ON WARPED PRODUCT MANIFOLDS ADMITTING $\tau$-QUASI RICCI-HARMONIC METRICS

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Abstract. In this paper, we study warped product manifolds admitting $\tau$-quasi Ricci-harmonic (RH) metrics. We prove that the metric of the fibre is harmonic Einstein when warped product metric is $\tau$-quasi RH metric. We also provide some conditions for $M$ to be a harmonic Einstein manifold. Finally, we provide necessary and sufficient conditions for a metric $g$ to be $\tau$-quasi RH metric by using a differential equation system.

Keywords: warped products, gradient Ricci-Harmonic soliton, $\tau$-quasi Ricci-Harmonic metric, Harmonic Einstein

1. Introduction

Various geometric flows have been studied recently and one of them is Ricci flow coupled with harmonic map flow (shortly RH for Ricci-harmonic), defined by Müller [14, 15]. Let $(M^n, g(t))$ and $(N^m, h)$ be smooth Riemannian manifolds and $\phi(t) : M \to N$ is a family of smooth maps between $(M^n, g(t))$ with the metric $g(t)$ evolving along the RH flow and a fixed Riemannian manifold $(N, h)$. The Ricci-harmonic flow is the coupled system

$$
\begin{cases}
\frac{\partial}{\partial t} g = -2 \text{Ric} + 2 c d\phi \otimes d\phi \\
\frac{\partial}{\partial t} \phi = \tau g \phi
\end{cases}
$$

where $c(t) > 0$ is a time dependent constant, $d\phi \otimes d\phi = \phi^* h$ is the pullback of $h$ via $\phi$ and $\tau g \phi = \text{tr} \nabla d\phi$ is the tension field of $\phi$. The RH flow behaves less singular than
Ricci flow and many fundamental results in Ricci flow have been extended to the RH flow. A self-similar solution to RH flow is defined by Müller [14, 15] as follows.

**Definition 1.1.** Let $(M, g)$ and $(N, h)$ be two smooth Riemannian manifolds and $\phi : M \rightarrow N$ be a smooth map. If there is a smooth function $f : M \rightarrow \mathbb{R}$ and constants $c \geq 0, \lambda$ such that the coupled system

\[
\begin{align*}
\text{Ric} + \nabla^2 f - cd\phi \otimes d\phi &= \lambda g, \\
\tau(\phi) - d\phi(\nabla f) &= 0,
\end{align*}
\]

is satisfied, then $(M, g, f, \phi, \lambda)$ is called as a gradient Ricci-harmonic soliton and $f$ is called the potential function. There have been many studies involving gradient Ricci-harmonic solitons such as [9, 18, 20, 21, 23]. When $f$ is a constant, gradient RH soliton is called harmonic Einstein, i.e.,

\[
\begin{align*}
\text{Ric} - cd\phi \otimes d\phi &= \lambda g, \\
\tau_g \phi &= 0.
\end{align*}
\]

It is well-known that for $\tau > 0$, the Bakry-Émery curvature is defined by

\[
\text{Ric}_{u,\tau} = \text{Ric} + \nabla^2 u - \frac{1}{\tau} du \otimes du,
\]

and $g$ is called a $\tau$-quasi Einstein metric for some constant $\tau$ if there is a constant $\lambda$ and a potential function $u$ such that

\[
(1.1) \quad \text{Ric}_{u,\tau} = \lambda g
\]

is satisfied. From this point of view, $\tau$-quasi Ricci-harmonic metric is defined in [20].

**Definition 1.2.** Let $(N, h)$ be a fixed Riemannian manifold. A metric $g$ of $M$ is called $\tau(>0)$-quasi RH (with respect to $h$), if for a map $\phi : M \rightarrow N$, potential function $u : M \rightarrow \mathbb{R}$ and constants $\alpha \geq 0, \lambda, g$ satisfies the coupled system

\[
\begin{align*}
\text{Ric} + \nabla^2 u - \frac{1}{\tau} du \otimes du - cd\phi \otimes d\phi &= \lambda g, \\
\tau(\phi) - d\phi(\nabla u) &= 0.
\end{align*}
\]

In [17], the authors studied a structure such that the warping function and the potential function are not the same. This idea provided interesting results and led a growing interest in warped products on Ricci solitons [1, 5, 6, 8, 11, 13, 17], almost Ricci solitons [7], Yamabe solitons [10, 19] and RH solitons [2].

In this paper, we will investigate a generalized version on the warped product manifolds which admits $\tau$-quasi RH metric. We prove that the metric of the fibre is harmonic Einstein when warped product metric is $\tau$-quasi RH metric. We also provide some conditions for $M$ to be a harmonic Einstein manifold. Finally, we provide necessary and sufficient conditions for a metric $g$ to be $\tau$-quasi RH metric by using a differential equation system.
2. Preliminaries

Our aim is to remind the warped product $M = B \times_f F$, and the notion of lift by following the notation and terminology of O’Neill [16].

**Definition 2.1.** Let $(B^n, g_B)$ and $(F^m, g_F)$ be two Riemannian manifolds, and $f$ be a positive smooth function on $B$. The warped product $M = B \times_f F$ is the product manifold $B \times F$ with the metric tensor $g$ defined by

$$g = \pi^*g_B + (f \circ \pi)^2\sigma^*g_F.$$  

Here $\pi$ and $\sigma$ are the projections of $B \times F$ onto $B$ and $F$ respectively. The function $f$ is called the warping function, $B$ is the base and $F$ is the fiber. When $f$ is a constant function, $M$ is simply a Riemannian product.

The lift of $V$ to $M$ is the unique element of $\mathfrak{X}(M)$ that is $\sigma$–related to $V$ and $\pi$–related to zero vector field on $B$. The set of all such vertical lifts $\tilde{V}$ is denoted by $\mathfrak{L}(F)$. The set of all horizontal lifts $\tilde{X}$ is denoted by $\mathfrak{L}(B)$. In the same way, functions defined on $B$ and $F$ can be lifted to $M$. Let $u_B$, $h_F$ be smooth functions on $B$ and $F$, respectively. The lift of $u_B$ to $M$ is the function $u = u_B \circ \pi$, and the lift of $h_F$ to $M$ is the function $h = h_F \circ \sigma$. Moreover, one can extend the idea to a mapping $\phi : M = B^n \times_f F^m \to N$ by component-wise and consider $\phi$ as $\phi = \phi_B \circ \pi$ or $\phi = \phi_F \circ \sigma$. Throughout this paper, we will use the same notation for a vector field (and for a function) and its lift for simplicity. We denote the Levi-Civita connections by $D$, $\nabla$ and $\nabla^F$; Ricci tensors by $\text{Ric}$, $B\text{Ric}$ and $F\text{Ric}$ of the $M$, $B$ and $F$, respectively.

Now, we recall the following propositions.

**Proposition 2.1.** On $M = B^n \times F^m$, if $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, then

1. $D_XY \in \mathfrak{L}(B)$ is the lift of $\nabla_XY$ on $B$,
2. $D_XV = D_VX = Xf\frac{V}{f}$,
3. nor $D_VW = -\frac{g(V, W)}{f}\nabla f$,
4. $\tan D_VW \in \mathfrak{L}(F)$ is the lift of $\nabla^F V$ on $F$.

**Proposition 2.2.** On a warped product $M = B^n \times F^m$ with $m > 1$, let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then,

1. $\text{Ric}(X, Y) = B\text{Ric}(X, Y) - m\frac{\nabla^2 f(X, Y)}{f}$,
2. $\text{Ric}(X, V) = 0$, 

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3. Ric(V, W) = \( F\text{Ric}(V, W) - \left( \frac{\Delta f}{f} + (m - 1) \frac{|\nabla f|^2}{f} \right) g(V, W) \).

In [12], the authors give the following corollary from Proposition 2.2.

**Corollary 2.1.** The warped product \( M = B^n \times_f F^m \) is Einstein with \( \text{Ric} = \lambda g \) if and only if

1. \( B\text{Ric} = \lambda g_B + \frac{m}{\tau} \nabla^2 f \),

2. \( (F, g_F) \) is Einstein with \( F\text{Ric} = \mu g_F \),

3. \( \lambda f^2 + f \Delta f + (m - 1)|\nabla f|^2 = \mu \).

**3. Main Results**

Inspiring from [17], we investigate the potential function \( u \) and conclude the next proposition.

**Proposition 3.1.** Let the metric \( g \) of warped product manifold \( M = B^n \times_f F^m \) be a \( \tau \)-quasi Ricci-harmonic metric. Then in a neighbourhood of a point \((p, q) \in B^n \times F^m \), the non-constant map \( \phi \) is \( \phi = \phi_B \circ \pi \) or \( \phi = \phi_F \circ \sigma \) if and only if the potential \( u \) is the lift of a function defined on \( B \).

**Proof.** Let \( X \in \mathfrak{X}(B) \) and \( V \in \mathfrak{X}(F) \). Assume that \( g \) be a \( \tau \)-quasi RH metric on \( M = B^n \times_f F^m \), then we have

\[
\text{Ric}(X, V) + \nabla^2 u(X, V) - \frac{1}{\tau} du \otimes du(X, V) - cd\phi \otimes d\phi(X, V) = \lambda g(X, V).
\]

Since \( \text{Ric}(X, V) = 0 \) and \( g(X, V) = 0 \), (3.1) becomes,

\[
\nabla^2 u(X, V) - \frac{1}{\tau} du \otimes du(X, V) - cd\phi \otimes d\phi(X, V) = 0.
\]

Now suppose that \( u \) is the lift of a function defined on \( F \), and therefore \( \nabla u \in \mathfrak{X}(F) \). Then the equation (3.2) is reduced to

\[
0 = \nabla^2 u(X, V) = \langle \nabla_X \nabla u, V \rangle = \frac{Xf}{f} \langle \nabla u, V \rangle
\]

meaning \( u \) is a constant which contradicts the hypothesis. As a result, \( u \) is the lift of a function defined on \( B \).

Conversely, suppose that \( u \) is a lift of a function defined on \( B \), and therefore \( \nabla u \in \mathfrak{X}(B) \). From Proposition 2.2, we have

\[
\nabla^2 u(X, V) - \frac{1}{\tau} du \otimes du(X, V) = 0.
\]
(3.3) \[ cd\phi(X)d\phi(V) = 0. \]

Since \( \phi \) is not constant from the hypothesis, there is a vector field \( W = X + V \) in \( M \) such that \( d\phi(W)d\phi(W) \neq 0 \) in a neighbourhood of \((p,q) \in M\). By taking square of both sides we have that

\[ (d\phi(X))^2 + 2\phi(X)d\phi(V) + (d\phi(V))^2 \neq 0. \]

Using (3.3) in the above, we can conclude that \((d\phi(X))^2 + (d\phi(V))^2 \neq 0\), hence \( d\phi(X) = 0 \) or \( d\phi(V) = 0 \).

**Remark 3.1.** Notice that the function \( f \) on the second line cannot be a constant because in that case \( M \) is simply a Riemannian product.

Now we can state our first theorem by using Proposition 3.1.

**Theorem 3.1.** The metric \( g \) of warped product \( M = B^n \times_f F^m \) is a \( \tau \)-quasi Ricci-harmonic metric if and only if

(i) If \( \phi = \phi_B \circ \pi \), then

\[ (3.4) \quad B\text{Ric} - \frac{m}{f} \nabla^2 f + \nabla^3 u - \frac{1}{\tau} du \otimes du - cd\phi \otimes d\phi = \lambda g_B, \]

and \( F \) is Einstein with \( F\text{Ric} = \mu g_F \).

(ii) If \( \phi = \phi_F \circ \sigma \), then

\[ (3.5) \quad B\text{Ric} - \frac{m}{f} \nabla^2 f + \nabla^3 u - \frac{1}{\tau} du \otimes du = \lambda g_B, \]

and \( F \) is harmonic Einstein with

\[ \begin{align*}
F\text{Ric} - cd\phi \otimes d\phi &= \mu g_F, \\
\tau g = 0.
\end{align*} \]

In both cases \( \mu \) is

(3.6) \[ \mu = f \Delta f + (m-1) |\nabla f|^2 + \lambda f^2 + f\nabla f(u). \]

**Proof.** Case (i): Let \( \phi = \phi_B \circ \pi \). Using Proposition 2.2 for \( X,Y \in \mathfrak{L}(B) \) in (1.2), we get (3.4). For \( V,W \in \mathfrak{L}(F) \), the equation (1.2) is

\[ \text{Ric}(V,W) + \nabla^2 u(V,W) - \frac{1}{\tau} du \otimes du(V,W) - cd\phi \otimes d\phi(V,W) = \lambda g(V,W). \]
From Proposition 3.1, we know that $u$ is lifted from $B$. So we can conclude that $du(V) = 0$ and similarly $d\phi(V) = 0$. Using Proposition 2.2 above we reach

\[ F \text{Ric}(V,W) = \left( \frac{\Delta f}{f} + (m-1)\frac{|\nabla f|^2}{f^2} \right) g(V,W) + \nabla^2 u(V,W) = \lambda g(V,W). \tag{3.7} \]

Using Proposition 2.1, we compute

\[
\begin{align*}
\nabla^2 u(V,W) &= g(\nabla_V \nabla u, W) \\
&= g \left( \frac{\nabla u(f)}{f} V, W \right) \\
&= fg f \nabla u(f)
\end{align*}
\]

and substitute the result in (3.7) so we get

\[ F \text{Ric}(V,W) = (f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2 + f\nabla f(u)) g_f(V,W) \]

which means $F$ is Einstein.

Case (ii): Assume that $\phi = \phi_F \circ \sigma$. Using Proposition 2.2 for $X, Y \in \mathfrak{L}(B)$ in (1.2), we get (3.5) since $d\phi(X) = 0$. For $V, W \in \mathfrak{L}(F)$, the equation (1.2) is

\[ \text{Ric}(V,W) + \nabla^2 u(V,W) - \frac{1}{\tau} du \otimes du(V,W) - cd\phi \otimes d\phi(V,W) = \lambda g(V,W). \]

Using Proposition 2.2, the fact that $du(V) = 0$ and (3.8) we get

\[ F \text{Ric}(V,W) - cd\phi \otimes d\phi(V,W) = \mu g(V,W). \]

Since $d\phi(\nabla u) = 0$, we can conclude that $F$ is harmonic Einstein. \qed

Remark 3.2. Theorem 3.1 is a generalization of Corollary 2.1 and Theorem 1.3 in [2].

In [4], if the equation (1.1) is satisfied for a smooth function $\lambda$, then the metric is called generalized $\tau$-quasi Einstein metric. Similarly, when $\lambda$ in the equation (1.2) is a function, the metric is called generalized $\tau$-quasi RH metric [22]. Under the assumption of the gradient of the warping function $f$ being a conformal vector field, we can conclude the following.

Corollary 3.1. Let the metric $g$ of warped product $M = B^n \times_f F^m$ be a $\tau$-quasi Ricci-harmonic metric and assume that $\nabla f$ is conformal vector field on $B$.

(i) If $\phi = \phi_B \circ \pi$, then the metric $g_B$ of $B$ is a generalized $\tau$-quasi RH metric.
(ii) If $\phi = \phi_F \circ \sigma$, then $B$ is generalized $\tau$-quasi Einstein manifold.

Theorem 3.2. Let the metric $g$ of warped product $M = B^n \times_f F^m$ be a $\tau$-quasi Ricci-harmonic metric with non-constant $\phi$. If $\lambda \geq 0$ and $\frac{m}{f} \Delta f \geq R$, then $u$ is a constant. Therefore, $M$ is harmonic Einstein.
Proof. Taking the trace of (3.4), we have
\[ \Delta_B u = \lambda u + m \int \Delta_B f - B R + \frac{1}{\tau} |\nabla u|^2 + \alpha |\nabla \phi|^2. \]
Using the hypothesis, we reach that \( \Delta_B u \geq 0 \), so we can use maximum principle to conclude that \( u \) is a constant on \( B \) and so is it’s lift. Hence \( M \) is harmonic Einstein. \( \square \)

The following results of this paper will be given under the assumption of the harmonic map \( \phi \) as a real valued function, i.e., \( \phi : M \rightarrow \mathbb{R} \). Our construction in Theorem 3.1 helps us to drop the restrictions the fiber manifold \( F \) which differs from [17].

**Theorem 3.3.** The metric \( g \) of warped product \( M = \mathbb{R}^n \times_f F^m \) is a \( \tau \)-quasi Ricci-harmonic metric with non-constant \( \phi \) and \( f = f \circ \xi, u = u \circ \xi, \phi = \phi \circ \xi \) defined in \((\mathbb{R}^n, \varphi^{-2} g_0)\) furnished with the metric tensor \( g = \varphi^{-2} g_0 + f^2 g_F \) if and only if the functions verify the system below:

\[
\begin{align*}
(3.9) & \quad (n-2) \frac{\varphi''}{\varphi} - m \frac{f''}{f} - 2m \frac{\varphi'}{\varphi} f' + u'' + 2 \frac{\varphi'}{\varphi} u' - \frac{1}{\tau} (u')^2 - c (\phi')^2 = 0, \\
(3.10) & \quad \left[ \frac{\varphi''}{\varphi} - (n-1) \left( \frac{\varphi'}{\varphi} \right)^2 + m \frac{\varphi'}{\varphi} f' - \frac{\varphi'}{\varphi} u' \right] ||\alpha||^2 = \frac{\lambda}{\varphi^2}, \\
(3.11) & \quad \left[ \frac{f''}{f} - (n-2) \frac{\varphi'}{\varphi} f' + (m-1) \left( \frac{f'}{f} \right)^2 - \frac{f'}{f} u' \right] ||\alpha||^2 = \frac{\mu}{f^2 \varphi^2} - \frac{\lambda}{\varphi^2}, \\
(3.12) & \quad \left[ \phi'' - (n-2) \frac{\varphi'}{\varphi} \phi' + m \phi' f' - \phi' u' \right] ||\alpha||^2 = 0.
\end{align*}
\]

Proof. The Theorem 3.1 gives us necessary and sufficient condition to the metric \( g \) of \( B^n \times_f F^m \) be a \( \tau \)-quasi Ricci-harmonic metric. By using invariant solution technique, we reach equations (3.9), (3.10), (3.11) and (3.12).

For an arbitrary choice of a nonzero vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \), consider \( \xi : \mathbb{R}^n \rightarrow \mathbb{R} \) given by \( \xi(x_1, \ldots, x_n) = \sum_{i=1}^n \alpha_i x_i \). Assume that \( \varphi(\xi), f(\xi), u(\xi) \) and \( \phi(\xi) \) are functions of \( \xi \), so we have
\[
\begin{align*}
\varphi_{,x_i} &= \varphi' \alpha_i, & f_{,x_i} &= f' \alpha_i, & u_{,x_i} &= u' \alpha_i, & \phi_{,x_i} &= \phi' \alpha_i, \\
\varphi_{,x_i x_j} &= \varphi'' \alpha_i \alpha_j, & f_{,x_i x_j} &= f'' \alpha_i \alpha_j, & u_{,x_i x_j} &= u'' \alpha_i \alpha_j, & \phi_{,x_i x_j} &= \phi'' \alpha_i \alpha_j.
\end{align*}
\]
Notice that the functions \( f, \varphi, u \) and \( \phi \) are lifted from \( B = (\mathbb{R}^n, \varphi^{-2} g_0) \).
For the conformal metric $g_B = \varphi^{-2}g_0$, the Ricci curvature is given by [3]:
\begin{equation}
\begin{align*}
&B\text{Ric} = \frac{1}{\varphi^2} \left\{ (n-2)\varphi^2 \text{Hess}_{g_0}(\varphi) + [\varphi \Delta_{g_0} \varphi - (n-1)|\nabla_{g_0} \varphi|^2]g_0 \right\}.
\end{align*}
\end{equation}

Since $(\text{Hess}_{g_0}(\varphi))_{i,j} = \varphi'' \alpha_i \alpha_j$, $\Delta_{g_0} \varphi = \varphi'' ||\alpha||^2$, and $|\nabla_{g_0} \varphi|^2 = \varphi' ||\alpha||^2$ we have
\begin{equation}
(3.13) \quad (B\text{Ric})_{i,j} = \frac{1}{\varphi} \left( (n-2) \varphi'' \alpha_i \alpha_j \right) \quad \forall i \neq j, 1, ..., n
\end{equation}
\begin{equation}
(3.14) \quad (B\text{Ric})_{i,i} = \frac{1}{\varphi^2} \left( (n-2) \varphi'' \alpha_i \alpha_i \right) = \frac{1}{\varphi^2} \left( (n-2) \varphi'' \alpha_i \alpha_i - (n-1)(\varphi')^2 ||\alpha||^2 \epsilon_i \right) \quad \forall i = 1, ..., n.
\end{equation}

For the metric $g_B$, $\text{Hess}(u)$ is
\begin{equation}
(\text{Hess}_{g_B}(u))_{ij} = u_{,i,x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k u_{,x_k},
\end{equation}
where the Christoffel symbol $\Gamma_{ij}^k$ for distinct $i, j, k$ are given by
\begin{equation}
\Gamma_{ij}^k = 0, \quad \Gamma_{ij}^i = \frac{\varphi_{,i}}{\varphi}, \quad \Gamma_{ij}^k = \epsilon_i \epsilon_k \frac{\varphi_{,x_k}}{\varphi} \quad \text{and} \quad \Gamma_{ii}^k = -\frac{\varphi_{,x_i}}{\varphi}.
\end{equation}

Hence,
\begin{equation}
(\text{Hess}_{g_B}(u))_{ij} = u_{,i,x_j} + \varphi^{-1}(\varphi_{,x_i}u_{,x_j} + \varphi_{,x_j}u_{,x_i}) - \delta_{ij} \epsilon_i \sum_k \epsilon_k \varphi^{-1}\varphi_{,x_k}u_{,x_k}
\end{equation}
\begin{equation}
(3.15) \quad = \alpha_i \alpha_j u'' + (2\alpha_i \alpha_j - \delta_{ij} \epsilon_i ||\alpha||^2) \varphi^{-1} \varphi' \varphi'.
\end{equation}

Clearly, the Laplacian $\Delta_{g_B} f = \sum_k \varphi^2 \epsilon_k (\text{Hess}_{g_B}(f))_{kk}$ of $f$ is
\begin{equation}
\Delta_{g_B} f = ||\alpha||^2 \varphi^2 (f'' - (n-2)\varphi^{-1} \varphi' \varphi').
\end{equation}

Since $g_B$ is a conformal metric, the terms $\nabla f(u)$, $|\nabla f|^2$ and $(\nabla \phi \otimes \nabla \phi)_{ij}$ can be given by
\begin{equation}
\nabla g_B f(u) = (\nabla g_B f, \nabla g_B u) = \varphi^2 \sum_k \epsilon_k f_{,x_k} u_{,x_k} = ||\alpha||^2 \varphi^2 f' u',
\end{equation}
\begin{equation}
(3.17) \quad |\nabla g_B f|^2 = \varphi^2 \sum_k \epsilon_k f_{,x_k}^2 = ||\alpha||^2 \varphi^2 (f')^2,
\end{equation}
\begin{equation}
(\nabla g_B \phi \otimes \nabla g_B \phi)_{ij} = \phi_{,i} \phi_{,x_j} = \alpha_i \alpha_j (\phi')^2.
\end{equation}

Plugging in (3.14), (3.15) and (3.17) for $i = j$ into (3.4) we get (3.10).

When $i \neq j$, substituting (3.13) and (3.15) into (3.4) we obtain
\begin{equation}
\alpha_i \alpha_j \left( (n-2) \varphi'' - m \frac{f''}{f} - 2m \frac{f'}{f} \varphi' + \frac{u''}{\varphi} + 2 \frac{\varphi'}{\varphi} u' \right) = 0.
\end{equation}
If there exist \( i, j, i \neq j \) such that \( \alpha_i \alpha_j \neq 0 \), we have the equation (3.9). If \( \alpha_i \alpha_j = 0, \forall i \neq j \), then consider for a fixed \( k_0 \neq k \) such that \( \alpha_{k_0} = 1, \alpha_k = 0 \). For \( i \neq k_0 \), substituting (3.14), (3.15) and (3.17) into (3.4) we get the equation (3.10), i.e., \( \alpha_i = 0 \). For \( i = k_0 \), we have the equation (3.9), i.e., \( \alpha_{k_0} = 1 \).

Similarly, we obtain (3.11) by substituting (3.16), (3.17) in (3.6). Considering (1.3) and the laplace of \( \phi \), which is lifted from base, we have

\[
\Delta \phi = \left[ \Delta_{g_B} \phi + \frac{m}{f} g_B(\nabla \phi, \nabla f) \right] = g_B(\nabla \phi, \nabla u).
\]

Using (3.16) and (3.17) in (3.18) we have (3.12) which completes the proof.

**Corollary 3.2.** Let \( f = f \circ \xi, u = u \circ \xi, \phi \circ \xi, \phi = \phi \circ \xi \) defined in \((\mathbb{R}^n, \varphi^{-2}g_0)\) and the metric \( g \) of warped product \((M = \mathbb{R}^n \times_f F^n, g = \varphi^{-2}g_0 + f^2g_F)\) be a \( \tau \)-quasi Ricci-harmonic metric with non-constant \( \phi \). If \( ||\alpha||^2 = 0 \), then \( \lambda = 0 \) and \( \mu = 0 \), i.e., \( F^n \) is Ricci flat.

**Example 3.1.** Let \( ||\alpha||^2 = 0 \) in Theorem 3.3. For simplicity, assume that \( c = 1, m = 4, n = 3, \tau = 1 \) and \( \varphi(\xi) = e^\xi, f(\xi) = e^\xi \) and \( \phi(\xi) = \xi \). Solving (3.9), we obtain

\[
u(\xi) = -\log \left( \cos(\sqrt{10}(c_1 + \xi)) \right) + \xi + c_2, \quad c_1, c_2 \in \mathbb{R}
\]

which defines a \( \tau \)-quasi RH metric on \( M \).

**Theorem 3.4.** The metric \( g \) of warped product \( M = B^n \times_f F^n \) is a \( \tau \)-quasi Ricci-harmonic metric with non-constant \( \phi \), \( f = f \circ \xi, u = u \circ \xi, \varphi \circ \xi, \phi = \phi \circ \xi \) defined in \((\mathbb{R}^n, \varphi^{-2}g_0)\) and \((\mathbb{R}^m, \psi^{-2}g_0)\), respectively, and furnished with the metric tensor \( g = \varphi^{-2}g_0 + f^2\psi^{-2}g_F \) if and only if the functions verify the system below:

\[
(n - 2) \frac{\psi''}{\psi} - m \frac{f''}{f} - 2m \frac{\varphi' f'}{\varphi f} + u'' + 2 \frac{\varphi'}{\varphi} u' - \frac{1}{\tau} (u')^2 = 0,
\]

\[
\left[ \frac{\varphi''}{\varphi} - (n - 1) \left( \frac{\varphi'}{\varphi} \right)^2 + m \frac{\varphi' f'}{\varphi f} - \frac{\varphi'}{\varphi} h' \right] ||\alpha||^2 = \frac{\lambda}{\varphi^2},
\]

\[
\left[ f'' \varphi^2 f - (n - 2) \varphi' \varphi f f' + (m - 1)(f')^2 \varphi^2 - f' \varphi^2 h' \right] ||\alpha||^2 + \lambda f^2
\]

\[
= \left[ \frac{\psi''}{\psi} - (m - 1) \left( \frac{\psi'}{\psi} \right)^2 \right] ||\beta||^2,
\]

\[
(m - 2) \frac{\psi''}{\psi} - c(\phi')^2 = 0,
\]

\[
[\psi^2 \phi'' - (m - 2) \psi \phi'] ||\beta||^2 = 0.
\]
Proof. We use the same technique as in the proof of the Theorem 3.3 for both the base and the fiber. When \( i \neq j \), substituting the equation (3.13) and (3.17) in (3.5) we have the equation (3.19) and when \( i = j \), plugging in (3.14) and (3.17) in (3.5) we get (3.20).

From Theorem 3.1, \( F \) is harmonic Einstein,

\[
F \text{Ric} - cd\phi \otimes d\phi = \mu g_F
\]

where \( c > 0 \) and

\[
\mu = f \Delta g_F f + (m-1)|\nabla f|^2 + \lambda f^2 + f\nabla f(u).
\]

For an arbitrary choice of a nonzero vector \( \beta = (\beta_1, \ldots, \beta_m) \), let \( \psi : \mathbb{R}^m \to \mathbb{R}^+ \) be the conformal factor of the fiber and \( \zeta : \mathbb{R}^m \to \mathbb{R} \) be the invariant function so that \( u(\zeta) \) is a function of \( \zeta \) which gives

\[
(3.26) \quad (\nabla g_F \phi_F \otimes \nabla g_F \phi_F)_{ij} = \phi_{,i}\phi_{,j}\beta_i\beta_j \quad \forall i, j = 1, \ldots, m.
\]

Using (3.17) in (3.25) we obtain

\[
(3.27) \quad \left[ f'' \varphi^2 f - (n-2) \varphi \varphi f f' + (m-1)(f')^2 \varphi^2 - f' \varphi^2 u' \right] |\alpha|^2 + \lambda f^2 = \mu.
\]

Replacing (3.13), (3.14), (3.26) and (3.27) in (3.24) we get the equations (3.21) and (3.22) for \( i = j \) or \( i \neq j \).

From (ii) of Theorem 3.1 we have \( \Delta g_F \phi = 0 \), by using (3.16), we obtain (3.23).

\[\square\]

REFERENCES


