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A THREE FRACTIONAL ORDER JERK EQUATION WITH ANTI PERIODIC CONDITIONS

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Abstract. We study a new Jerk equation involving three fractional derivatives and anti periodic conditions. By Banach contraction principle, we present an existence and uniqueness result for the considered problem. Utilizing Krasnoselskii fixed point theorem we prove another existence result governing at least one solution. We provide an illustrative example to claim our established results. At the end, an approximation for Caputo derivative is proposed and some chaotic behaviours are discussed by means of the Runge Kutta 4th order method.
Keywords: Jerk equation, fractional derivatives, anti periodic conditions, fixed point theorem.

1. Introduction

Fractional differential equations theory has been a powerful tool for modeling several phenomena in applied sciences and engineering, such as, visco-elasticity, chemistry, fluid flow, electrical networks, electrical circuits, optics, chaotic phenomena in dynamical systems, so on. For more details, one can consult the papers [2, 3, 8, 4, 10, 11, 13, 15, 16, 19, 25]. In this paper, we are concerned with some applications of dynamical systems of chaotic behaviours. To do this, we begin by

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recalling some papers that have motivated the present work. We begin by the references [22], where J.C. Sportt discovered some new chaotic systems through an extensive computer search on this phenomenon, with five terms and two quadratic nonlinearities or six terms and a single quadratic nonlinearity. Then, H.P.W. Gottlieb [7] studied the following problem of chaotic type:

$$x''' = -x's + x''(x + x'')/x'$$

which he called a Jerk equation. Then, J.C. Sportt [23] discovered the following particular case of Jerk equation with chaotic behaviours:

$$x''' + ax'' \pm x'^2 + x = 0,$$

which has only three terms in its Jerk representation or five terms in its dynamical system representation with a single quadratic nonlinearity and a single parameter a. We cite also the work of Z. Fu and J. Heidel [5], where the authors proved that there can be no simpler system with a quadratic nonlinearity. Then, B. Munmuangsaen et al. [17] studied several simple chaotic systems of the form:

$$x''' + x'' + x = h(x').$$

Other papers dealing with Jerk equations and systems for chaotic behaviours can be found in [1, 9, 12, 14, 18, 20, 24].

In this work, we try to find a suitable fractional presentation for a simple Jerk circuit that allows us to study some chaotic behaviours.

So, we consider the following problem:

$$\begin{cases} D^{\alpha}[D^{\beta}(D^{\gamma} + \lambda)]y(t) = f(t, y(t), D^{\gamma}y(t)), t \in J, \\ y(0) + y(T) = 0, (D^{\gamma})y(0) + (D^{\gamma})y(T) = 0, \\ D^{\gamma}D^{\gamma}y(0) + D^{\gamma}D^{\gamma}y(T) = 0. \end{cases} \tag{1.1}$$

In (1), the derivatives $D^{\alpha}, D^{\beta}, D^{\gamma}$ are taken in the sense of Caputo, with $0 \leq \gamma \leq \beta \leq \alpha \leq 1$, $\alpha + \beta \notin]0, 1), \beta + \gamma \notin]0, 1)$, $J := [0, T]$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function.

It is to important to note that in the present work:

1. We take the derivatives of Caputo in both sides of the problem.

2. We consider three parameters of derivation α, β and γ which allow us to be concerned with a sequential Jerk problem without commutativity and semi group properties.

3. The above sequential processes with the anti-periodic conditions on the problem allow us to consider a new type of Jerk problem of three fractional order.

4. Also, it is important to note that Eq. (1.1) is general enough to describe many problems that arise in mathematical physics, and depending on the values of the constants and functions involved in (1.1), there are several particular types of equations with important practical applications. For example, Eq. (1.1) includes the standard Jerk equation of Gottlieb [7] as a particular case. Also, it includes the above two models of Sportt [23] and Munmuangsaen et al. [17].

To the best of our knowledge, this is the first time in the literature where such problem is considered.

The organization of the paper is as follows: In the second section, we recall some fundamental results about fractional calculus. In the third section, the theoretical main results are obtained by using Banach contraction mapping principle and Krasnoselskii fixed point theorem. An illustrative example is discussed in the fourth section. Moreover, some applied results are discussed in section five; an approximation for Caputo derivative is proposed and some numerical simulations and some chaotic behaviours for problem (1.1) are discussed in this section. A conclusion follows at the last section.

2. Preliminaries on Fractional Calculus

We recall some definitions and lemmas that will be used later. For more details, we refer to [10, 6, 21].

Definition 2.1. Let $\alpha > 0$ and $f : [0,T] \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order α is defined by:

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ \alpha > 0\ ,\ 0 < t < T.$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du.$

Definition 2.2. For a function $f \in C^n([0,T],\mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo fractional derivative is defined by:

$$\begin{aligned} D^{\alpha}f(t) &= J^{n-\alpha}\frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds. \end{aligned}$$

In order to study the problem (1.1), we need the following lemmas:

Lemma 2.1. *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. The general solution of $D^{\alpha}y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1.$

Lemma 2.2. *Let $n \in \mathbb{N}^*$, $n-1 < \alpha < n$. Then*

$$I^\alpha D^\alpha y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$.

Lemma 2.3. (Krasnoselskii Theorem) *Let Ω be a closed convex and nonempty subset of Banach space $\mathcal{H}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two operators such that:*

1. *$\mathcal{H}_1 y_1 + \mathcal{H}_2 y_2 \in \Omega,\ \forall y_1, y_2 \in \Omega$.*
2. *$\mathcal{H}_1$ is compact and continuous.*
3. *$\mathcal{H}_2$ is a contraction mapping.*

Then there exists $y_3 \in \Omega$ such that $\mathcal{H}_1 y_3 + \mathcal{H}_2 y_3 = y_3$.

Lemma 2.4. *Let $G \in C(J)$. Then, the integral representation problem of the following problem*

$$\begin{cases} D^\alpha[D^\beta(D^\gamma + \lambda)]y(t) = G(t),\ t \in J, \\ y(0) + y(T) = 0, (D^\gamma)y(0) + (D^\gamma)y(T) = 0, \\ D^\gamma D^\gamma y(0) + D^\gamma D^\gamma y(T) = 0, \end{cases}$$

is given by

$$\begin{aligned} y(t) &= \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)}\ J^{\alpha+\beta-\gamma}G(T)\left(\frac{T^\gamma\ t^\gamma}{2\Gamma(\gamma+1)} - \frac{T^{2\gamma}}{4\Gamma(\gamma+1)}\right) \\ &+ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)}\ J^{\alpha+\beta-\gamma}G(T)\left(\frac{t^{\beta+\gamma}}{T^{\beta-\gamma}} - \frac{T^{2\gamma}}{2}\right) \\ &+ J^{\alpha+\beta}G(T)\ \left(\frac{T^\gamma}{4\Gamma(\gamma+1)} - \frac{t^\gamma}{2\Gamma(\gamma+1)}\right) \\ &+ J^{\alpha+\beta+\gamma}G(t) - \tfrac{1}{2}\ J^{\alpha+\beta+\gamma}G(T) \\ &- \lambda J^\gamma y(t) + \frac{\lambda}{2}\ J^\gamma y(T). \end{aligned}$$

Proof. Thanks to Lemma 2.2, we can written:

$$[D^\beta(D^\gamma + \lambda)]y(t) = J^\alpha G(t) - c_0.$$

With the the same idea, we can write

$$(D^\gamma + \lambda)y(t) = J^{\alpha+\beta}G(t) - c_0 J^\beta(T) - c_1.$$

Hence, it yields that

$$y(t)=J^{\alpha+\beta+\gamma}G(t)-c_0\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}-c_1\frac{t^{\gamma}}{\Gamma(\gamma+1)}-\lambda J^{\gamma}y(t)-c_2.$$

Thanks to the anti-periodic conditions, we observe that

$$\begin{cases} -2c_2+J^{\alpha+\beta+\gamma}G(T)-c_0J^{\beta+\gamma}(T)-c_1J^{\gamma}(T)-\lambda J^{\gamma}y(T)=0,\\ -2c_1+J^{\alpha+\beta}G(T)-c_0J^{\beta}(T)=0,\\ J^{\alpha+\beta-\gamma}G(T)-c_0J^{\beta-\gamma}(T)=0. \end{cases}$$

Solving the above system, we get:

$$\begin{cases} c_0 &= \Gamma(\beta-\gamma+1)J^{\alpha+\beta-\gamma}G(T),\\ 2c_1 &= J^{\alpha+\beta}G(T)-\Gamma(\beta-\gamma+1)J^{\alpha+\beta-\gamma}G(t)J^{\beta}(T),\\ 2c_2 &= J^{\alpha+\beta+\gamma}G(T)-\Gamma(\beta-\gamma+1)J^{\alpha+\beta-\gamma}G(T)J^{\beta+\gamma}(T)\\ &- \frac{1}{2}\Big(J^{\alpha+\beta}G(T)-\Gamma(\beta-\gamma+1)J^{\alpha+\beta-\gamma}G(T)J^{\beta}(1)\Big)J^{\gamma}(T)-\lambda J^{\gamma}y(T). \end{cases}$$

The lemma is thus proved. □

3. Main Results

Now, we introduce the Banach space:

$$X:=\{x\in C(J,\mathbb{R}),D^{\gamma}x\in C(J,\mathbb{R})\},$$

endowed with the norm:

$$\|x\|_X=\|x\|_\infty+\|D^{\gamma}x\|_\infty,$$

where,

$$\|x\|_\infty=\sup_{t\in J}|x(t)|\ ,\ \|D^{\gamma}x\|_\infty=\sup_{t\in J}|D^{\gamma}x(t)|.$$

Then, we consider the notations:

$$\alpha_1=\alpha+\beta+\gamma,\ \alpha_2=\alpha+\beta-\gamma,\ \alpha_3=\alpha+\beta,\ \alpha_4=\beta+\gamma.$$

Over the above Banach space, we define the nonlinear operator $\mathcal{H}$: $X\to X$ by:

$$\mathcal{H}y(t)=\mathcal{H}_1y(t)+\mathcal{H}_2y(t),$$

where,

$\mathcal{H}_1y(t)$

$$
\begin{aligned}
&= \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} f(\tau, y(\tau), D^\gamma y(\tau))d\tau \ \left(\frac{T^\gamma\ t^\gamma}{2\Gamma(\gamma+1)} - \frac{T^{2\gamma}}{4\Gamma(\gamma+1)}\right) \\
&+ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} f(\tau, y(\tau), D^\gamma y(\tau))d\tau \ \left(\frac{t^{\beta+\gamma}}{T^{\beta-\gamma}} - \frac{T^{2\gamma}}{2}\right) \\
&+ \int_0^T \frac{(T-\tau)^{\alpha_3-1}}{\Gamma(\alpha_3)} f(\tau, y(\tau), D^\gamma y(\tau))d\tau \ \left(\frac{T^\gamma}{4\Gamma(\gamma+1)} - \frac{t^\gamma}{2\Gamma(\gamma+1)}\right) \\
&+ \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} f(\tau, y(\tau), D^\gamma y(\tau))d\tau \\
&- \frac{1}{2} \int_0^T \frac{(T-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} f(\tau, y(\tau), D^\gamma y(\tau))d\tau,
\end{aligned}
$$

and

$$
\mathcal{H}_2 y(t) = -\lambda \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(t)d\tau + \frac{\lambda}{2} \int_0^T \frac{(T-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(t)d\tau.
$$

Note that $0 \leq \gamma \leq \beta \leq \alpha \leq 1$, $\alpha+\beta \notin]0,1)$, $\beta+\gamma \notin]0,1)$.
We need also to consider the following hypotheses:

(H1): There exists $K_1, K_2 > 0$, such that for all $t \in [0,T]$ and $u_i, v_i \in \mathbb{R}, i = 1,2$, we have

$$
|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq K_1|u_1 - v_1| + K_2|u_2 - v_2|, \ K := max(K_1, K_2).
$$

(H2): Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function.

(H3): There exists a continuous function ψ defined over $[0,T]$; such that for all $t \in [0,T]$ and $u_i \in \mathbb{R}, i = 1,2$, we have

$$
|f(t, u_1, u_2)| \leq \psi(t); \sup_{t\in[0,T]} = \|\psi\|.
$$

For computation convenience, we define the function $F_y : [0,T] \to \mathbb{R}$ by:

$$
F_y(t) := f(t, y(t), D^\gamma y(t)),
$$

and

$$
\begin{aligned}
\mathcal{I}_1 :=& \frac{3T^{\alpha_2+2\gamma}}{2\Gamma(\alpha_2+1)} \ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)} + \frac{3T^{\alpha_2+2\gamma}}{4\Gamma(\gamma+1)\Gamma(\alpha_2+1)} \ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \\
&+ \frac{3T^{\alpha_3+\gamma}}{4\Gamma(\alpha_3+1)\Gamma(\gamma+1)} + \frac{3T^{\alpha_1}}{2\Gamma(\alpha_1+1)}.
\end{aligned}
$$

$$\mathcal{L}_1 := \frac{T^{\alpha_2+\gamma}}{\Gamma(\alpha_2+1)} \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} + \frac{T^{\alpha_2+\gamma}}{2\Gamma(\alpha_2+1)} \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} + \frac{T^{\alpha_3}}{2\Gamma(\alpha_3+1)} + \frac{T^{\alpha_1-\gamma}}{\Gamma(\alpha_1-\gamma+1)}.$$

$$\mathcal{I}_2 := \frac{3|\lambda|}{2\Gamma(\gamma+1)}, \; \mathcal{L}_2 := |\lambda|.$$

Also, we take:

$$\Delta := K \; (\mathcal{I}_1 + \mathcal{L}_1) + (\mathcal{I}_2 + \mathcal{L}_2).$$

Now, we are ready to study the above problem by means of the fixed point theory.

3.1. A Unique Solution Via Banach Contraction

Theorem 3.1. *Assume that* (H1) *is valid and* $\Delta < 1$. *Then, the problem* (1.1) *has a unique solution on* $[0, T]$.

Proof. We show that the operator $\mathcal{H}$ is contractive. Let $y_i \in X, i = 1, 2$. Then, for each $t \in [0, T]$, we have

$$\begin{aligned}
&\left|\mathcal{H}_1 y_1(t) - \mathcal{H}_1 y_2(t)\right| \\
= \;& \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} |F_{y_1}(\tau) - F_{y_2}(\tau)| \, d\tau \; \left| \frac{T^\gamma \; t^\gamma}{2\Gamma(\gamma+1)} + \frac{T^{2\gamma}}{4\Gamma(\gamma+1)} \right| \\
+ \;& \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} |F_{y_1}(\tau) - F_{y_2}(\tau)| \, d\tau \; \left| \frac{t^{\beta+\gamma}}{T^{\beta-\gamma}} + \frac{T^{2\gamma}}{2} \right| \\
+ \;& \int_0^T \frac{(T-\tau)^{\alpha_3-1}}{\Gamma(\alpha_3)} |F_{y_1}(\tau) - F_{y_2}(\tau)| \, d\tau \; \left| \frac{T^\gamma}{4\Gamma(\gamma+1)} + \frac{t^\gamma}{2\Gamma(\gamma+1)} \right| \\
+ \;& \int_0^t \frac{(T-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} |F_{y_1}(\tau) - F_{y_2}(\tau)| \, d\tau \\
+ \;& \frac{1}{2} \int_0^T \frac{(T-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} |F_{y_1}(\tau) - F_{y_2}(\tau)| \, d\tau.
\end{aligned}$$

Thanks to (H1), yields

$$\begin{aligned}
&\left|\mathcal{H}_1 y_1(t) - \mathcal{H}_1 y_2(t)\right| \\
\leq \;& \frac{K\|y_1 - y_2\|_X \; T^{\alpha_2}}{\Gamma(\alpha_2+1)} \left| \frac{T^\gamma \; t^\gamma}{2\Gamma(\gamma+1)} + \frac{T^{2\gamma}}{4\Gamma(\gamma+1)} \right| \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{K\|y_1-y_2\|_X \; T^{\alpha_2}}{\Gamma(\alpha_2+1)} \left|\frac{t^{\beta+\gamma}}{T^{\beta-\gamma}}+\frac{T^{2\gamma}}{2}\right| \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)} \\
&+ \quad \frac{K\|y_1-y_2\|_X \; T^{\alpha_3}}{\Gamma(\alpha_3+1)} \left|\frac{T^{\gamma}}{4\Gamma(\gamma+1)}+\frac{T^{\gamma}}{2\Gamma(\gamma+1)}\right| \\
&+ \quad \frac{K\|y_1-y_2\|_X \; T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{K\|y_1-y_2\|_X}{2} \; \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\Big|\mathcal{H}_2 y_1(t)-\mathcal{H}_2 y_2(t)\Big| \quad &\leq \quad |\lambda| \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} |y_1(\tau)-y_2(\tau)| \; d\tau \\
&\quad + \frac{|\lambda|}{2} \int_0^T \frac{(T-\tau)^{\gamma-1}}{\Gamma(\gamma)} |y_1(\tau)-y_2(\tau)| \; d\tau.
\end{aligned}
$$

$$
\|\mathcal{H}_2 y_1-\mathcal{H}_2 y_2\|_\infty \quad \leq \quad \mathcal{I}_2 \|y_1-y_2\|_\infty \quad \leq \quad \mathcal{I}_2 \|y_1-y_2\|_X. \tag{3.1}
$$

Therefore,

$$
\|\mathcal{H} y_1-\mathcal{H} y_2\|_\infty \quad \leq \quad (\mathcal{I}_1 K+\mathcal{I}_2) \; \|y_1-y_2\|_X. \tag{3.2}
$$

In the same manner, we can write

$$
\begin{aligned}
&\Big|D\mathcal{H}_1 y_1(t)-D\mathcal{H}_1 y_2(t)\Big| \\
&\leq \quad \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} |F_{y_1}(\tau)-F_{y_2}(\tau)| \; d\tau \; \left|\frac{\gamma T^{\gamma} \; t^{\gamma-1}}{2\Gamma(\gamma+1)}\right| \\
&+ \quad \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} |F_{y_1}(\tau)-F_{y_2}(\tau)| \; d\tau \; \left|\frac{(\beta+\gamma) \; t^{\beta+\gamma-1}}{T^{\beta-\gamma}}\right| \\
&+ \quad \int_0^T \frac{(T-\tau)^{\alpha_3-1}}{\Gamma(\alpha_3)} |F_{y_1}(\tau)-F_{y_2}(\tau)| \; d\tau \; \left|\frac{\gamma t^{\gamma-1}}{2\Gamma(\gamma+1)}\right| \\
&+ \quad \int_0^t \frac{(T-\tau)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} |F_{y_1}(\tau)-F_{y_2}(\tau)| \; d\tau \; .
\end{aligned}
$$

Hence, we have

$$
|D^{\gamma}\mathcal{H}_1 y_1(t)-D^{\gamma}\mathcal{H}_1 y_2(t)|
$$

$$
\begin{aligned}
&\leq \quad \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} |F_{y_1}(\tau)-F_{y_2}(\tau)|\, d\tau \left| \frac{\Gamma(\beta+\gamma+1)\ t^{\beta}}{T^{\beta-\gamma}\ \Gamma(\beta+1)} \right| \\
&+ \quad \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \int_0^T \frac{(T-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} |F_{y_1}(\tau)-F_{y_2}(\tau)|\, d\tau \left| \frac{T^{\gamma}}{2} \right| \\
&+ \quad \frac{1}{2} \int_0^T \frac{(T-\tau)^{\alpha_3-1}}{\Gamma(\alpha_3)} |F_{y_1}(\tau)-F_{y_2}(\tau)|\, d\tau \\
&+ \quad \int_0^t \frac{(T-\tau)^{\alpha_1-\gamma-1}}{\Gamma(\alpha_1-\gamma)} |F_{y_1}(\tau)-F_{y_2}(\tau)|\, d\tau\ .
\end{aligned}
$$

Consequently,

$$\left| D^{\gamma}\mathcal{H}_1 y_1(t) - D^{\gamma}\mathcal{H}_1 y_2(t) \right| \leq \mathcal{L}_1 \|y_1 - y_2\|_X,$$

and

$$\|D^{\gamma}\mathcal{H}_2 y_1 - D^{\gamma}\mathcal{H}_2 y_2\|_{\infty} \quad \leq \quad \mathcal{L}_2 \|y_1 - y_2\|_X. \tag{3.3}$$

This amounts to say that

$$\|D^{\gamma}\mathcal{H} y_1 - D^{\gamma}\mathcal{H} y_2\|_{\infty} \quad \leq \quad (\mathcal{L}_1 K + \mathcal{L}_2)\|y_1 - y_2\|_X. \tag{3.4}$$

Thanks to (3.2)-(3.4), we get

$$\|\mathcal{H}y_1 - \mathcal{H}y_2\|_X \quad \leq \quad \Big((\mathcal{I}_1 + \mathcal{L}_1)K + (\mathcal{I}_2 + \mathcal{L}_2)\Big)\ \|y_1 - y_2\|_X.$$

By Theorem 3.1, we deduce that $\mathcal{H}$ is contractive. As a consequence of Banach contraction principle, we conclude that $\mathcal{H}$ has a unique fixed point which is the solution of (1.1). $\square$

3.2. Existence via Krasnoselskii Theorem

Theorem 3.2. *Assume that* (H2) *and* (H3) *are valid and* $\Delta < 1$. *Then, the problem* (1.1) *has at least one solution* $y(t), t \in [0, T]$ *that satisfies:*

$$\|y\|_X \leq \epsilon$$

where ϵ *is an infinitesimal that satisfies*

$$\epsilon \geq \frac{\|\psi\|\ (\mathcal{I}_1 + \mathcal{L}_1)}{1 - (\mathcal{I}_2 + \mathcal{L}_2)}.$$

Proof. Let us consider the following closed convex subset:

$$B(\epsilon) := \{y \in X, \|y\|_X \leq \epsilon\} \subset X,$$

with

$$\epsilon \geq \frac{\|\psi\| \ (\mathcal{I}_1 + \mathcal{L}_1)}{1 - (\mathcal{I}_2 + \mathcal{L}_2)}.$$

Our first claim is to prove that for any $y_1, y_2 \in B(\epsilon)$, we have $\mathcal{H}B(\epsilon) \subset B(\epsilon)$. Let $y_1, y_2 \in B(\epsilon)$. Then, by (H3), it follows that

$$\|\mathcal{H}_1 y_1 + \mathcal{H}_2 y_2\|_\infty \leq \mathcal{I}_1 \|\psi\| + \mathcal{I}_2 \epsilon.$$

On the other hand, we have

$$\|D^\gamma \mathcal{H}_1 y_1 + D^\gamma \mathcal{H}_2 y_2\|_\infty \leq \mathcal{L}_1 \|\psi\| + \mathcal{L}_2 \epsilon.$$

Therefore,

$$\|\mathcal{H}_1 y_1 + \mathcal{H}_2 y_2\|_X \leq \epsilon.$$

Furthermore, $\mathcal{H}_2$ satisfies the Banach contraction principle (as it has been shown in the inequalities (3.1)-(3.3)). In fact, we have

$$\|\mathcal{H}_2 y_1 - \mathcal{H}_2 y_2\|_X \leq (\mathcal{I}_2 + \mathcal{L}_2) \|y_1 - y_2\|_X.$$

The last step in this proof is to show that $\mathcal{H}_1$ is compact and continuous. The continuity of the operator $\mathcal{H}_1$ is due to the continuity of f (see (H2)). Moreover, $\mathcal{H}_1$ is bounded on $B(\epsilon)$. Indeed, for any $y \in B(\epsilon)$, we have:

$$\|\mathcal{H}_1 y\|_X \leq (\mathcal{I}_1 + \mathcal{L}_1) \ \|\psi\|.$$

Next, we shall prove that $\mathcal{H}_1$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then, we have:

$$\begin{aligned}
&\Big|\mathcal{H}_1 y(t_1) - \mathcal{H}_1 y(t_2)\Big| \\
\leq\ & \frac{\Gamma(\beta - \gamma + 1)}{\Gamma(\beta + 1)} \int_0^T \frac{(T-\tau)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} |F_y(\tau)| \, d\tau \ \left| \frac{T^\gamma \ (t_1^\gamma - t_2^\gamma)}{2\Gamma(\gamma + 1)} \right| \\
+\ & \frac{\Gamma(\beta - \gamma + 1)}{\Gamma(\beta + \gamma + 1)} \int_0^T \frac{(T-\tau)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} |F_y(\tau)| \, d\tau \ \left| \frac{t_1^{\beta+\gamma} - t_2^{\beta+\gamma}}{T^{\beta-\gamma}} \right| \\
+\ & \int_0^T \frac{(T-\tau)^{\alpha_3 - 1}}{\Gamma(\alpha_3)} |F_y(\tau)| \, d\tau \ \ \left| \frac{(t_1^\gamma - t_2^\gamma)}{2\Gamma(\gamma + 1)} \right| \\
+\ & \int_0^{t_1} \frac{\left|(t_1 - \tau)^{\alpha_1 - 1} - (t_2 - \tau)^{\alpha_1 - 1}\right|}{\Gamma(\alpha_1)} |F_y(\tau)| \, d\tau
\end{aligned}$$

$$+ \int_{t_1}^{t_2} \frac{(t_2-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} |F_y(\tau)|\, d\tau.$$

Thanks to (H3), we can write

$$\begin{aligned}
&\Big|\mathcal{H}_1 y(t_1) - \mathcal{H}_1 y(t_2)\Big| \\
\leq &\int_0^{t_1} \frac{\big|(t_1-\tau)^{\alpha_1-1} - (t_2-\tau)^{\alpha_1-1}\big|}{\Gamma(\alpha_1)} |F_y(\tau)|\, d\tau + \int_{t_1}^{t_2} \frac{(t_2-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} |F_y(\tau)|\, d\tau \\
&+\frac{\|\psi\|\ (T)^{\alpha_2+\gamma}}{\Gamma(\alpha_2+1)}\ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)}\ \left|\frac{(t_1^{\gamma}-t_2^{\gamma})}{2\Gamma(\gamma+1)}\right| + \frac{\|\psi\|\ (T)^{\alpha_3}}{\Gamma(\alpha_3+1)}\ \left|\frac{(t_1^{\gamma}-t_2^{\gamma})}{2\Gamma(\gamma+1)}\right| \\
&+\frac{\|\psi\|\ (T)^{\alpha_2}}{\Gamma(\alpha_2+1)}\ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)}\ \left|\frac{t_1^{\beta+\gamma}-t_2^{\beta+\gamma}}{T^{\beta-\gamma}}\right|. \qquad (3.5)
\end{aligned}$$

With the same arguments as before, we have

$$\begin{aligned}
&\Big|D^{\gamma}\mathcal{H}_1 y(t_1) - D^{\gamma}\mathcal{H}_1 y(t_2)\Big| \\
\leq &\int_0^{t_1} \frac{\big|(t_1-\tau)^{\alpha_1-1} - (t_2-\tau)^{\alpha_1-\gamma-1}\big|}{\Gamma(\alpha_1)} |F_y(\tau)|\, d\tau \\
&+\int_{t_1}^{t_2} \frac{(t_2-\tau)^{\alpha_1-\gamma-1}}{\Gamma(\alpha_1)} |F_y(\tau)|\, d\tau + \frac{\|\psi\|\ (T)^{\alpha_2}}{\Gamma(\alpha_2+1)}\ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\beta+\gamma+1)}\ \left|\frac{t_1^{\beta}-t_2^{\beta}}{T^{\beta-\gamma}}\right|. \qquad (3.6)
\end{aligned}$$

The right hand sides of (3.5) and (3.6) tend to zero as $t_1 \to t_2$. Then, $\mathcal{H}_1$ is equicontinuous. Hence by the Arzela-Ascoli theorem, the operator $\mathcal{H}_1$ is compact on $B(\epsilon)$.

Thus, thanks to Krasnoselskii theorem, the problem (1.1) has at least one solution $y; \|y\|_X \leq \epsilon$, with ϵ satisfies the condition in Theorem 8. $\square$

4. An Illustrative Example

We consider the following problem:

(4.1)

$$\begin{cases} D^{\alpha}[D^{\beta}(D^{\gamma}+\lambda)]y(t) = f(t, y(t), D^{\gamma}y(t)),\ t\in[0,1],\ 0<\alpha,\beta,\gamma\leq 1. \\ y(0)+y(1)=0,\ (D^{\gamma})y(0)+(D^{\gamma})y(1)=0,\ D^{\gamma}D^{\gamma}y(0)+D^{\gamma}D^{\gamma}y(1)=0. \end{cases}$$

Here, we take

$$f(t, y(t), D^{\alpha}y(t)) = \pm\sinh(D^{\alpha}y(t)) - 0.95\ D^{\alpha}y(t) - 0.08\ y(t),$$

$$T=1, \alpha=0.9, \beta=0.75, \gamma=0.6, \lambda=0.1.$$

For all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, and $t \in [0.1]$, we have:

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \quad \leq \quad 0.08 \, \Big(|x_1 - y_1| + |x_2 - y_2|\Big).$$

We have also

$$\mathcal{I}_1 = 4.263, \mathcal{I}_2 = 0.167, \mathcal{L}_1 = 4.486, \mathcal{L}_2 = 0.1, \Delta = 0.967.$$

Thanks to Theorem 3.1, we can state that the problem (4.1) has a unique solution on $[0, 1]$. where

$$\begin{aligned}
& y(t) \\
= \; & 1.0152 \int_0^1 \frac{(1-\tau)^{-0.25}}{1.2254} \Big[\pm \sinh(D^{0.6} y(\tau)) - 0.95 \; D^{0.6} y(\tau) - 0.08 \; y(\tau) \Big] d\tau \; \left(\frac{t^{0.6}}{1.7870} - \frac{1}{3.5741} \right) \\
& +0.8785 \int_0^1 \frac{(1-\tau)^{-0.25}}{1.2254} \Big[\pm \sinh(D^{0.6} y(\tau)) - 0.95 \; D^{0.6} y(\tau) - 0.08 \; y(\tau) \Big] d\tau \; \left(t^{\beta+\gamma} - \frac{1}{2} \right) \\
& + \int_0^1 \frac{(1-\tau)^{0.35}}{0.8912} \Big[\pm \sinh(D^{0.6} y(\tau)) - 0.95 \; D^{0.6} y(\tau) - 0.08 \; y(\tau) \Big] d\tau \; \left(\frac{1}{3.5741} - \frac{t^{0.6}}{1.7870} \right) \\
& + \int_0^t \frac{(t-\tau)^{0.95}}{0.9799} \Big[\pm \sinh(D^{0.6} y(\tau)) - 0.95 \; D^{0.6} y(\tau) - 0.08 \; y(\tau) \Big] d\tau \\
& -\frac{1}{2} \int_0^1 \frac{(1-\tau)^{0.95}}{0.9799} \Big[\pm \sinh(D^{0.6} y(\tau)) - 0.95 \; D^{0.6} y(\tau) - 0.08 \; y(\tau) \Big] d\tau, \\
& -\lambda \int_0^t \frac{(t-\tau)^{-0.4}}{1.4892} y(\tau) d\tau \; + \frac{\lambda}{2.9784} \int_0^1 (1-\tau)^{-0.4} y(\tau) d\tau.
\end{aligned}$$

5. Existence of Fractional Chaotic Behaviours

In the following two subsections, we present a numerical approach for the Caputo derivative. Then, in order to study dynamic behavior of the above fractional Jerk problem, we present a reduced fractional differential system that is equivalent to our studied problem. For this reduction, we will show sensitive attractors to initial anti-periodic conditions in phase space, which is one of the properties of certain chaotic behaviours.

The numerical simulation of the fractional system is done by the fourth-order Runge-Kutta applied to Caputo derivative.

5.1. Numerical Approach for Caputo Derivative

Since the Caputo derivative has wide applications, so we put our attention to the numerical approach of this derivative. To do this, we begin this section by recalling the following theorem [11] in which the authors presented an important numerical approach for the Riemann-Liouville fractional integral.

Theorem 5.1. *Assume that $y \in \mathcal{C}^1([0,T],\mathbb{R})$. The numerical approach for fractional integral is given by:*

$$J^\alpha y(t_i) \simeq \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{i} K_j y_j, \qquad i = 0,\ldots,n+1,\ y_0 \textit{ initial condition},$$

where

$$K_j := \sigma_j(\alpha) = \begin{cases} (n+2-j)^{(\alpha+1)} + (n-j)^{(\alpha+1)} - 2(n-j+1)^{(\alpha+1)},\ j = 1\ldots i-1. \\ (n)^{(\alpha+1)} - (n-\alpha)(n+1)^\alpha,\ j=0, \quad 1,\ j=i. \end{cases}$$

Based on Theorem 5.1, we propose the following approximation for Caputo derivative:

Theorem 5.2. *Assume that $y \in \mathcal{C}^1([0,T],\mathbb{R})$ and $0 < \alpha \leq 1$. Then, we have:*

$$D^\alpha y(t_i) \simeq \frac{h^{1-\alpha}}{\Gamma(1-\alpha+2)} \sum_{j=0}^{i} L_j y^{(j)}(t_j), \qquad i = 0,\ldots,n.$$

Where,

$$L_j := \sigma_j(1-\alpha).$$

and

$$y^{(j)} = \begin{cases} \frac{y_1 - y_0}{h},\ j=0, \quad \frac{y_{j+1} - y_{j-1}}{2h},\ j = 1\ldots i-1, \quad \frac{y_i - y_{i-1}}{h},\ j = i. \end{cases}$$

Proof. Let $y \in \mathcal{C}^1([0,T],\mathbb{R}), 0 < \alpha \leq 1, 0 \leq t \leq T$ and setting $h := \frac{T}{n}; n \in \mathbb{N}^*$.

Then, thanks to Theorem 9 and using the above definition of Caputo derivative, we obtain:

$$D^\alpha y(t_n) = J^{1-\alpha}\ Dy(t_n) \approx \frac{h^{1-\alpha}}{\Gamma(1-\alpha+2)} \sum_{j=0}^{n} L_j Dy(t_j).$$

By finite difference scheme, instead of using *"central difference scheme"* for $j = 0,\ldots,n$, we use *"forward difference scheme"* for $j=0$, *"backward difference scheme"* for $j=n$, and *" central difference scheme"* for $j = 1,\ldots n-1$.

By substitution in the above formula, we obtain:

$$D^{\alpha}y(t_n)$$
$$\approx \quad \frac{h^{1-\alpha}}{\Gamma(1-\alpha+2)}\left(\sum_{j=1}^{n-1} L_j\ (\frac{y_{j+1}-y_{j-1}}{2h}) + L_0\ (\frac{y_1-y_0}{h}) + L_n\ (\frac{y_n-y_{n-1}}{h})\right).$$

Theorem 10 is thus proved and a Caputo derivative approximation is obtained. □

5.2. Simulation for Chaotic Behaviours

We note that the problem (1.1) can reduced to the following system:

$$\begin{aligned} D^{\gamma}y(t) &= z(t) - \lambda\ y(t) \\ D^{\beta}z(t) &= w(t) \\ D^{\alpha}w(t) &= f(t, y(t), D^{\gamma}y(t)), \end{aligned}$$

that is

$$\begin{aligned} Dy(t) &= D^{1-\gamma}(z(t) - \lambda y(t)) \\ Dz(t) &= D^{1-\beta}\ w(t) \\ Dw(t) &= D^{1-\alpha}\ f(t, y(t), D^{\gamma}y(t)). \end{aligned} \tag{5.1}$$

(**a:**) As a first simulation, we consider the case where f is given by:

$$f(t, y(t), D^{\gamma}y(t)) = \pm((D^{\alpha}y(t))^2 \pm 2)^{-2} - \ y(t), \tag{5.2}$$

with initial conditions $(0.0945, 0, -0.0945)$, $\lambda = 0.7$ and $h = 0.005$. The integration of system (5.1) is carried out by the fourth-order Runge-Kutta method and the Caputo approach.

- For incommensurate order $(\gamma, \beta, \alpha) = (0.8, 0.85, 0.9)$, we get

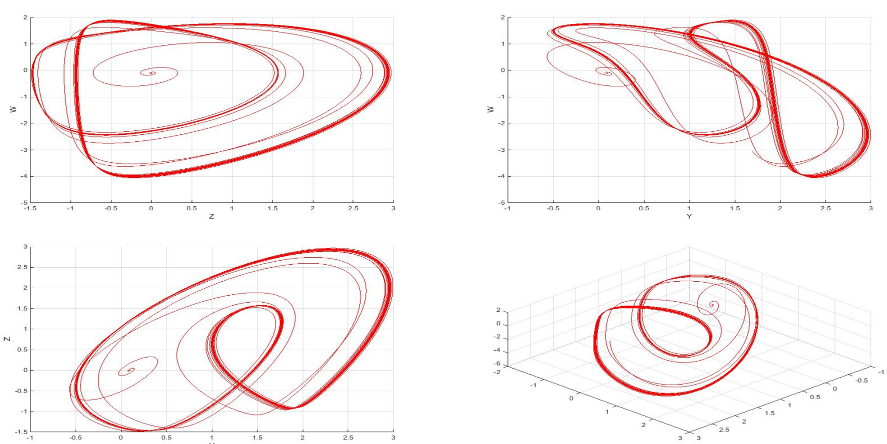

FIG. 5.1: Different phase portrait of incommensurate order of system(5.1) for (5.2)

- For commensurate order $(\gamma,\beta,\alpha)=(0.9,0.9,0.9)$, so with the same data as above, we have

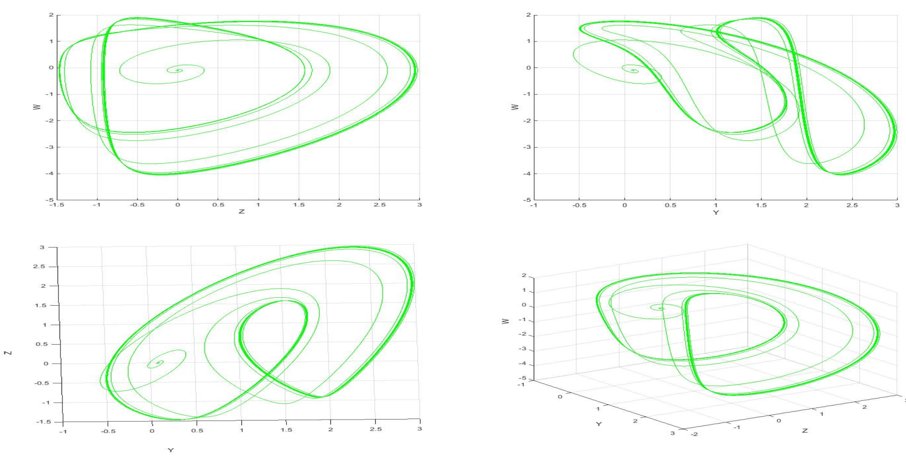

FIG. 5.2: Different phase portrait of commensurate order of system(5.1) for (5.2)

- For $\gamma=\beta=\alpha=1$, the incorporation of system (5.1) is carried out by the $4th$ Runge-Kutta method, we obtain

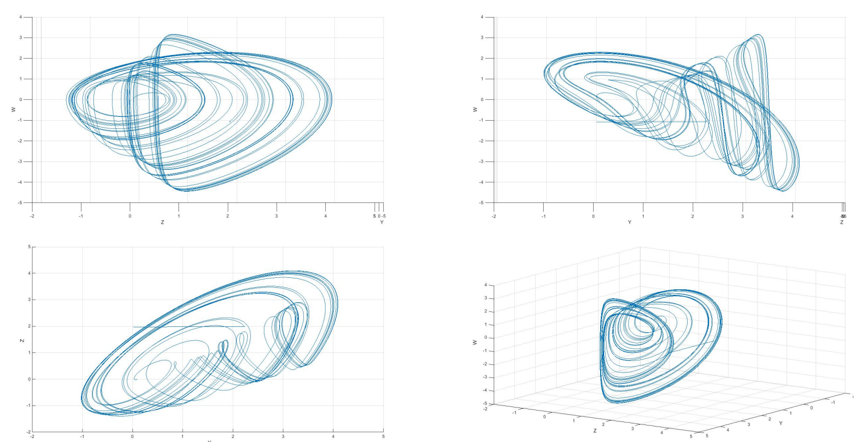

FIG. 5.3: Different phase portrait of $\gamma=\beta=\alpha=1$, of system (5.1) for (5.2)

(**b:**) As a second simulation, we consider the case where f is given by:

$$f(t,y(t),D^{\gamma}y(t))=\pm 0.1\exp(\mp D^{\alpha}y(t))-\ y(t), \tag{5.3}$$

with initial conditions $(0.1021, 0, -0.1021)$, $\lambda = 0.7$ and $h = 0.005$, the integration of system (5.1) is carried out by the fourth-order Runge-Kutta method and Caputo approach.

- For incommensurate order $(\gamma, \beta, \alpha) = (0.85, 0.9, 0.9)$, we get graphical illustrations.

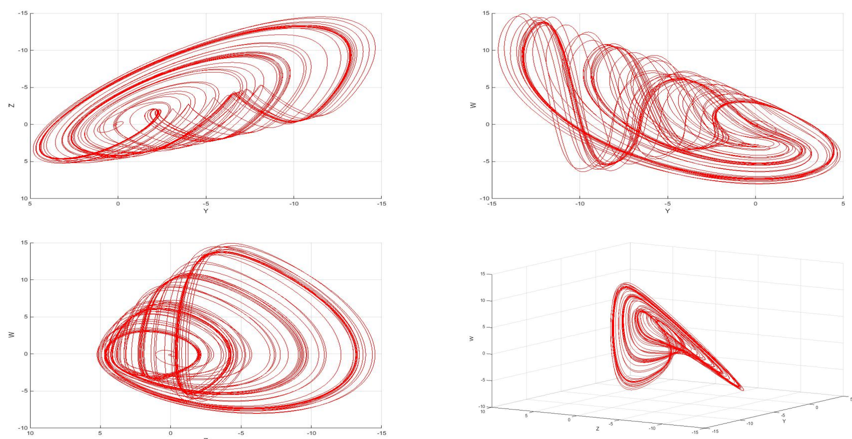

FIG. 5.4: Different phase portrait of incommensurate order of system(5.1) for (5.3)

- For commensurate order $(\gamma, \beta, \alpha) = (0.9, 0.9, 0.9)$, with the same data as above, we obtain

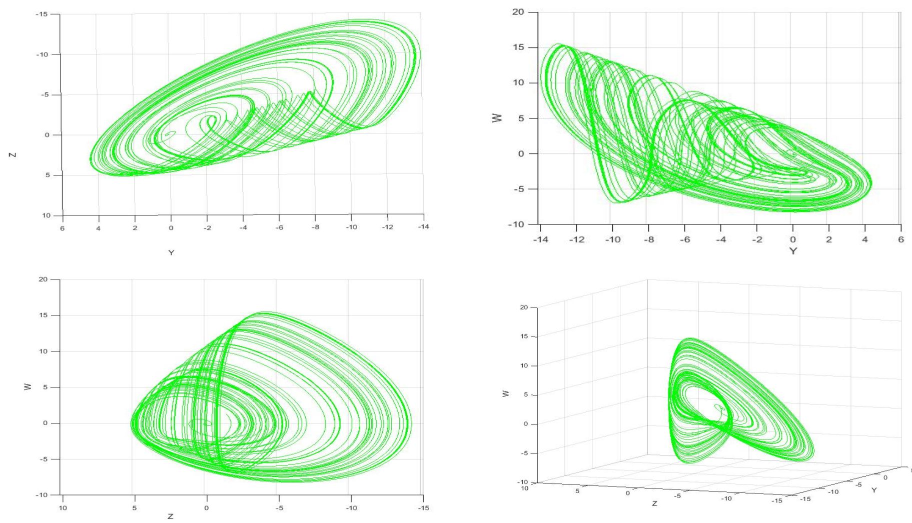

Fig. 5.5: Different phase portrait of commensurate order of system (5.1) for (5.3)

- For $\gamma = \beta = \alpha = 1$, the incorporation of system (5.1) is carried out only by the fourth-order Runge-Kutta method.

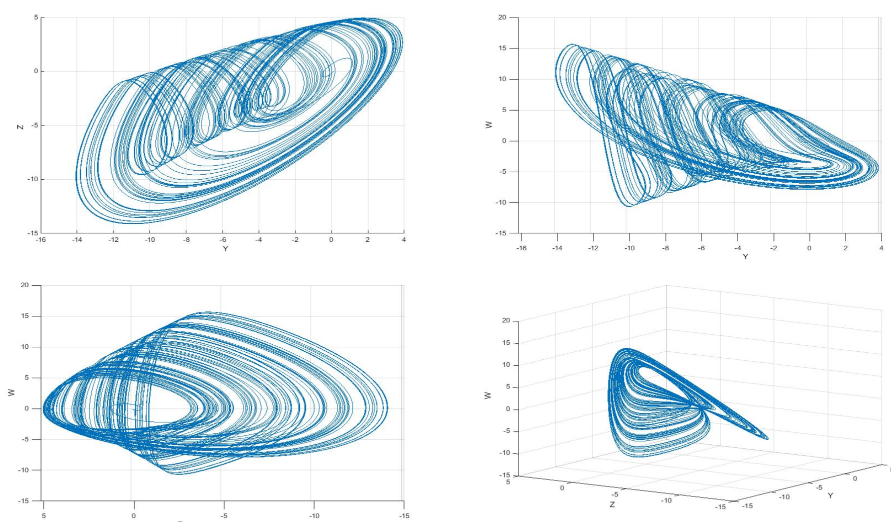

Fig. 5.6: Different phase portrait of $\gamma = \beta = \alpha = 1$, of system (5.1) for (5.3)

Remark 5.1.

- Numerical simulation comparisons revealed a strong correlation for specific parameters. Unfortunately, it is not the same in all cases.
- According to numerical simulations, these fractional order systems may coexist with strange attractors.
- The incorporation has indicated a high qualitative agreement between chaotic systems for $\alpha, \beta, \gamma \longrightarrow 1$.
- Commensurate and incommensurate orders illustrate the influence of the fractional order on chaotic systems.

6. Conclusion

We have introduced a new problem for Jerk circuits of chaotic phenomena by means of fractional derivatives. An existence and uniqueness result has been established by means of Banach contraction principle. Then, using Krasnoselskii fixed point theorem, another main result for the existence of one solution has also been discussed. An illustrative example has been presented to show the applicability of our main result. At the end, an approximation for Caputo derivative has been proved and some numerical solutions having chaotic behaviours have been illustrated and discussed.

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