FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 1–20
https://doi.org/10.22190/FUMI210510002G
Original Scientific Paper

NONLOCAL BOUNDARY VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION ON THE HALF-LINE

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Abstract. This paper aims to investigate a class for nonlocal fractional boundary value problem on an infinite interval due to its importance in provide a powerful tool for mathematical modeling of complex phenomena in science. New existence results are acquired for the given problem by using the Krasnosel'skii's fixed point theorem. Moreover, sufficient conditions are obtained as well as a modified compactness criterion that guarantees the existence of at least one solution. In addition, an illustrative example is given in the final part of the paper.
Keywords: Boundary value problem, infinite interval, fractional differential equation, nonlocal condition, fixed point theorem.

1. Introduction

Fractional calculus is a generalization of classical integer-order calculus and has been studied for more than for several years ago. Unlike integer-order derivatives, the fractional differential equations provide a powerful tool for mathematical modeling of complex phenomena in science, engineering practice and processes in the fields of physics, chemistry, electrical circuits, biology, and so on.

This is the main advantage of fractional differential equations in comparison with classical integer-order models. Further, the concept of nonlocal boundary conditions has been introduced to extend the study of classical boundary value problems. This

Received May 10, 2021, revised: November 23, 2023, accepted: November 27, 2023
Communicated by Hijaz Ahmad
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2010 *Mathematics Subject Classification.* 34B10, 34B15, 34B40.

notion is more precise for describing natural phenomena than the classical notion because additional information is taken into account.

Recently, several papers have studied questions of existence of solutions for some classes of bvps for fractional differential equations on finite intervals, see, e g., [2, 3, 4, 5, 8, 9, 11, 18, 19] and references therein. Different methods have been employed. However, research works on the existence of multiple solutions for fractional differential equations with nonlocal boundary condition on infinite intervals are few, we refer to [6, 7, 12, 15, 16, 17] and references therein.

In this paper, we will consider the boundary value problem (bvp for short)

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t)), \qquad t \in (0, +\infty), \\ u(0) = 0 = D_{0^+}^{\alpha-2} u(0) = 0, \quad \lim\limits_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = N(u), \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$ and $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $N : Y \to \mathbb{R}$ are given functions such that Y is a suitable Banach space. $D_{0^+}^{\alpha}$ refers to the standard Riemann-Liouville fractional derivative and $I_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional integral.

By using the famous Leray-Schauder Nonlinear Alternative theorem, Y. Gholami [6] obtained an unbounded solution for the following multi-point bvp in unbounded interval

$$\begin{cases} D_{0^+}^{\alpha} u(t) + a(t) f(t, u(t), u'(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) = u'(0) = 0, \quad \lim\limits_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = \sum\limits_{i=1}^{m} \beta_i D_{0^+}^{\alpha-1} u(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $a \in C([0, +\infty), [0, +\infty))$, $0 < \xi_1 < \xi_2 < \ldots < \xi_m < +\infty$, $\beta_i \in \mathbb{R}$ with $\sum\limits_{i=1}^{m} \beta_i < 1$.
In [15], Shen, Zhou and Yang established the existence of positive solutions for the bvp

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t), D_{0^+}^{\alpha-1} u(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D_{0^+}^{\alpha-1} u(+\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha \leq 3$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Gamma(\alpha) - \sum\limits_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} \neq 0$. with a suitable growth condition imposed on the nonlinear term. By using Schauder fixed point theorem, they proved the existence of at least one solution.
Ghanbari, Gholami [7] discussed the existence and multiplicity of positive solutions for a m-point nonlinear fractional bvp on an infinite interval

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda a(t) f(t, u(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) + u'(0) = 0, \quad \lim\limits_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = \sum\limits_{i=1}^{m-2} \beta_i u'(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, $f \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, $a \in C([0,+\infty), [0,+\infty))$, λ is a positive parameter and $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < +\infty$, $\beta_i \in [0,+\infty)$ with $0 < \sum_{i=1}^{m-2}(\alpha-1)\beta_i\xi_i^{\alpha-1} < \Gamma(\alpha)$.

Motivated by the above works and by recent studies of nonlocal boundary value problems of fractional order, we consider a more general problem of fractional differential equations of arbitrary order with nonlocal boundary conditions. Precisely, we investigate the problem (1.1).

The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models, in which N is a mapping defined on a proposed space consisting of certain functions which represent the solutions to the problem proposed in this paper. Then we give a model of the function g in the form of a linear combination of the solution at some points in the example proposed in this paper to confirm our results.

The work presented in this paper is a continuation of previous works and is concerned with a bvp of fractional order set on the half-axis. The main difficulty in treating this class of the fractional differential equations is the possible lack of compactness due to the infinite interval. In order to overcome these difficulties, we use a special Banach space in which similar inequalities as finite interval can be established. The main tool used in this paper is Krasnosel'skii' s fixed point theorem (nonlinear alternative). Under a compactness criterion, the existence of solutions is established.

The plan of the paper is as follows. In Section 2, we outline some basic concepts of fractional calculus. We prove some technical lemmas which are needed later in Section 3. Section 4 is devoted to our main existence results. In Section 5, an example of applications is supplied to illustrate our theoretical results.

2. Preliminaries

We start with some definitions and lemmas on the fractional calculus (see [10], [13]).

One of the basic tools of the fractional calculus is the Gamma function which extends the factorial to positive real numbers (and even complex numbers with positive real parts).

Definition 2.1. For $\alpha > 0$, the Euler Gamma function is defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt.$$

Proposition 2.1. *Let $\alpha > 0$, $p > 0$, $q > 0$ and n a positive integer. Then*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\Gamma(\alpha+n)=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\Gamma(\alpha).$$

In particular

$$\Gamma(1)=\int_0^{+\infty}e^{-t}dt=1,\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},$$
$$\Gamma(n+1)=n!,\quad \Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}(2n)!}{2^{2n}n!}.$$

Definition 2.2. The fractional integral of order $\alpha>0$ for function h is defined by

$$I_{0^+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds,$$

provided the right hand side is point-wise defined on $(0,+\infty)$.

Definition 2.3. For a given function h defined on the interval $[0,+\infty)$, the Riemann-Liouville fractional derivative of order $\alpha>0$ is defined by

$$D_{0^+}^{\alpha}h(t)=\left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{h(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n=[\alpha]+1$.

Lemma 2.1. *([10]) Let $\alpha>0$, then*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+...+c_nt^{\alpha-n},$$

for some $c_i\in\mathbb{R}$, $i=1,2,...,n$, $n=[\alpha]+1$.

Proposition 2.2. *[13] The following composition relations hold:*
(a) $D_{0^+}^{\alpha}I_{0^+}^{\alpha}h(t)=h(t)$, $\quad\alpha>0$, $\quad h\in L^1[0,+\infty)$.
(b) $D_{0^+}^{\alpha}I_{0^+}^{\gamma}h(t)=I_{0^+}^{\gamma-\alpha}h(t)$, $\quad\gamma>\alpha>0$, $\quad h\in L^1[0,+\infty)$.
(c) $I_{0^+}^{\alpha}I_{0^+}^{\gamma}h(t)=I_{0^+}^{\alpha+\gamma}h(t)$, $\quad\alpha>0$, $\quad\gamma>0$, $\quad h\in L^1[0,+\infty)$.
(d) $D_{0^+}^{\alpha}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}$, for $\lambda>-1$, in particular for $D_{0^+}^{\alpha}t^{\alpha-m}=0$, $m=1,2,\dots,N$, where N is the smallest integer greater than or equal to α.
(e) $I_{0^+}^{\alpha}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}t^{\alpha+\lambda}$, $\quad\alpha>0$, $\quad\lambda>-1$.

The following result is needed to prove our main existence result. This is a nonlinear alternative for Krasnosel'skii' s fixed point theorem [1].

Theorem 2.1. *([1]) Let U be an open set in a closed, convex set C of a Banach space E. Assume $0\in U$, $T(\overline{U})$ bounded and $T:\overline{U}\to C$ is given by $T=T_1+T_2$, where*
$T_1:\overline{U}\to E$ is continuous and completely continuous and $T_2:\overline{U}\to E$ is contraction (i.e., there exists a constant $0<l<1$, such that $\|T_2(x)-T_2(y)\|\leqslant l\|x-y\|$, for all $x,y\in\overline{U}$). Then either,
(a) T has a fixed point in $\overline{U}$, or
(b) There is a point $u\in\partial U$ and $\lambda\in(0,\ 1)$ with $u=\lambda T(u)$.

3. Related Lemmas

Consider the Banach spaces X, Y defined by

$$X = \left\{ u \in C([0, +\infty), \mathbb{R}), \quad \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty \right\}$$

with the norm

$$\|u\|_X = \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}}$$

and

$$\begin{aligned} Y \quad = \quad & \Bigg\{ u \in X,\ D_{0^+}^{\alpha-2} u,\ D_{0^+}^{\alpha-1} u \in C([0, +\infty), \mathbb{R}), \\ & \sup_{t \geq 0} \frac{|D_{0^+}^{\alpha-2} u(t)|}{1 + t} < +\infty,\ \sup_{t \geq 0} |D_{0^+}^{\alpha-1} u(t)| < +\infty \Bigg\} \end{aligned}$$

with the norm

$$\|u\|_Y = \max \left\{ \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}},\ \sup_{t \geq 0} \frac{|D_{0^+}^{\alpha-2} u(t)|}{1 + t},\ \sup_{t \geq 0} |D_{0^+}^{\alpha-1} u(t)| \right\}.$$

Now, we list some conditions in this paper for convenience:
$(H1)$ The function $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory, i.e., $f(t, u, v, w)$ is Lebesgue measurable in t for all $(u, v, w) \in \mathbb{R}^3$, and continuous in (u, v, w) for a.e. $t \in [0, +\infty)$.
$(H2)$ There exist nonnegative functions $(1 + t^{\alpha-1})\varphi(t),\ \psi(t),\ (1 + t)\mu(t),\ \phi(t) \in L^1[0, +\infty)$ such that
$|f(t, x, y, z)| \leqslant \varphi(t)|x| + \psi(t)|y| + \mu(t)|z| + \phi(t)$ for all $x,\ y,\ z \in \mathbb{R}$ and $t \in [0, +\infty)$.
$(H3)$ There exists a positive constant l such that $0 < l < \Gamma(\alpha)$ and

$$|N(u) - N(v)| \leqslant \frac{l}{\Gamma(\alpha)} \|u - v\|_Y \text{ for all } u,\ v \in Y.$$

$(H4)$ $N(0) = 0$.
$(H5)$ There exists $\rho > 0$ such that

$$\rho > 2 \int_0^{+\infty} \big(\rho((1 + s^{\alpha-1})\varphi(s) + \psi(s) + (1 + s)\mu(s)) + \phi(s) \big)\, ds + \frac{l\rho}{\Gamma(\alpha)}.$$

Lemma 3.1. *Let $h \in L^1[0, +\infty)$, then the bvp*

$$\left\{ \begin{array}{ll} D_{0^+}^{\alpha} u(t) = h(t), & t \in (0, +\infty), \\ u(0) = D_{0^+}^{\alpha-2} u(0) = 0, & \lim\limits_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = N(u), \end{array} \right. \tag{3.1}$$

has a unique solution given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s) ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} h(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} g(u).$$

Proof. By Lemma 2.1 and from $D^{\alpha}_{0+}u(t) = h(t)$, we have

$$u(t) = I^{\alpha}_{0+}h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \text{ for some constants } c_1,\, c_2,\, c_3 \in \mathbb{R}.$$

So the solution of (3.1) can be written as

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

From $u(0) = 0$ we get

$$t^{\alpha-3}\left(c_1 t^2 + c_2 t + c_3\right) = 0,$$

we known that $c_3 = 0$.
On the other hand, we have

$$\begin{aligned} D^{\alpha-2}_{0+}u(t) &= D^{\alpha-2}_{0+}I^{\alpha}_{0+}h(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \\ &= I^{2}_{0+}h(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \\ &= \int_0^t (t-s)h(s)ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1). \end{aligned}$$

From $D^{\alpha-2}_{0+}u(0) = 0$ we known that $c_2 = 0$.
Moreover

$$\begin{aligned} D^{\alpha-1}_{0+}u(t) &= D^{\alpha-1}_{0+}I^{\alpha}_{0+}h(t) + c_1\Gamma(\alpha) \\ &= I^{1}_{0+}h(t) + c_1\Gamma(\alpha) \\ &= \int_0^t h(s)ds + c_1\Gamma(\alpha). \end{aligned}$$

From $\lim\limits_{t\to+\infty} D^{\alpha-1}_{0+}u(t) = N(u)$, we get $c_1 = \frac{1}{\Gamma(\alpha)}N(u) - \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds$.
Therefore, the unique solution of fractional bvp (3.1) is

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}N(u).$$

□

Now, define the following operators $T_1,\ T_2,\ T$ on Y by

$$\begin{aligned} (T_1u)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds, \\ (T_2u)(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}N(u), \\ (Tu)(t) &= (T_1u)(t) + (T_2u)(t). \end{aligned}$$

Bvp (1.1) has a solution u if and only if u solves the operator equation $u = Tu$.
We will prove the existence of a fixed point of T. For this we verify that the operator T satisfies all conditions of Theorem 2.1.
Since the Arzela-Ascoli theorem fails to work in the space Y, we need a modified compactness criterion to prove T_1 is compact.

Lemma 3.2. *([14]) Let $Z = \{u \in Y,\ \|u\|_Y < l\}$ such that $l > 0$, $Z_1 = \{\frac{u(t)}{1+t^{\alpha-1}},\ u \in Z\}$,*
$Z_2 = \{D_{0^+}^{\alpha-1}u(t),\ u \in Z\}$ and $Z_3 = \left\{\frac{D_{0^+}^{\alpha-2}u(t)}{1+t},\ u \in Z\right\}$. Then Z is relatively compact on Y if Z_1, Z_2 and Z_3 are equicontinuous on any compact intervals of $[0,+\infty)$ and are equiconvergent at infinity.

Definition 3.1. Z_1, Z_2 and Z_3 are called equiconvergent at infinity if and only if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon,\ \left|D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)\right| < \varepsilon \text{ and}$$
$$\left|\frac{D_{0^+}^{\alpha-2}u(t_1)}{1+t_1} - \frac{D_{0^+}^{\alpha-2}u(t_2)}{1+t_2}\right| < \varepsilon,$$

for any $t_1,\ t_2 > \delta$ and $u \in Z$.

Let $\Omega_r = \{u \in Y,\quad \|u\|_Y < r\},\ (r > 0)$ be the open ball of radius r in Y.

Lemma 3.3. *If $(H1)-(H4)$ hold, then $T(\overline{\Omega}_r)$ is a bounded set.*

Proof. We have

$$\begin{aligned}
&\sup_{t\geq 0}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right. \\
& + \frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
& \left. + \frac{t^{\alpha-1}}{1+t^{\alpha-1}}|N(u)|\right) \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds + \frac{lr}{\Gamma(\alpha)}\right).
\end{aligned}$$

In addition

$$\begin{aligned}
&\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \\
\leqslant\ & 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds + |N(u)| \\
\leqslant\ & 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds + \frac{lr}{\Gamma(\alpha)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| \\
&\leqslant\ 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)| \\
&\leqslant\ 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds+\frac{lr}{\Gamma(\alpha)}.
\end{aligned}$$

So

$$\|Tu\|_Y<+\infty,\ \text{for } u\in\overline{\Omega}_r.$$

□

Lemma 3.4. *If* $(H1)$, $(H2)$ *hold, then* $T_1:\overline{\Omega}_r\to Y$ *is completely continuous.*

Proof. We firstly verify that the set $T_1(\overline{\Omega}_r)$ is bounded.
By definition of the operator T_1 we have that, for any $u\in\overline{\Omega}_r$,

$$\begin{aligned}
\left|\frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| &\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\
&\quad\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\
&\leqslant \frac{2}{\Gamma(\alpha)}\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

In addition

$$\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_1u(t)\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

Also

$$\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

Hence

$$\|T_1u\|_Y\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds,\ \text{for } u\in\overline{\Omega}_r.$$

Now, we divide the proof into three steps.
- Step 1: We show that T_1 is continuous.
Let $u_n \to u$ as $n \to +\infty$ in $\overline{\Omega}_r$, we have

$$\begin{aligned}
&\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s))|ds \\
&+\frac{2}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} (\|u_n\|_Y((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds \\
&+\frac{2}{\Gamma(\alpha)}\int_0^{+\infty} (\|u\|_Y((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

So

$$\begin{aligned}
&\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{4}{\Gamma(\alpha)}\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

Using the continuity of f, we obtain that

$$|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))| \to 0, \text{ as } n \to +\infty,$$

which implies

$$\|T_1u_n - T_1u\|_X = \sup_{t\geq 0}\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \to 0,$$

uniformly as $n \to +\infty$.
Moreover

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}T_1u_n(t) - D_{0^+}^{\alpha-1}T_1u(t)\right| \\
\leqslant\ & 2\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & 4\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

Also

$$\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u_n(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right|\\
\leqslant\ & 2\int_0^{+\infty}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s))\\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
\leqslant\ & 4\left(\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

Using again the continuity of f, we get

$$\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}T_1u_n(t)-D_{0^+}^{\alpha-1}T_1u(t)\right|\to 0,\ \sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}T_1u_n(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right|\to 0,$$

uniformly as $n\to+\infty$.
We conclude

$$\|T_1u_n-T_1u\|_Y\to 0,\ \text{uniformly as } n\to+\infty,\ \text{as claimed.}$$

- Step 2: We show that $T_1:\overline{\Omega}_r\to X$ is relatively compact.
According to the above $T_1(\overline{\Omega}_r)$ is uniformly bounded. We show that functions from $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$ and functions from $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equicontinuous on any compact intervals of $[0,+\infty)$.
Let $I\subset[0,+\infty)$ be a compact interval, then for any $t_1,\ t_2\in I$ such that $t_1<t_2$, and for $u\in\overline{\Omega}_r$, we have

$$\begin{aligned}
&\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|\\
=\ & \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&-\int_0^{+\infty}\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\left.+\int_0^{+\infty}\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\left(\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\right.\\
&\left.-\int_0^{t_2}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|
\end{aligned}$$

$$\begin{aligned}
&+\left|\int_0^{t_2}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\left.-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\left.+\int_0^{+\infty}\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\
&+\int_0^{t_2}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\left.+\int_0^{+\infty}\frac{\left|t_1^{\alpha-1}-t_2^{\alpha-1}\right|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right.\\
+\ &\int_0^{t_2}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\\
+\ &\left.\int_0^{+\infty}\frac{\left|t_1^{\alpha-1}-t_2^{\alpha-1}\right|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

The last term converges to 0 uniformly as $|t_1-t_2|\to 0$.
Moreover

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right|\\
=\ &\left|\int_0^{t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\left.-\int_0^{t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right|\\
\leqslant\ &\int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds,
\end{aligned}$$

which converges to 0 uniformly as $|t_1-t_2|\to 0$. Also

$$\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right|\\
=\ &\left|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\left.+\frac{t_2-t_1}{(1+t_1)(1+t_2)}\int_0^{+\infty}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left|\int_0^{t_1} \frac{t_1-s}{1+t_1} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \right.\\
&\quad \left. -\int_0^{t_2} \frac{t_1-s}{1+t_1} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\quad +\left|\int_0^{t_2} \frac{t_1-s}{1+t_1} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \right.\\
&\quad \left. -\int_0^{t_2} \frac{t_2-s}{1+t_2} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\quad +\frac{|t_2-t_1|}{(1+t_1)(1+t_2)}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds.\\
&\leqslant \int_{t_1}^{t_2} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\\
&\quad +\frac{2|t_1-t_2|}{(1+t_1)(1+t_2)}\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)\\
&\quad +(1+s)\mu(s))+\phi(s))ds,
\end{aligned}
$$

which converges to 0 uniformly as $|t_1-t_2|\to 0$.
Then, for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$
\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|<\varepsilon,\ \left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right|<\varepsilon
$$

and

$$
\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right|<\varepsilon,
$$

for all $u\in\overline{\Omega}_r$, if $|t_1-t_2|<\delta$, $t_1,\ t_2\in I$.
Showing that, the functions belonging to $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$ and the functions belonging to $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and to $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are locally equicontinuous on $[0,+\infty)$.

- Step 3: We show that the functions from $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$, $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equiconvergent at infinity.

For any $u\in\overline{\Omega}_r$, we have

$$
\int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))\right|ds<+\infty.
$$

Considering condition $(H2)$, for given $\varepsilon>0$, there exists a constant $L>0$ such that

$$
\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds<\varepsilon.
$$

On the other hand, since $\lim\limits_{t\to+\infty}\frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and $\lim\limits_{t\to+\infty}\frac{t-L}{1+t}=1$, there exists a constant $\delta>L>0$ such that for any t_1, $t_2\geq\delta$ and $0\leq s\leq L$, we have

$$\begin{aligned}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| &= \left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-1+1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\\ &\leqslant \left|1-\frac{(t_1-L)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|1-\frac{(t_2-L)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|<\varepsilon\end{aligned}$$

and

$$\begin{aligned}\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right| &= \left|\frac{t_1-s}{1+t_1}-1+1-\frac{t_2-s}{1+t_2}\right|\\ &\leqslant \left|1-\frac{t_1-L}{1+t_1}\right|+\left|1-\frac{t_2-L}{1+t_2}\right|<\varepsilon.\end{aligned}$$

Thus, for any t_1, $t_2\geq\delta>L>0$, we get

$$\begin{aligned}&\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|\\ =\ &\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\ &-\int_0^{+\infty}\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\ &-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\ &\left.+\int_0^{+\infty}\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|.\end{aligned}$$

So

$$\begin{aligned}&\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_0^L\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\ &+\int_L^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\ &+\int_L^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\ &\left.+2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_0^L\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^L |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&+4\int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\Bigg) \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\Bigg(\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L\varepsilon \\
&+2\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L+4\varepsilon\Bigg).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right| \\
=\ & \left|\int_{t_1}^{t_2} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right| \\
\leqslant\ & \int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds<\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right| \\
=\ & \Bigg|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \\
&-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \\
&-\frac{t_1}{1+t_1}\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \\
&+\frac{t_2}{1+t_2}\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\Bigg|.
\end{aligned}
$$

So

$$
\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right| \\
\leqslant\ & \int_0^L\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&+\int_L^{t_1}\frac{t_1-s}{1+t_1}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&+\int_L^{t_2}\frac{t_2-s}{1+t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
\leqslant\ &\int_0^{L}\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&+2\int_0^{L}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&+4\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
\leqslant\ &\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L\varepsilon\\
&+2\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L+4\varepsilon.
\end{aligned}
$$

Which yield that the functions from $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$, $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equiconvergent at infinity. According to Lemma 3.2, it follows that $T_1(\overline{\Omega}_r)$ is relatively compact, ending the proof of the Lemma. □

Lemma 3.5. *If $(H3)$ holds. Then $T_2:\overline{\Omega}_r\to Y$ is a contraction mapping.*

Proof. We have

$$
\begin{aligned}
\left|\frac{T_2u(t)}{1+t^{\alpha-1}}-\frac{T_2v(t)}{1+t^{\alpha-1}}\right| &\leqslant \frac{1}{\Gamma(\alpha)}\left|\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\right||N(u)-N(v)|\\
&\leqslant \frac{1}{\Gamma(\alpha)}|N(u)-N(v)|\\
&\leqslant \frac{l}{(\Gamma(\alpha))^2}\|u-v\|_Y.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_2u(t)-D_{0^+}^{\alpha-1}T_2v(t)\right| &= |N(u)-N(v)|\\
&\leqslant \frac{l}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}
$$

Also

$$
\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_2u(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_2v(t)}{1+t}\right| &= \left|\frac{t}{1+t}(N(u)-N(v))\right|\\
&\leqslant \frac{l}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}
$$

We conclude

$$
\|T_2u-T_2v\|_Y \leqslant \frac{l}{\Gamma(\alpha)}\|u-v\|_Y.
$$

From $(H3)$, we infer that T_2 is a contraction mapping. □

4. Main results

Theorem 4.1. *If assumptions $(H1)-(H5)$ hold, then the problem (1.1) has at least one solution.*

Proof. Consider the parameterized bvp

$$\text{(4.1)}\qquad \begin{cases} D_{0^+}^{\alpha}u(t)=\lambda f(t,u(t),D_{0^+}^{\alpha-1}u(t),D_{0^+}^{\alpha-2}u(t)), & t\in(0,+\infty),\\ u(0)=D_{0^+}^{\alpha-2}u(0)=0, \quad \lim\limits_{t\to+\infty}D_{0^+}^{\alpha-1}u(t)=\lambda N(u), & \end{cases}$$

for $\lambda\in(0\ ,\ 1)$.
Solving problem (4.1) is equivalent to solving the fixed point of equation $u=\lambda Tu$. Let

$$\Omega_\rho=\{u\in Y,\quad \|u\|_Y<\rho\}.$$

From Lemma 3.3, the set $T(\overline{\Omega}_\rho)$ is bounded and by Lemma 3.4, the operator $T_1:\overline{\Omega}_\rho\to Y$ is completely continuous, while Lemma 3.5 implies that the operator $T_2:\overline{\Omega}_\rho\to Y$ is contractive. So it remains to prove that $u\neq\lambda Tu$ for $u\in\partial\Omega_\rho$ and $\lambda\in(0\ ,\ 1)$.
Arguing by contradiction, if there exists $u\in\partial\Omega_\rho$ with $u=\lambda Tu$, then for $\lambda\in(0\ ,\ 1)$ we have

$$\begin{aligned}
&\sup_{t\geq0}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|\\
\leqslant\ &\sup_{t\geq0}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right|\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\
&\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}|N(u)|\right).\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)-N(0)|+|N(0)|\right).
\end{aligned}$$

So

$$\begin{aligned}
&\sup_{t\geq0}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}\right).
\end{aligned}$$

In addition

$$\begin{aligned}
&\sup_{t\geq0}\left|D_{0^+}^{\alpha-1}u(t)\right|\\
=\ &\sup_{t\geq0}\left|\lambda D_{0^+}^{\alpha-1}Tu(t)\right|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \\
&\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)| \\
&\leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}u(t)}{1+t}\right| \\
=\ &\sup_{t\geq 0}\left|\lambda\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| \\
\leqslant\ &\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| \\
\leqslant\ &\int_0^t\frac{t-s}{1+t}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&+\frac{t}{1+t}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}|g(u)| \\
\leqslant\ &2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}.
\end{aligned}$$

So

$$\|u\|_Y \leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}$$

and thus

$$\rho \leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}.$$

This implies that

$$\frac{\rho}{2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}} \leqslant 1,$$

contradicting condition $(H5)$. With theorem 2.1 we conclude that bvp (1.1) has at least one solution. □

5. Example

Example 5.1. Consider the bvp on infinite interval

$$(5.1)\quad \begin{cases} D_{0^+}^{\frac{5}{2}}u(t) = \frac{e^{-30t}}{1+\sqrt{t^3}}u(t) + \frac{D_{0^+}^{\frac{3}{2}}u(t)}{(50+t)^2} + \frac{D_{0^+}^{\frac{1}{2}}u(t)}{50(1+t)^3} + e^{-t}, \quad t \in (0,+\infty), \\ u(0) = D_{0^+}^{\frac{1}{2}}u(0) = 0, \quad \lim_{t\to+\infty} D_{0^+}^{\frac{3}{2}}u(t) = \frac{1}{10}u(1) + \frac{1}{20}u(4). \end{cases}$$

In this case, $\alpha = \frac{5}{2}$, $\Gamma(\frac{5}{2}) \approx 1.329340388$, $N(u) = \frac{1}{10}u(1) + \frac{1}{20}u(4)$, it's mean $c_1 = \frac{1}{10}$, $c_2 = \frac{1}{20}$, $\xi_1 = 1$, $\xi_2 = 4$.
We will apply Theorem 4.1 to show that problem (5.1) has at least a solution.
Let

$$f(t,x,y,z) = \frac{e^{-30t}}{1+\sqrt{t^3}}x + \frac{y}{(50+t)^2} + \frac{z}{50(1+t)^3} + e^{-t}.$$

Choose

$$\rho > \frac{600\Gamma(\frac{5}{2})}{266\Gamma(\frac{5}{2}) - 195}.$$

Then
$(H1)$ $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \times\mathbb{R}$ is Carathéodory.
$(H2)$ $|f(t,x,y,z)| \leqslant \frac{e^{-30t}}{1+\sqrt{t^3}}|x| + \frac{1}{(50+t)^2}|y| + \frac{1}{50(1+t)^3}|z| + e^{-t}$. So we may take

$$\varphi(t) = \frac{e^{-30t}}{1+\sqrt{t^3}},\ \psi(t) = \frac{1}{(50+t)^2},\ \mu(t) = \frac{1}{50(1+t)^3},\ \phi(t) = e^{-t}$$

and note that $(1+\sqrt{t^3})\varphi(t)$, $\psi(t)$, $(1+t)\mu(t)$, $\phi(t) \in L^1[0,+\infty)$ such that

$$\begin{aligned} \int_0^{+\infty} \left(1+s^{\frac{3}{2}}\right)\varphi(s)ds &= \frac{1}{30},\ \int_0^{+\infty} \psi(s)ds = \frac{1}{50}, \\ \int_0^{+\infty} (1+s)\mu(s)ds &= \frac{1}{50},\ \int_0^{+\infty} \phi(s)ds = 1. \end{aligned}$$

$(H3)$ Choose $l = c_1(1+\sqrt{\xi_1^3}) + c_2(1+\sqrt{\xi_2^3}) = \frac{9}{10}$ verify $0 < l < \Gamma(\frac{5}{2})$ with $|N(u)-N(v)| \leqslant \frac{l}{\Gamma(\frac{5}{2})}\|u-v\|_Y$ for all $u,\ v \in Y$.
$(H4)$ $N(0) = 0$.
$(H5)$

$$\begin{aligned} & \frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s) + \psi(s) + (1+s)\mu(s)) + \phi(s)\right)ds + \frac{l\rho}{\Gamma(\alpha)}} \\ = \ & \frac{\rho}{\frac{34\Gamma(\frac{5}{2})+195}{300\Gamma(\frac{5}{2})}\rho + 2} \\ = \ & \frac{300\Gamma(\frac{5}{2})}{34\Gamma(\frac{5}{2}) + 195 + \frac{600\Gamma(\frac{5}{2})}{\rho}} \\ > \ & 1. \end{aligned}$$

Which implies

$$\rho > 2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s) + \psi(s) + (1+s)\mu(s)) + \phi(s)\right)ds + \frac{l\rho}{\Gamma(\alpha)}.$$

Hence, all conditions of Theorem 4.1 are satisfied, we deduce that the bvp (5.1) has at least one solution.

6. Conclusion

In this work, we considered a class of fractional differential equation with nonlocal boundary conditions on an infinite interval. With the aid of the Krasnosel'skii's fixed point theorem, we have obtained existence results for the proposed problem in this paper. An example was presented to illustrate the main results. The boundary value problem of fractional differential equations on an infinite interval have been widely discussed in recent years. The examples of this is establishing the existence of solutions for fractional differential equations with multi-point boundary conditions, as well as the existence of positive solutions for fractional boundary value problem on an infinite interval.

Acknowledgements

The authors thank the anonymous referees for their thorough review.

REFERENCES

1. R. P. Agarwal, M. Meehan and D. O'Regan: *Fixed point theory and applications.* Cambridge Tracts in Mathematics **141**. Cambridge University Press, Cambridge, 2001.
2. I. Ahmad, H. Ahmad, Mustafa INC, S. W. Yao and B. Almohsen: *Application of local meshless method for the solution of two term time fractional-order multi-dimensional PDE arising in heat and mass transfer* Thermal Science **24(1)** (2020), 95-105.
3. H. Ahmad, T. A. Khan, I. Ahmad, P. S. Stanimirović and Y. M. Chu: *A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations.* Results in Physics **19** (2020) 103462.
4. I. Ahmad, Siraj-ul-Islam, Mehnaz and S. Zaman: *Local meshless differential quadrature collocation method for time-fractional PDEs.* Discrete and Continuous Dynamical Systems-S **13(10)** (2020), 2641-2654.
5. Z. Bai and Y. Zhang: *The existence of solutions for a fractional multi-point boundary value problem.* Computers and Mathematics with Applications **60** (2010), 2364-2372.
6. Y. Gholami: *Existence of an unbouded solution for multi-point boundary value problems of fractional differential equations on an infinite domain, class of Riemann-Liouville fractional differential equations.* Fractional Differential Calculus **4(2)** (2014), 125-136.
7. K. Ghanbari and Y. Gholami: *Existence and multiplicity of positive solutions for m-point nonlinear fractional differential equations on the half-line.* Electronic Journal of Differential Equations **2012(238)** (2012), 1-15.
8. J. He, X. Zhang, L. Liu, Y. Wu and Y. Gui: *Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions.* Boundary Value Problems (2018) 2018:189.
9. R. A. Khan and H. Khan: *On existence of solution for multi-points boundary value problem.* Journal of Fractional Calculus and Applications **5** (2014), 121-132.

10. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204. Elsevier Science B. V., Amsterdam, 2006.

11. B. Li, S. Sun, Y. Li and P. Zhao: *Multi-point boundary value problems for a class of Riemann-Liouville fractional differential equations*. Advances in Difference Equations 2014, 2014:151.

12. S. Liang and J. Zhang: *Existence of multiple positive solutions for m-point fractional boundary value problems of on an infinite interval*. Mathematical and Computer Modelling **54** (2011) 1334-1346.

13. K. Oldham and J. Spanier: *The fractional calculus*. Theory and applications of differentiation and integration to arbitrary order, (AP, 1974)(T)(ISBN 0-12-525550-0)(240s)-MCat.

14. X. Su and S. Zhang: *Unbounded solutions to a boundary value problem of fractional order on the half-line*. Computers and Mathematics with Applications **61** (2011), 1079-1087.

15. C. Shen, H. Zhou and L. Yang: *On the existence of solution to a boundary value problem of fractional differential equation on the infinite interval*. Boundary Value Problems (2015), 2015:241.

16. P. Thiramanus, S.K. Ntouyas and J. Tariboon: *Existence of solutions for Riemann-Liouville fractional differential equations with nonlocal Erdelyi-Kober integral boundary conditions on the half-line*. Boundary Value Problems (2015), 2015:196.

17. G. Wang: *Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval*. Applied Mathematics Letters, 47 (2015) 1-7.

18. Y. Yang and F. Meng: *Positive solutions for nonlocal boundary value problems of fractional differential equation*. WSEAS TRANSACTIONS on MATHEMATICS, 12 (2013) 1154-1163.

19. X. Zhang and Q. Zhong: *Multiple positive solutions for nonlocal boundary value problems of singular fractional differential equations*. Boundary Value Problems (2016) 2016:65.