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ON THE BI-P-HARMONIC MAPS AND THE CONFORMAL MAPS

Seddik Ouakkas and Abderrazak Halimi

Laboratory of Geometry, Analysis, Control and Applications, University of Saida, Dr Moulay Tahar, BP 138 EN-NASR, 20000 Saida, Algeria

Abstract. The objective of this paper is to study the bi-p-harmonicity of a conformal maps. We establish necessary and sufficient condition for a conformal map to be bi-p-harmonic and we construct several examples of this type of maps.
Keywords: p-harmonic map, bi-p-harmonic map, conformal map.

1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations given as follows:

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ. The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g.$$

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Corresponding Author: Seddik Ouakkas, Laboratory of Geometry, Analysis, Control and Applications, University of Saida, Dr Moulay Tahar, BP 138 EN-NASR, 20000 Saida, Algeria | E-mail: seddik.ouakkas@univ-saida.dz
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The biharmonicity of ϕ is characterized by the following equation:

$$\tau_2(\phi) = -Tr_g\left(\nabla^\phi\right)^2\tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on M, then

$$Tr_g\left(\nabla^\phi\right)^2\tau(\phi) = \left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla^M_{e_i}e_i}\right)\tau(\phi).$$

We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. A generalization of harmonic and biharmonic maps, p-harmonic and bi-p-harmonic maps are defined as follows : Let $p \geq 2$, the p-energy functional of ϕ is defined by

$$E_p(\phi) = \frac{1}{p}\int_M |d\phi|^p dv_g.$$

ϕ is said to be p-harmonic if it is a critical point of the p-energy functional (with respect to any variation of compact support). Equivalently, ϕ is p-harmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_p(\phi) = |d\phi|^{p-2}\left\{\tau(\phi) + (p-2)\, d\phi\left(grad \ln|d\phi|\right)\right\} = 0,$$

$\tau_p(\phi)$ is called the p-tension field of ϕ, one can refer to [1], [12] and [15] for more details on p-harmonic maps. The bi-p-energy of ϕ is defined by (see [4]) :

$$E_{2,p}(\phi) = \frac{1}{2}\int_M |\tau_p(\phi)|^2 dv_g.$$

Equivalently, ϕ is bi-p-harmonic if it satisfies the following equation:

$$\begin{aligned}\tau_{2,p}(\phi) = &-Tr_g\nabla^\phi |d\phi|^{p-2}\nabla^\phi\tau_p(\phi) - |d\phi|^{p-2} Tr_g R^N(\tau_p(\phi), d\phi)\, d\phi \\ &- (p-2)\, Tr_g\nabla^\phi\left(\langle\nabla\tau_p(\phi), d\phi\rangle |d\phi|^{p-4} d\phi\right) = 0,\end{aligned} \tag{1.1}$$

where

$$Tr_g\nabla^\phi |d\phi|^{p-2}\nabla^\phi\tau_p(\phi) = \nabla^\phi_{e_i}|d\phi|^{p-2}\nabla^\phi_{e_i}\tau_p(\phi) - |d\phi|^{p-2}\nabla^\phi_{\nabla_{e_i}e_i}\tau_p(\phi)$$

and

$$\begin{aligned}Tr_g\nabla^\phi\left(\langle\nabla\tau_p(\phi), d\phi\rangle |d\phi|^{p-4} d\phi\right) = &\nabla^\phi_{e_i}|d\phi|^{p-4}\langle\nabla\tau_p(\phi), d\phi\rangle\, d\phi(e_i) \\ &- |d\phi|^{p-4}\langle\nabla\tau_p(\phi), d\phi\rangle\, d\phi(\nabla_{e_i}e_i).\end{aligned}$$

$\tau_{2,p}(\phi)$ is called the bi-p-tension of ϕ. Following Jiang's notion (see [9]), we define stress bi-p-energy tensor associated to the bi-p-energy functionals by varying the functionals with respect to the metric on the domain (see [11]). For any $X,Y \in \Gamma(TM)$, we have

$$
\begin{aligned}
S_{2,p}(\phi)(X,Y) &= \frac{1}{2}|\tau_p(\phi)|^2 g(X,Y) + |d\phi|^{p-2}\left\langle d\phi, \nabla^\phi \tau_p(\phi)\right\rangle g(X,Y) \\
&\quad - |d\phi|^{p-2}\left\{h\left(d\phi(X), \nabla^\phi_Y \tau_p(\phi)\right) + h\left(d\phi(Y), \nabla^\phi_X \tau_p(\phi)\right)\right\} \\
&\quad - (p-2)|d\phi|^{p-4}\left\langle d\phi, \nabla^\phi \tau_p(\phi)\right\rangle h(d\phi(X), d\phi(Y)).
\end{aligned}
\tag{1.2}
$$

The stress bi-p-energy tensor of ϕ satisfies the following relationship

$$divS_{2,p}(\phi) = -h(\tau_{2,p}(\phi), d\phi).$$

The notion of bi-p-harmonic maps was introduced by A.M.Cherif [4] where he gave the Euler-Lagrange equations associated with the bi-p-energy and he proved a Liouville type theorem for this class of maps. It is important to recall that the p-biharmonic maps are the critical points of the p-bi-energy functional

$$E_{p,2}(\phi) = \frac{1}{p}\int_M |\tau(\phi)|^p dv_g,$$

and this type of maps was studied in [3], [5] and [8]. This paper is a continuation of Cherif's work [4] on bi-p-harmonic maps where we study the bi-p-harmonicity of a conformal map $\phi : (M^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$, we calculate $\tau_{2,p}(\phi)$ and we prove that any conformal map is bi-p-harmonic if and only if the gradient of its dilation satisfies a certain second-order elliptic partial differential equation. From these results, we construct new examples of bi-p-harmonic maps.

2. The main results

In the first we give the relation between $\tau_{2,p}(\phi)$ and $\tau_p(\phi)$.

Proposition 2.1. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map, then the relation between $\tau_{2,p}(\phi)$ and $\tau_p(\phi)$ is given by the following equation*

$$
\begin{aligned}
\tau_{2,p}(\phi) &= -|d\phi|^{p-2}\left(Tr_g\left(\nabla^\phi\right)^2 \tau_p(\phi) + Tr_g R^N(\tau_p(\phi), d\phi)\, d\phi\right) \\
&\quad + (p-2)|d\phi|^{p-4}\langle \nabla\tau_p(\phi), d\phi\rangle\, d\phi\left(grad\left(\ln|d\phi|^2\right)\right) \\
&\quad - (p-2)|d\phi|^{p-4}\, d\phi\left(grad\langle \nabla\tau_p(\phi), d\phi\rangle\right) \\
&\quad - (p-2)|d\phi|^{-2}\langle \nabla\tau_p(\phi), d\phi\rangle\, \tau_p(\phi) \\
&\quad - \frac{(p-2)}{2}|d\phi|^{p-2}\, \nabla^\phi_{grad\left(\ln|d\phi|^2\right)}\tau_p(\phi).
\end{aligned}
\tag{2.1}
$$

Proof of Proposition 2.1. Let us choose $\{e_i\}_{1\leq i\leq m}$ to be an orthonormal frame on (M,g). By definition, we have

$$
\begin{aligned}
\tau_{2,p}(\phi) &= -Tr_g\nabla^\phi |d\phi|^{p-2}\, \nabla^\phi \tau_p(\phi) - |d\phi|^{p-2}\, Tr_g R^N(\tau_p(\phi), d\phi)\, d\phi \\
&\quad - (p-2)\, Tr_g\nabla^\phi\left(\langle \nabla\tau_p(\phi), d\phi\rangle |d\phi|^{p-4}\, d\phi\right).
\end{aligned}
\tag{2.2}
$$

For the term $Tr_g \nabla^{\phi} |d\phi|^{p-2} \nabla^{\phi} \tau_p(\phi)$, we obtain

$$Tr_g \nabla^{\phi} |d\phi|^{p-2} \nabla^{\phi} \tau_p(\phi) = \nabla^{\phi}_{e_i} |d\phi|^{p-2} \nabla^{\phi}_{e_i} \tau_p(\phi) - |d\phi|^{p-2} \nabla^{\phi}_{\nabla_{e_i} e_i} \tau_p(\phi),$$

a simple calculation gives us

$$\begin{aligned}\nabla^{\phi}_{e_i} |d\phi|^{p-2} \nabla^{\phi}_{e_i} \tau_p(\phi) &= |d\phi|^{p-2} \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} \tau_p(\phi) + e_i\left(|d\phi|^{p-2}\right) \nabla^{\phi}_{e_i} \tau_p(\phi) \\ &= |d\phi|^{p-2} \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} \tau_p(\phi) + \frac{(p-2)}{2} |d\phi|^{p-2} \nabla_{grad\left(\ln|d\phi|^2\right)} \tau_p(\phi),\end{aligned}$$

then

$$\begin{aligned}Tr_g \nabla^{\phi} |d\phi|^{p-2} \nabla^{\phi} \tau_p(\phi) &= |d\phi|^{p-2} Tr_g \left(\nabla^{\phi}\right)^2 \tau_p(\phi) \\ &+ \frac{(p-2)}{2} |d\phi|^{p-2} \nabla^{\phi}_{grad\left(\ln|d\phi|^2\right)} \tau_p(\phi).\end{aligned} \tag{2.3}$$

We will develop the term $Tr_g \nabla^{\phi}\left(\langle \nabla \tau_p(\phi), d\phi\rangle |d\phi|^{p-4} d\phi\right)$, we have

$$\begin{aligned}&Tr_g \nabla^{\phi}\left(\langle \nabla \tau_p(\phi), d\phi\rangle |d\phi|^{p-4} d\phi\right) \\ &= \nabla^{\phi}_{e_i} |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle d\phi(e_i) - |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle d\phi(\nabla_{e_i} e_i) \\ &= |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle \nabla^{\phi}_{e_i} d\phi(e_i) + e_i\left(|d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle\right) d\phi(e_i) \\ &- |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle d\phi(\nabla_{e_i} e_i) \\ &= |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle \nabla^{\phi}_{e_i} d\phi(e_i) - |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle d\phi(\nabla_{e_i} e_i) \\ &+ |d\phi|^{p-4} e_i(\langle \nabla \tau_p(\phi), d\phi\rangle) d\phi(e_i) + \langle \nabla \tau_p(\phi), d\phi\rangle e_i\left(|d\phi|^{p-4}\right) d\phi(e_i) \\ &= |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle \tau(\phi) + |d\phi|^{p-4} d\phi(grad \langle \nabla \tau_p(\phi), d\phi\rangle) \\ &+ \frac{p-4}{2} |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle d\phi\left(grad\left(\ln|d\phi|^2\right)\right).\end{aligned}$$

Using the fact that

$$\tau(\phi) = |d\phi|^{-p+2} \tau_p(\phi) - \frac{(p-2)}{2} d\phi\left(grad\left(\ln|d\phi|^2\right)\right),$$

it follows that

$$\begin{aligned}Tr_g \nabla^{\phi}\left(\langle \nabla \tau_p(\phi), d\phi\rangle |d\phi|^{p-4} d\phi\right) &= |d\phi|^{-2} \langle \nabla \tau_p(\phi), d\phi\rangle \tau_p(\phi) \\ &+ |d\phi|^{p-4} d\phi(grad \langle \nabla \tau_p(\phi), d\phi\rangle) \\ &- |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi\rangle d\phi\left(grad\left(\ln|d\phi|^2\right)\right).\end{aligned} \tag{2.4}$$

By replacing (2.3) and (2.4) in (2.2), we deduce that

$$\begin{aligned}\tau_{2,p}(\phi) =& -|d\phi|^{p-2}\left(Tr_g\left(\nabla^{\phi}\right)^2\tau_p(\phi) + Tr_g R^N\left(\tau_p(\phi), d\phi\right)d\phi\right)\\ &+(p-2)|d\phi|^{p-4}\langle\nabla\tau_p(\phi), d\phi\rangle d\phi\left(grad\left(\ln|d\phi|^2\right)\right)\\ &-(p-2)|d\phi|^{p-4}d\phi\left(grad\langle\nabla\tau_p(\phi), d\phi\rangle\right)\\ &-(p-2)|d\phi|^{-2}\langle\nabla\tau_p(\phi), d\phi\rangle\tau_p(\phi)\\ &-\frac{(p-2)}{2}|d\phi|^{p-2}\nabla^{\phi}_{grad\left(\ln|d\phi|^2\right)}\tau_p(\phi).\end{aligned}$$

Theorem 2.1. *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ , then the bi-p-tension of ϕ is given by*

$$\tau_{2,p}(\phi) = (n-p)\, n^{p-3}\lambda^{2p-4}d\phi\left(H(\lambda, n, p)\right),$$

where

$$\begin{aligned}H(\lambda, n, p) =& (n+p-2)\, grad\,(\Delta\ln\lambda)\\ &-\frac{\left(n^2-5np+4n-2p^2+8p-8\right)}{2}grad\left(|grad\ln\lambda|^2\right)\\ &-(p-1)\left(n^2-3np+4n-2p^2+8p-8\right)|grad\ln\lambda|^2\, grad\ln\lambda\\ &-2\left(n-p^2+3p-2\right)(\Delta\ln\lambda)\, grad\ln\lambda + 2nRicci\,(grad\ln\lambda).\end{aligned}$$

Lemma 2.1. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map. For any vector filed X and for any smooth function f on M, we have*

$$Tr_g\left(\nabla^{\phi}\right)^2 f d\phi(X) = fTr_g\left(\nabla^{\phi}\right)^2 d\phi(X) + 2\nabla^{\phi}_{gradf}d\phi(X) + (\Delta f)\, d\phi(X).$$

Proof of Theorem 2.1. The fact that the map ϕ is conformal of dilation λ gives us

$$\tau(\phi) = (2-n)\, d\phi\,(grad\ln\lambda), \qquad |d\phi|^2 = n\lambda^2, \qquad |d\phi|^{p-2} = n^{\frac{p-2}{2}}\lambda^{p-2}$$

and

$$grad\left(\ln|d\phi|^2\right) = 2grad\ln\lambda.$$

Then

$$\tau_p(\phi) = (p-n)\, n^{\frac{p-2}{2}}\lambda^{p-2}d\phi\,(grad\ln\lambda).$$

By replacing the expression of $\tau_p(\phi)$ in (2.1), we obtain

$$\begin{aligned}\tau_{2,p}(\phi) =& -(p-n)\, n^{p-2}\lambda^{p-2}Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\,(grad\ln\lambda)\\ &-(p-n)\, n^{p-2}\lambda^{p-2}Tr_g R^N\left(\lambda^{p-2}d\phi\,(grad\ln\lambda), d\phi\right)d\phi\\ &-(p-2)(p-n)\, n^{p-2}\lambda^{p-2}\nabla^{\phi}_{grad\ln\lambda}\lambda^{p-2}d\phi\,(grad\ln\lambda)\\ &-(p-2)(p-n)^2 n^{p-3}\lambda^{p-4}\left\langle\nabla\lambda^{p-2}d\phi\,(grad\ln\lambda), d\phi\right\rangle d\phi\,(grad\ln\lambda)\\ &-(p-2)(p-n)\, n^{p-3}\lambda^{p-4}d\phi\left(grad\left\langle\nabla\lambda^{p-2}d\phi\,(grad\ln\lambda), d\phi\right\rangle\right)\\ &+2(p-2)(p-n)\, n^{p-3}\lambda^{p-4}\left\langle\nabla\lambda^{p-2}d\phi\,(grad\ln\lambda), d\phi\right\rangle d\phi\,(grad\ln\lambda).\end{aligned} \tag{2.5}$$

We will simplify the terms of this last equation.
For the term $Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right)$, we have

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right) &= \lambda^{p-2}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\lambda\right)\\ &+ 2\nabla^{\phi}_{grad\lambda^{p-2}}d\phi\left(grad\ln\lambda\right)\\ &+ \left(\Delta\lambda^{p-2}\right)d\phi\left(grad\ln\lambda\right).\end{aligned}$$

The fact that ϕ is conformal gives us (see [13])

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\lambda\right) &= d\phi\left(grad\Delta\ln\lambda\right) + 2d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right)\\ &- (n-2)\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\ &- \left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right) + d\phi\left(Ricci\left(grad\ln\lambda\right)\right)\end{aligned}$$

and

$$\begin{aligned}2\nabla^{\phi}_{grad\lambda^{p-2}}d\phi\left(grad\ln\lambda\right) &= 2\left(p-2\right)\lambda^{p-2}\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\ &+ \left(p-2\right)\lambda^{p-2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right).\end{aligned}$$

A simple calculation gives

$$\Delta\lambda^{p-2} = \left(p-2\right)\lambda^{p-2}\left(\Delta\ln\lambda + \left(p-2\right)\left|grad\ln\lambda\right|^2\right),$$

then

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right) &= \lambda^{p-2}d\phi\left(grad\Delta\ln\lambda\right)\\ &+ p\lambda^{p-2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right)\\ &- \left(n-p^2+2p-2\right)\lambda^{p-2}\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\ &+ \left(p-3\right)\lambda^{p-2}\left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right)\\ &+ \lambda^{p-2}d\phi\left(Ricci\left(grad\ln\lambda\right)\right).\end{aligned} \tag{2.6}$$

The fact that ϕ conformal also gives us the following formulas (see [13])

$$\begin{aligned}Tr_gR^N\left(d\phi\left(grad\ln\lambda\right),d\phi\right)d\phi &= -\frac{n-2}{2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right)\\ &- \left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right)\\ &+ d\phi\left(Ricci\left(grad\ln\lambda\right)\right)\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\nabla^{\phi}_{grad\ln\lambda}\lambda^{p-2}d\phi\left(grad\ln\lambda\right) &= \left(p-1\right)\lambda^{p-2}\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\ &+ \frac{1}{2}\lambda^{p-2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right).\end{aligned} \tag{2.8}$$

For the term $\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle$, we have

$$\begin{aligned}\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle &= Tr_g h\left(\nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right)\\ &= h\left(\nabla_{e_i} \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\left(e_i\right)\right)\\ &= \lambda^{p-2} h\left(\nabla_{e_i} d\phi\left(grad \ln \lambda\right), d\phi\left(e_i\right)\right)\\ &+ e_i\left(\lambda^{p-2}\right) h\left(d\phi\left(grad \ln \lambda\right), d\phi\left(e_i\right)\right)\\ &= \lambda^{p-2}\left(\lambda^2 \Delta \ln \lambda + n\lambda^2 \left|grad \ln \lambda\right|^2\right)\\ &+ (p-2)\lambda^{p-2}\lambda^2 \left|grad \ln \lambda\right|^2 .\end{aligned}$$

Then

$$\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle = \lambda^p \left(\Delta \ln \lambda + (n+p-2)\left|grad \ln \lambda\right|^2\right). \tag{2.9}$$

Finally, using the following formulas

$$grad\left(\lambda^p\left(\Delta \ln \lambda\right)\right) = \lambda^p grad \Delta \ln \lambda + p\lambda^p \left(\Delta \ln \lambda\right) grad \ln \lambda$$

and

$$grad\left(\lambda^p \left|grad \ln \lambda\right|^2\right) = \lambda^p grad\left(\left|grad \ln \lambda\right|^2\right) + p\lambda^p \left|grad \ln \lambda\right|^2 grad \ln \lambda,$$

we obtain

$$\begin{aligned}grad\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle &= \lambda^p grad \Delta \ln \lambda + p\lambda^p \left(\Delta \ln \lambda\right) grad \ln \lambda\\ &+ \lambda^p (n+p-2)\, grad\left(\left|grad \ln \lambda\right|^2\right)\\ &+ p(n+p-2)\lambda^p \left|grad \ln \lambda\right|^2 grad \ln \lambda\end{aligned} \tag{2.10}$$

If we replace (2.6), (2.7), (2.8),(2.9) and (2.10) in (2.5), we conclude that

$$\tau_{2,p}(\phi) = (n-p)\, n^{p-3} \lambda^{2p-4} d\phi\left(H(\lambda, n, p)\right),$$

where

$$\begin{aligned}H(\lambda, n, p) &= (n+p-2)\, grad\left(\Delta \ln \lambda\right)\\ &- \frac{\left(n^2 - 5np + 4n - 2p^2 + 8p - 8\right)}{2} grad\left(\left|grad \ln \lambda\right|^2\right)\\ &- (p-1)\left(n^2 - 3np + 4n - 2p^2 + 8p - 8\right)\left|grad \ln \lambda\right|^2 grad \ln \lambda\\ &- 2\left(n - p^2 + 3p - 2\right)\left(\Delta \ln \lambda\right) grad \ln \lambda + 2nRicci\left(grad \ln \lambda\right).\end{aligned}$$

Theorem 2.2. *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ , then ϕ is bi-p-harmonic if and only if*

$$\begin{aligned}&(n+p-2)\, grad\left(\Delta \ln \lambda\right) - \frac{\left(n^2 - 5np + 4n - 2p^2 + 8p - 8\right)}{2} grad\left(\left|grad \ln \lambda\right|^2\right)\\ &- (p-1)\left(n^2 - 3np + 4n - 2p^2 + 8p - 8\right)\left|grad \ln \lambda\right|^2 grad \ln \lambda\\ &- 2\left(n - p^2 + 3p - 2\right)\left(\Delta \ln \lambda\right) grad \ln \lambda + 2nRicci\left(grad \ln \lambda\right) = 0.\end{aligned}$$

If we consider a conformal map $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ where we suppose that the dilation λ is radial, then the bi-p-harmonicity of ϕ is equivalent to an ordinary differential equation.

Corollary 2.1. *Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ where we suppose that the dilation λ is radial $(\lambda = \lambda(r), r = |x|)$. By setting $\beta = (\ln \lambda)'$, we get (see [13])*

$$grad \ln \lambda = \beta \frac{\partial}{\partial r}, \qquad |grad \ln \lambda|^2 = \beta^2, \qquad grad\left(|grad \ln \lambda|^2\right) = 2\beta\beta' \frac{\partial}{\partial r}$$

and

$$\Delta \ln \lambda = \beta' + \frac{n-1}{r}\beta, \qquad grad \Delta \ln \lambda = \left(\beta'' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta\right)\frac{\partial}{\partial r}.$$

Using Theorem 2.2, we deduce that ϕ is bi-p-harmonic if and only if β satisfies the following differential equation :

(2.11)

$$\begin{aligned}&(n+p-2)\beta'' - \left(n^2 - 5np + 6n - 4p^2 + 14p - 12\right)\beta\beta' + \frac{(n+p-2)(n-1)}{r}\beta' \\ &- \frac{(n+p-2)(n-1)}{r^2}\beta + \frac{2\left(p^2 - 3p - n + 2\right)(n-1)}{r}\beta^2 \\ &+ (p-1)\left(-n^2 + 3np - 4n + 2p^2 - 8p + 8\right)\beta^3 = 0.\end{aligned}$$

To solve equation (2.11) , we will study two types of solutions. In the first case, we look at the solutions which are written in the form $\beta = \frac{a}{r}, a \in \mathbb{R}^*$, we obtain the following result.

Corollary 2.2. *Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ where we suppose that $(\ln \lambda)' = \beta = \frac{a}{r}, a \in \mathbb{R}^*$. Then ϕ is bi-p-harmonic if and only if a is solution of the following algebraic equation :*

(2.12)

$$\begin{aligned}&a^2n^2p - a^2n^2 - 3a^2np^2 + 7a^2np - 4a^2n - 2a^2p^3 + 10a^2p^2 - 16a^2p + 8a^2 + an^2 \\ &- 2anp^2 + 11anp - 12an + 6ap^2 - 20ap + 16a + 2n^2 + 2np - 8n - 4p + 8 = 0.\end{aligned}$$

Remark 2.1. Equation (2.12) leads us to two types of solutions

1.
$$a = -\frac{2(n-2)\left(n + \sqrt{n(17n-16)}\right)}{(3n^2 - 6n + 4)\sqrt{n(17n-16)} - 13n^3 + 42n^2 - 28n}$$

   and
$$p = \frac{1}{4}\sqrt{n(17n-16)} - \frac{3}{4}n + 2,$$

   where $n \geq 3$.

2.
$$a=\frac{A\left(n,p\right)-12n-20p-2np^{2}+11np+n^{2}+6p^{2}+16}{8n+32p+6np^{2}-2n^{2}p-14np+2n^{2}-20p^{2}+4p^{3}-16}$$
or
$$a=-\frac{A\left(n,p\right)+12n+20p+2np^{2}-11np-n^{2}-6p^{2}-16}{8n+32p+6np^{2}-2n^{2}p-14np+2n^{2}-20p^{2}+4p^{3}-16},$$
where
$$A\left(n,p\right)=\sqrt{\begin{array}{c}4\left(n-1\right)^{2}p^{4}-4\left(n-1\right)\left(n-4\right)p^{3}+\left(12n^{3}-35n^{2}+8n+16\right)p^{2}\\ -2n\left(4n^{3}-3n^{2}-16n+16\right)p+n^{2}\left(3n-4\right)^{2}\end{array}}$$
and
$$p\neq\frac{1}{4}\sqrt{n\left(17n-16\right)}-\frac{3}{4}n+2$$

Remark 2.1 allows us to study the following examples. The examples to be cited correspond to the cases where $a=-2$ and $a=-1$.

Example 2.1. We consider the inversion $\phi:\mathbb{R}^{n}\backslash\{0\}\longrightarrow\mathbb{R}^{n}\backslash\{0\}$ $(n\geq3)$ defined by $\phi\left(x\right)=\frac{x}{|x|^{2}}$. ϕ is a conformal map with dilation $\lambda=\frac{1}{r^{2}}$. We deduce that ϕ is bi-p-harmonic if and only if
$$p=-\frac{1}{2}n+\frac{1}{4}\sqrt{-20n+12n^{2}+9}+\frac{5}{4},\qquad n\geq4$$
or
$$p=-\frac{3}{4}n+\frac{1}{4}\sqrt{n\left(17n-16\right)}+2,\qquad n\geq3.$$

Example 2.2. Let $\phi:\mathbb{R}^{n}\backslash\{0\}\longrightarrow mathbbR\times S^{n-1}$ given in polar coordinates by
$$\phi\left(r\theta\right)=\left(\ln r,\theta\right),\qquad r>0,\qquad\theta\in S^{n-1}\subset\mathbb{R}^{n}.$$
ϕ is a conformal map with dilation $\lambda=\frac{1}{r}$. We conclude that ϕ is bi-p-harmonic if and only if
$$p=\frac{n}{2},\qquad n\geq4$$
or
$$p=-\frac{3}{4}n+\frac{1}{4}\sqrt{n\left(17n-16\right)}+2,\qquad n\geq3.$$

As a second particular case, we will look for the solutions of the form $\beta=\frac{a}{1+r^{2}},a\in\mathbb{R}^{*}$.

Corollary 2.3. *Let $\phi:(\mathbb{R}^{n},g)\longrightarrow(N^{n},h)$ $(n\geq3)$ be a conformal map of dilation λ where we suppose that $\left(\ln\lambda\right)'=\beta=\frac{a}{1+r^{2}},a\in\mathbb{R}^{*}$. Then ϕ is bi-p-harmonic if and only if a is solution of the following system:*

$$(2.13)\qquad\left\{\begin{array}{c}n^{5}p+2n^{5}-3n^{4}p^{2}-6n^{4}p-4n^{4}+n^{3}p^{3}+6n^{3}p^{2}\\ +14n^{3}p-4n^{3}+3n^{2}p^{4}-6n^{2}p^{3}-12n^{2}p^{2}+4n^{2}p\\ +8n^{2}-2np^{5}+4np^{4}-2np^{3}+16np^{2}-24np\\ +4p^{4}-16p^{3}+16p^{2}=0\\ \textit{and}\\ 3an^{2}-2anp^{2}+anp-2ap^{2}+8ap-8a+2n^{2}+2np+4p-8=0\end{array}\right.$$

Remark 2.2. To solve this system, we distingue three cases

1. $$p = n, \qquad a = \frac{2}{n-2}, \qquad n \geq 3.$$

   In this case, the conformal map is n-harmonic so bi-n-harmonic.

2. $$p = \frac{n}{2}, \qquad a = \frac{6n-8}{n^2-8n+8}, \qquad n \geq 4.$$

   Then ϕ is bi-p-harmonic non-p-harmonic.

3. $$p = \frac{1}{2n}\left(\sqrt{-16n+4n^2+8n^3+n^4+4}+n^2+2\right)$$

   and

   $$a = -\frac{2n^2+2np+4p-8}{3n^2-2np^2+np-2p^2+8p-8}, \qquad n \geq 3.$$

   Then ϕ is bi-p-harmonic non-p-harmonic.

As the last result of this paper, we calculate the stress bi-p-energy tensor for a conformal map.

Theorem 2.3. *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map of dilation λ, then we have*

$$\begin{aligned} & S_{2,p}(\phi)(X,Y) \\ & = \frac{p-n}{2} n^{p-3}\lambda^{2p-2}\left(n(n+p-4)-2(p-2)^2\right)|grad\ln\lambda|^2 g(X,Y) \\ & + (p-n)(n-p+2)n^{p-3}\lambda^{2p-2}(\Delta\ln\lambda)g(X,Y) \\ & - 2(p-n)n^{p-2}\lambda^{2p-2}(\nabla d\ln\lambda(X,Y)-(p-2)X(\ln\lambda)Y(\ln\lambda)), \end{aligned} \tag{2.14}$$

and the trace of $S_{2,p}(\phi)$ is given by

$$\begin{aligned} & Tr_g S_{2,p}(\phi) \\ & = \frac{p-n}{2} n^{p-2}\lambda^{2p-2}(n(n+p-4)-2(p-2)(p-4))|grad\ln\lambda|^2 \\ & - (p-n)^2 n^{p-2}\lambda^{2p-2}(\Delta\ln\lambda). \end{aligned} \tag{2.15}$$

By using the fact that

$$\Delta\lambda^k = k\lambda^k\left(\Delta\ln\lambda + k|grad\ln\lambda|^2\right),$$

we obtain the following corollary :

Corollary 2.4. *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map of dilation λ where $n \neq p$, then*

$$Tr_g S_{2,p}(\phi) = -(p-n)^2 n^{p-2}\lambda^{2p-2}T(\lambda),$$

where

$$T(\lambda) = \Delta \ln \lambda + \frac{n(n+p-4) - 2(p-2)(p-4)}{2(n-p)} |grad \ln \lambda|^2$$

and

$$Tr_g S_{2,p}(\phi) = 0 \text{ if and only if the function } \lambda^{\frac{n(n+p-4)-2(p-2)(p-4)}{2(n-p)}} \text{ is harmonic.}$$

Remark 2.3. Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$, $(n \neq p)$ be a conformal map of dilation λ where we suppose that the dilation λ is radial. By setting $\beta = (\ln \lambda)'$, we deduce that the trace of $S_{2,p}(\phi)$ is zero if if and only if β satisfies the following differential equation :

$$\beta' + \frac{n-1}{r}\beta + \frac{n+2p-4}{2}\beta^2 = 0. \tag{2.16}$$

The general solution of this equation is given by :

$$\beta = \begin{cases} \frac{2(n-2)}{A(n-2)r^{n-1} - (n+2p-4)r}, & n \neq 2, \quad A \in \mathbb{R} \\ \frac{2}{(n+2p-4)r \ln r + Ar}, & n = 2, \quad A \in \mathbb{R}. \end{cases}$$

Remark 2.4. Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$, $(n \neq p, \quad n \neq 2)$ be a conformal map of dilation λ where we suppose that the dilation λ is radial. we will look for the solutions of the form $\beta = \frac{a}{r}, a \in \mathbb{R}^*$. we deduce that the trace of $S_{2,p}(\phi)$ is zero if if and only if

$$a = -\frac{2(n-2)}{n+2p-4}, \qquad n + 2p - 4 \neq 0. \tag{2.17}$$

For example, if we consider the conformal map $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R} \times S^{n-1}$ given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$, we conclude that for this map ϕ the trace of $S_{2,p}(\phi)$ is zero if if and only if $n = 2p$.

Proof of Theorem 2.3. Let us choose $\{e_i\}_{1 \leq i \leq n}$ to be an orthonormal frame on (M, g). By definition, we have

$$\begin{aligned} S_{2,p}(\phi)(X,Y) &= \frac{1}{2} |\tau_p(\phi)|^2 g(X,Y) + |d\phi|^{p-2} \left\langle d\phi, \nabla^\phi \tau_p(\phi) \right\rangle g(X,Y) \\ &- |d\phi|^{p-2} h\left(d\phi(X), \nabla^\phi_Y \tau_p(\phi)\right) - |d\phi|^{p-2} h\left(d\phi(Y), \nabla^\phi_X \tau_p(\phi)\right) \\ &- (p-2) |d\phi|^{p-4} \left\langle d\phi, \nabla^\phi \tau_p(\phi) \right\rangle h(d\phi(X), d\phi(Y)). \end{aligned} \tag{2.18}$$

Using the fact that

$$\tau_p(\phi) = (p-n) n^{\frac{p-2}{2}} \lambda^{p-2} d\phi(grad \ln \lambda),$$

we obtain

$$|\tau_p(\phi)|^2 = (p-n)^2 n^{p-2} \lambda^{2p-2} |grad \ln \lambda|^2. \tag{2.19}$$

For the term $\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle$, we have

$$\begin{aligned}\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle &= h\left(d\phi(e_i), \nabla^{\phi}_{e_i}\tau_p(\phi)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}} h\left(d\phi(e_i), \nabla^{\phi}_{e_i}\lambda^{p-2} d\phi(grad \ln\lambda)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}}\lambda^{p-2} h\left(d\phi(e_i), \nabla^{\phi}_{e_i} d\phi(grad \ln\lambda)\right)\\ &+ (p-n)\, n^{\frac{p-2}{2}} e_i\left(\lambda^{p-2}\right) h\left(d\phi(e_i), d\phi(grad \ln\lambda)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\left(\Delta\ln\lambda + n\left|grad\ln\lambda\right|^2\right)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}}\lambda^{p}\left|grad\ln\lambda\right|^2.\end{aligned}$$

It follows that

$$\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle = (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\left(\Delta\ln\lambda + (n+p-2)\left|grad\ln\lambda\right|^2\right). \tag{2.20}$$

It remains to simplify $h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right)$ and $h\left(d\phi(Y), \nabla^{\phi}_{X}\tau_p(\phi)\right)$, we have

$$\begin{aligned}h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right) &= (p-n)\, n^{\frac{p-2}{2}} h\left(d\phi(X), \nabla^{\phi}_{Y}\lambda^{p-2} d\phi(grad\ln\lambda)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\nabla d\ln\lambda(X,Y)\\ &+ (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}}\lambda^{p} X(\ln\lambda)\, Y(\ln\lambda),\end{aligned}$$

which gives us

$$\begin{aligned}h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right) &= (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\nabla d\ln\lambda(X,Y)\\ &+ (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}}\lambda^{p} X(\ln\lambda)\, Y(\ln\lambda).\end{aligned} \tag{2.21}$$

A similar calculation gives

$$\begin{aligned}h\left(d\phi(Y), \nabla^{\phi}_{X}\tau_p(\phi)\right) &= (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\nabla d\ln\lambda(X,Y)\\ &+ (p-n)\, n^{\frac{p-2}{2}}\lambda^{p}\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}}\lambda^{p} X(\ln\lambda)\, Y(\ln\lambda).\end{aligned} \tag{2.22}$$

By substituting (2.19), (2.20), (2.21) and (2.22) in (2.18) and using the fact that

$$\left|d\phi\right|^{p-2} = n^{\frac{p-2}{2}}\lambda^{p-2}, \quad \left|d\phi\right|^{p-4} = n^{\frac{p-4}{2}}\lambda^{p-4},$$

we deduce that

$$\begin{aligned}&S_{2,p}(\phi)(X,Y)\\ &= \frac{p-n}{2} n^{p-3}\lambda^{2p-2}\left(n(n+p-4) - 2(p-2)^2\right)\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(n-p+2)\, n^{p-3}\lambda^{2p-2}(\Delta\ln\lambda)\, g(X,Y)\\ &- 2(p-n)\, n^{p-2}\lambda^{2p-2}\left(\nabla d\ln\lambda(X,Y) - (p-2) X(\ln\lambda)\, Y(\ln\lambda)\right).\end{aligned}$$

To complete the proof, let's calculate the trace of $S_{2,p}(\phi)$, we have

$$\begin{aligned}
&Tr_g S_{2,p}(\phi) = S_{2,p}(\phi)(e_i, e_i) \\
&= \frac{p-n}{2} n^{p-3} \lambda^{2p-2} \left(n(n+p-4) - 2(p-2)^2 \right) |grad \ln \lambda|^2 g(e_i, e_i) \\
&+ (p-n)(n-p+2) n^{p-3} \lambda^{2p-2} (\Delta \ln \lambda) g(e_i, e_i) \\
&- 2(p-n) n^{p-2} \lambda^{2p-2} (\nabla d \ln \lambda (e_i, e_i) - (p-2) e_i(\ln \lambda) e_i(\ln \lambda)),
\end{aligned}$$

then

$$\begin{aligned}
Tr_g S_{2,p}(\phi) = &\frac{p-n}{2} n^{p-2} \lambda^{2p-2} (n(n+p-4) - 2(p-2)(p-4)) |grad \ln \lambda|^2 \\
&- (p-n)^2 n^{p-2} \lambda^{2p-2} (\Delta \ln \lambda).
\end{aligned}$$

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