for all $n \in \mathbb{N}$.*

2. Main results

In this section, we prove a fixed point result in partially ordered Banach spaces. Let E be a partially ordered Banach space, P be a normal cone and the partial order $\preccurlyeq$ on E be induced by the cone P. We denote by $\mathcal{F}$, the set of all functions $F : P - \{\theta\} \to \mathbb{R}$ that satisfying the following conditions:
(F_1') F is strictly increasing, i.e., for all $x, y \in P$ such that $x \prec y, F(x) < F(y)$ or $x \preccurlyeq y$ and $x \neq y$ yields $F(x) \leq F(y)$ and $F(x) \neq F(y)$.
(F_2') For each sequence $\{x_n\}$ in P,

$$\lim_{n\to+\infty} x_n = \theta \ if\, and\, only\, if \ \lim_{n\to+\infty} F(x_n) = -\infty.$$

(F_3') There exists $k \in (0, +\infty)$ such that $\lim_{x\to\theta} ||x||^k F(x) = 0$.
Our new result is the following:

Theorem 2.1. *Let $X \subseteq E$ be a closed set, $P \subseteq X$ and let $T : X \to X$ be a self-mapping on X. Suppose that the following hypotheses hold:*
(i) X is a lattice;
(ii) T is a decreasing operator, i.e., $x \preccurlyeq y$ implies $Tx \succcurlyeq Ty$;
(iii) there exsits $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(Tu - Tv) \leq F(v - u), \tag{2.1}$$

for all $u, v \in X$, where $u \preccurlyeq v$ and $Tu \neq Tv$. Then, T has a unique fixed point $p \in X$.

Proof. Let $x_0 \in X$ be arbitrary. If $Tx_0 = x_0$ the proof is finished, that is T has a fixed point x_0. Let $Tx_0 \neq x_0$ and we consider the following two case.
Case1. Let x_0 is comparable with Tx_0. Without loss of generality, we suppose that $x_0 \prec Tx_0$. Since T is decreasing, we get $Tx_0 \succcurlyeq T^2x_0$. We can easily check that T^2 is increasing. From (2.1), we have

$$\tau + F(Tx_0 - T^2x_0) \leq F(Tx_0 - x_0).$$

Then, we get

$$F(Tx_0 - T^2x_0) \leq F(Tx_0 - x_0).$$

Since, F is strictly increasing, we get

$$Tx_0 - T^2x_0 \quad \preccurlyeq Tx_0 - x_0.$$

Then, we have

$$x_0 \preccurlyeq T^2x_0. \tag{2.2}$$

Using (2.1), we obtain

$$\begin{aligned} \tau + F(T^2v - T^2u) &\leq F(Tu - Tv) \\ &\leq F(v-u) - \tau \\ &< F(v-u), \end{aligned} \tag{2.3}$$

for all $u, v \in X$, where $u \prec v$ or $u \preccurlyeq v$ and $u \neq v$. Let $Sx = T^2x$ for all $x \in X$. Then, from (2.3), we have

$$\tau + F(Sv - Su) \leq F(v-u), \tag{2.4}$$

for all $u, v \in X$, where $u \prec v$ or $u \preccurlyeq v, u \neq v$ and $F \in \mathcal{F}$. Also, from (2.2) we have $x_0 \preccurlyeq Sx_0$. Now, we show that S has a unique fixed point. Consider the iterated sequence $\{x_n\}$, where $x_{n+1} = Sx_n$ for $n = 0, 1, 2, \ldots$. Since S is increasing, we have $x_{n+1} \preccurlyeq x_n$ for all $n = 0, 1, 2, \ldots$. Using (2.4), we have

$$F(x_{n+1} - x_n) \leq F(x_n - x_{n-1}) - \tau \leq \ldots \leq F(x_1 - x_0) - n\tau. \tag{2.5}$$

Letting $n \to +\infty$ above inequality, we obtain

$$\lim_{n \to +\infty} F(x_{n+1} - x_n) = -\infty.$$

Using F_2', we get $\alpha_n = x_{n+1} - x_n \to \theta$ as $n \to +\infty$. This implies that

$$\lim_{n \to +\infty} ||\alpha_n|| = 0. \tag{2.6}$$

From (F_3'), there exsits $k \in (0,1)$ such that

$$\lim_{n \to +\infty} ||\alpha_n||^k F(\alpha_n) = 0. \tag{2.7}$$

From, (2.5) we have

$$(||\alpha_n||^k F(\alpha_n) - ||\alpha_n||^k F(\alpha_0)) \leq ||\alpha_n||^k (F(\alpha_0) - n\tau) - ||\alpha_n||^k F(\alpha_0) = -||\alpha_n||^k n\tau \leq 0.$$

Using (2.6) and (2.7) and letting $n \to +\infty$ in above inequality, we get

$$\lim_{n \to +\infty} n||\alpha_n||^k = 0. \tag{2.8}$$

It follows from (2.8), there exists $N \in \mathbb{N}$, such that

$$||\alpha_n|| \leq \frac{1}{n^{\frac{1}{k}}}, \tag{2.9}$$

for all $n > N$. Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose $m, n \in \mathbb{N}$ and $m > n > N$.

$$||x_m - x_n|| \leq ||\alpha_{m-1}|| + ||\alpha_{m-2}|| + \ldots + ||\alpha_n|| \leq \sum_{i=n}^{+\infty} ||\alpha_i|| \leq \sum_{i=n}^{+\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Then $||x_m - x_n|| \to 0$ as $m, n \to +\infty$, which implies $\{x_n\}$ is a Cauchy sequence. Since X is closed, then there exists point p in X such that $\lim_{n \to +\infty} x_n = p$. Using Lemma 1.2, we get $x_n \preccurlyeq p$ for all $n \in \mathbb{N}$. From (2.4), we have

$$F(Sx_n - Sp) \leq F(x_n - p) - \tau \leq F(x_n - p).$$

Since F is strictly increasing, we have

$$Sx_n - Sp \prec x_n - p, \tag{2.10}$$

for all $n \in \mathbb{N}$. From Lemma (1.1) exists a norm $||.||_1$ such that is equivalent with $||.||$ and

$$||Sx_n - Sp||_1 \leq ||x_n - p||_1, \tag{2.11}$$

for all $n \in \mathbb{N}$. Using (2.11), we obtain

$$\begin{aligned} ||p - Sp||_1 &\leq ||p - x_{n+1}||_1 + ||x_{n+1} - Sp||_1 \\ &\leq ||p - x_{n+1}||_1 + ||x_n - p||_1, \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in above inequality, we get $||p - Sp||_1 = 0$, which implies $Sp = p$. To see the uniqueness of the fixed point, let us consider p and q be two distinct fixed points of S, that is, $Sp = p \neq q = Sq$. If q comparable with p, without loss of generality, we suppose that $q \preccurlyeq p$. Then, by (2.4), we obtain

$$\tau \leq F(p - q) - F(Sp - Sq) = 0, \tag{2.12}$$

which is a contradiction. Now, suppose p is not comparable to q. Since X is a lattice, there exists $r \in X$ such that $r = inf\{p, q\}$, which implies $r \preccurlyeq p$ and $r \preccurlyeq q$. Since S is increasing, we have $S^n r \preccurlyeq S^n p$ and $S^n r \preccurlyeq S^n q$. Using (2.4) we obtain,

$$F(p - S^n r) = F(S^n p - S^n r) \leq F(S^{n-1} p - S^{n-1} r) - \tau \leq \ldots \leq F(p - r) - n\tau,$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in above inequality, we have $\lim_{n \to +\infty} F(p - S^n r) = -\infty$ that together with (F_2') gives $lim_{n \to +\infty}(p - S^n r) = \theta$. This implies that $\lim_{n \to +\infty} S^n r = p$. Similarly, $\lim_{n \to +\infty} S^n r = q$. So, $p = q$ that is S has a unique fixed point p. Now, we show that the unique fixed point of S is also the unique fixed point of T. Since S has a fixed point p, we have

$$S(Tp) = T^2(Tp) = T(T^2p) = T(Sp) = Tp. \tag{2.13}$$

From the uniqueness of the fixed point of S, we know $Tp = p$.
Case2. Suppose x_0 is not comparable to Tx_0. Since X is a lattice, there exists $y \in X$ such that $y = inf\{x_0, Tx_0\}$, which implies $y \preccurlyeq x_0$ and $y \preccurlyeq Tx_0$. Since T is decreasing, we have $Tx_0 \preccurlyeq Ty$, which implies $y \preccurlyeq Ty$. Similarly to the proof of case 1, we can show T has a unique fixed point.

□

Example 2.1. Let $E = R \times R$ endowed with the norm $||.||_1$ which is defined as follows $||(x_1, x_2)||_1 = |x_1| + |x_2|, x_1, x_2 \in \mathbb{R}$. Also, we define a partial order on $\mathbb{R}^2$ as follows

$$(a, b) \preccurlyeq (c, d) \ if\, and\, only\, if \ a \leq c, b \leq d.$$

Then $(X, ||.||, \preccurlyeq)$ is a partially ordered Banach space. Suppose $X = [0, +\infty) \times [0, +\infty)$, $P = \{(\alpha, 0) : \alpha \geq 0\}$ and $F : P - \{\theta\} \to \mathbb{R}$ by $F\alpha = ln\alpha$. Define $T = (T_1, T_2)$ where $T_i : [0, +\infty) \to \mathbb{R}, i = 1, 2$ and $T_1(a) = e^{-\tau}\frac{-a}{1+a}, T_2(b) = e^{-\tau}\frac{2}{1+b}$,

$$T(a, b) = (T_1(a), T_2(b)) = (e^{-\tau}\frac{-a}{1+a}, e^{-\tau}\frac{2}{1+b}),$$

for all $a, b \in [0, +\infty)$ where $\tau > 0$. It is clear that both $T_i, i = 1, 2$ are strictly decreasing, so, T is decreasing. We show that T is F-contraction. Indeed, let $u = (x_1, y_1) \preccurlyeq v = (x_2, y_2)$, we have

$$\begin{aligned} Tu - Tv &= e^{-\tau}(\frac{-x_1}{2+x_1}, \frac{2}{1+y_1}) - e^{-\tau}(\frac{-x_2}{2+x_2}, \frac{2}{1+y_2}) \\ &= e^{-\tau}(\frac{-2x_1 - x_1x_2 + 2x_2 + x_1x_2}{4 + 2x_1 + 2x_2 + x_1x_2}, \frac{2 + 2y_2 - 2 - 2y_1}{1 + y_2 + y_1 + y_1y_2}) \\ &\leq e^{-\tau}(x_2 - x_1, y_2 - y_1) \\ &= e^{-\tau}(v - u). \end{aligned}$$

Which implies that

$$\tau + ln(Tu - Tv) \leq ln(v - u).$$

Then, all the conditions of Theorem 2.1 are satisfied and so T has a unique fixed point $(0, \frac{-1+\sqrt{1+8e^{-\tau}}}{2})$, where τ is given.

Example 2.2. Let $E = \mathbb{R}, X = [0, +\infty), P = [0, +\infty)$ and $F : P\backslash\{0\} \to \mathbb{R}$ with $F(r) = -\frac{1}{r}$. Define the mapping $T : X \to X$ by $Tx = \frac{1}{1+x}$. . It is clear that the all conditions of Theorem 2.1 are satisfied. The condition (2.1) is true i.e. exists $\tau > 0$ such that

$$\tau + F(Tu - Tv) \leq F(v - u).$$

Indeed, for $v > u$, we obtain

$$\begin{aligned} F(v-u) - F(Tu - Tv) &= -\frac{1}{v-u} + \frac{1}{\frac{1}{1+u} - \frac{1}{1+v}} \\ &= -\frac{1}{v-u} + \frac{(1+v)(1+u)}{v-u} \\ &= -\frac{1}{v-u} + \frac{1+u+v+vu}{v-u} \\ &= \frac{u+v+vu}{v-u} \\ &\geq \frac{u+v}{v-u} \\ &\geq \frac{v-u}{v-u} = 1. \end{aligned}$$

Hence, for any $\tau \in (0,1]$, we have

$$\tau + F(Tu - Tv) \leq F(v-u).$$

Thus, T has a unique fixed point $u_0 = \frac{\sqrt{5}-1}{2}$.

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