INTERSECTIONS OF SURFACES OF REVOLUTION

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Abstract. In this paper, we deal with surfaces of revolution and their intersections. We start with the surfaces of revolution $RS$ that have their axis along the $x^3$-axis and find intersections with a line, a plane, and then intersection of two such $RS$. Furthermore, we apply formulas for the intersection with a line to determine the visibility of $RS$. Later we develop formulas for the intersection of two surfaces of revolution that have their axis along different arbitrary straight lines, and, as a special case, the intersections of two spheres and intersections of general surface of revolution with a sphere and a surface given by an equation. We apply our own software to the graphical representation of all the results we present.

Keywords: intersections, surfaces of revolution, visualization, visibility.

1. Introduction
Surfaces of revolution are created by rotating a planar curve about an axis in the plane. They are easy to understand and deal with, so they play important role and are widely used in various fields of mathematics, physics and engineering.

Nevertheless, they are still interesting for research. Some papers that explore their properties are [1, 2, 3, 6]. Some of the papers study the intersections of surfaces of revolution are [8, 9, 21]. Some problems related to finding the intersection of two surfaces are discussed in [7]. Determining surfaces of revolution from some of their properties is considered in the papers [10, 11, 22].
Visualization strongly supports the understanding of mathematical concepts. We developed our own software for the visualization of mathematical objects and relations between them. We use the approach of vector graphics, perform calculations analytically, display the surfaces by families of curves on them without any approximating body of the polygon mesh type, so we get figures of high precision. We use a small number of curves on the surfaces, so that our figures look clean, without unnecessary details, so the parametrization of surfaces and the lines of intersection can be easily recognized. In order to emphasize the lines of intersection, which are most important in this paper, we emphasize them with thicker lines.

The basic version of our software is covered in detail in the book [13]. Later, the software was much improved and applied for the visualization in various fields of mathematics, for example in topology [18] and functional analysis [15, 16, 17, 20], but also in other sciences such as physics [12] and crystallography [14, 19].

In this paper we study intersections of surfaces of revolution $RS$, first of those which have their axis along the $x^3$–axis, and later of those with an arbitrary axis. In Section 2, we start with the intersection of $RS$ with a straight line and apply it to the solution of the visibility problem for $RS$. In the special cases of a sphere, cylinder and cone, the solution of the visibility problem reduces considerably. The special conditions for the intersection of $RS$ with a plane are determined. This section concludes with the intersection of two $RS$ with their axes along the $x^3$–axis. Section 3. is dedicated to the surfaces of revolution with arbitrary axes, so we refer to them as general surfaces of revolution. The solutions of the problems of visibility and contour line that arise in the graphical representation can be reduced to the methods described in detail in the previous section. For the intersections, first we study special cases of the intersection of two spheres, then intersections of general $RS$ with a sphere, intersections of $RS$ and a surface given by an equation and, in the end, intersections of two general surfaces of revolution.

We developed our own software to visualize all the presented results. All the geometrical figures in this paper have been created by our software package.

2. Surface of Revolution with Axis Along $x^3$–axis
A surface of revolution is generated by rotating a planar curve $\gamma$ about an axis in the plane. As far as the study of geometrical properties is concerned, we may assume that the curve $\gamma$ is in the $x^1x^3$–plane and the axis of rotation is the $x^3$–axis. Later, we will consider surfaces of revolution generated by rotations about arbitrary axes in three–dimensional space.

Let $I \subset \mathbb{R}$ be an interval, $r, h \in C^r(I)$, where $r \in \mathbb{N}$ is chosen to need. We assume that $\gamma$ is given by a parametric representation

$$\vec{x}(t) = (r(t), 0, h(t)) \text{ for } t \in I \text{ where } r(t) > 0 \text{ and } |r'(t)| + |h'(t)| \neq 0 \text{ on } I.$$ 

Writing $u^1 = t$ and $u^2$ for the polar angle in the $x^1x^2$–plane, we obtain

$$\vec{x}(u^1) = (r(u^1) \cos u^2, r(u^1) \sin u^2, h(u^1)) \text{ for } (u^1, u^2) \in D \subset I \times (0, 2\pi)$$

$$\vec{x}(u^{i_1}, u^{i_2}) = (r(u^{i_1}) \cos u^{i_2}, r(u^{i_1}) \sin u^{i_2}, h(u^{i_1})) \text{ for } (u^{i_1}, u^{i_2}) \in D \subset I \times (0, 2\pi)$$
as a parametric representation for the surface of revolution $RS(\gamma)$ generated by the curve $\gamma$.

### 2.1. The Intersection with a Line

We start by finding the intersection of surface of revolution $RS$ with a parametric representation (2.1) where $D = I_1 \times I_2$ and $I_2 \subset (0, 2\pi)$, and a straight line $L$, given by a parametric representation $\vec{y} = \vec{p} + t\vec{v}$ ($t \in \mathbb{R}$), that is, we have to find $(u^1, u^2) \in D$ and $t \in \mathbb{R}$ such that

$$x(u^i) = \vec{p} + t\vec{v}. \tag{2.2}$$

Thus, writing

$$\vec{u} = \vec{u}(u^2) = (\cos u^2, \sin u^2, 0) \text{ and } \vec{e}^3 = (0, 0, 1),$$

we have to find the solutions $(u^1, u^2) \in D = I_1 \times I_2$ and $t \in \mathbb{R}$ of the equations

$$r(u^1)\vec{u}(u^2) + h(u^1)\vec{e}^3 - (\vec{p} + t\vec{v}) = \vec{0}. \tag{2.3}$$

This means in particular

$$h(u^1) - (p^3 + tv^3) = 0. \tag{2.4}$$

**Case 1.** First we consider the case $v^3 \neq 0$ when $L$ is not orthogonal to the axis of rotation of $RS$. Then (2.4) implies

$$t = t(u^1) = \frac{h(u^1) - p^3}{v^3} \tag{2.5}.$$

We put

$$\vec{a} = \vec{p} - \frac{p^3}{v^3} \cdot \vec{v}, \quad \vec{b} = \frac{1}{v^3} \cdot \vec{v} \tag{2.6}$$

and obtain, squaring (2.3) and substituting (2.5),

$$r^2(u^1) + h^2(u^1) = \left(\vec{p} - \frac{p^3}{v^3} \cdot \vec{v} + h(u^1) \cdot \frac{1}{v^3} \cdot \vec{v}\right)^2 = (\vec{a} + h(u^1) \cdot \vec{b})^2.$$  

Thus we have to find the zeros $u_0^1 \in I_1$ of

$$f(u^1) = r^2(u^1) + h^2(u^1) - (\vec{a} + h(u^1) \cdot \vec{b})^2 \tag{2.7}.$$  

For each zero $u_0^1$ of (2.7), we compute the value $t_0 = t(u_0^1)$ from (2.5) and finally the values $u_0^2 \in I_2$ from

$$\cos u_0^2 = \frac{p^1 + t_0v^1}{r(u_0^1)} \quad \text{and} \quad \sin u_0^2 = \frac{p^2 + t_0v^2}{r(u_0^1)}. \tag{2.8}$$
We remark that since
\[ r^2(u^1) = (p^1 + tv^1)^2 + (p^2 + tv^2)^2 \geq \max\{|p^1 - tv^1|, |p^2 - tv^2|\}, \]
and \( r^2(u^1) > 0 \) for all \( u^1 \), the equations in (2.8) always have a unique solution in the interval \((0, 2\pi)\).

**Case 2.** Now we consider the case \( v^3 = 0 \) when \( L \) is orthogonal to the axis of rotation of \( RS \). Then it follows from (2.4) that
\[ (2.9) \quad f(u^1) = h(u^1) - p^3 = 0. \]

Now we find the solutions \( u^1_0 \in I_1 \) of (2.9). Furthermore, squaring (2.3) leads to the quadratic equation
\[ (2.10) \quad t^2 \vec{v}^2 + 2t \vec{p} \cdot \vec{v} + \vec{p}^2 - (r^2(u^1_0) + h^2(u^1_0)) = 0. \]

For each such \( u^1_0 \) there are at most two points of intersection with the corresponding \( u^2 \)-line and we obtain the \( t \)-parameters \( t_0 = t(u^1_0) \) of these points of intersection from (2.10). Finally we find the values \( u^2_0 \in I_2 \) in the same way as in the Case 1, from (2.8).

Figure 2.1 shows intersections of a surface of revolution with straight lines; the figure on the right hand side shows its intersection with its axis, a case which mathematically cannot happen since \( r(u^1) > 0 \).

![Fig. 2.1: Intersections of a surface of revolution and straight lines](image)

### 2.1.1. Visibility of Surfaces of Revolution

The visibility of points on a surface of revolution is determined analytically. To check the visibility of a point \( P \) we choose the straight line \( L \) to be the projection ray. Let \( C \) be the centre of projection and \( \vec{p} \) denote the position vector of a point \( P \) then we put \( \vec{v} = \overrightarrow{PC} \) in the equations above. Now \( P \) is hidden by \( RS \) if and only if there is a solution \( u^1_0 \in I_1 \), for \( v^3 \neq 0 \) of (2.7) with corresponding \( t_0 \) from (2.5), or
for $v^3 = 0$ of (2.9) with corresponding $t_0 > 0$ from (2.10), and $u_0^2 \in I_2$ from (2.8).

The same argument applies for the visibility of a point $P$ on $RS$ with respect to $RS$ itself; now we observe that $P \in RS$ implies $\vec{p}^2 - (r^2(u^1) + h^2(u^1)) = 0$ and the quadratic equation (2.10) reduces to $t = -2\vec{p} \cdot \vec{v}/\vec{v}^2$.

We have seen above that intersecting a surface of revolution and a straight line involves finding the zeros of the real valued function $f$ in (2.7) or (2.9). An algorithm for this and its implementation can be found in [13, Section 6.1, pp. 502–511].

### 2.1.2. Visibility in Special Cases: Sphere, Cylinder and Cone

Finally we consider the special cases when the surface of revolution is a sphere, cylinder or cone. Then the solution of the visibility problem reduces considerably.

A sphere $Sph$ with radius $r$ and its centre in the origin has a parametric representation (2.1) with $r(u^1) = r \cos u^1$ and $h(u^1) = r \sin u^1$, and is given by the equation

$$\tag{2.11} (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2.$$  

Substituting the parametric representation (2.2) of a straight line $L$ in (2.11), we obtain

$$\tag{2.12} (t\vec{v} + \vec{p})^2 = t^2\vec{v}^2 + 2t\vec{v} \cdot \vec{p} + \vec{p}^2 = r^2.$$  

Thus the $t$–parameters along $L$ of the points of intersection are the solutions of the quadratic equation

$$\tag{2.12} at^2 + bt + c = 0$$

with $a = \vec{v}^2$, $b = 2\vec{v} \cdot \vec{p}$ and $c = \vec{p}^2 - r^2$.

We observe that if we check the visibility of a point $P$ on $Sph$ with respect to $Sph$ itself then $\vec{p} = r^2$, and the quadratic equation (2.12) reduces to $at = b$.

A circular cylinder $Cyl$ with radius $r$ and its axis along the $x^3$–axis has a parametric representation (2.1) with $r(u^1) = r$ and $h(u^1) = u^1$, and is given by the equation

$$\tag{2.13} (x^1)^2 + (x^2)^2 = r^2.$$  

Now the $t$–parameters along $L$ of the points of intersection are the solutions of the quadratic equation (2.12) with

$$a = (v^1)^2 + (v^2)^2, \quad b = 2(v^1p^1 + v^2p^2) \quad \text{and} \quad c = (p^1)^2 + (p^2)^2 - r^2$$

which again reduces to $at = b$ when we check the visibility of a point $P$ on $Cyl$ with respect to $Cyl$ itself.

A cone $Cone$ with its axis along the $x^3$-axis, its vertex in the origin and an angle of $2\beta \in (0, \pi)$ at its vertex has a parametric representation (2.1) with $r(u^1) = u^1 \sin \beta$ and $h(u^1) = u^1 \cos \beta$, and is given by the equation

$$\tag{2.14} (x^1)^2 + (x^2)^2 - \tan^2 \beta = 0.$$
Now the $t$–parameters along $L$ of the points of intersection are the solutions of the quadratic equation (2.12) with

$$a = (v^1)^2 + (v^2)^2 - (v^3 \tan \beta)^2, \quad b = 2 \left(v^1 p^1 + v^2 p^2 - \beta u^3 p^3 \tan \beta\right)$$

and $c = (p^1)^2 + (p^2)^2 - (p^3 \tan \beta)^2$

which again reduces to $at = b$ when we check the visibility of a point $P$ on $Cone$ with respect to $Cone$ itself.

2.2. The Intersections of Surfaces of Revolution and Planes

Let $RS$ be a surface of revolution given by a parametric representation (2.1) and $Pl$ be a plane through a point $P$ and orthogonal to a vector $\vec{N}_{Pl} = \{n^1_{Pl}, n^2_{Pl}, n^3_{Pl}\}$. Then the intersection $IS = RS \cap Pl$ of $RS$ and $Pl$ is given by the solution of

$$(r(u^1)\vec{u}(u^2) + h(u^1)\vec{e}^3 - \vec{p}) \cdot \vec{N}_{Pl} = 0.$$

In view of the symmetry of rotation, we may assume $n^2_{Pl} = 0$ and apply the same argument as in Subsection 2.1.2. to treat the general case. Writing $a_0 = \vec{p} \cdot \vec{N}_{Pl}$, we have to solve

$$n^1_{Pl}r(u^1) \cos u^2 + n^3_{Pl}h(u^1) - a_0 = 0. \quad (2.15)$$

If $g_2(u^1) = n^1_{Pl}r(u^1) = 0$ then $r(u^1) \neq 0$ implies $n^1_{Pl} = 0$ and consequently $\vec{N}_{Pl}$ is parallel to the axis of rotation. Now lines of intersection are the parts $u^2 \in I_2$ of the $u^2$–lines that correspond to solutions $u^1_0 \in I_1$ of $g_1(u^1) = n^3_{Pl}h(u^1) - a_0 = 0$. If $g_2(u^1) \neq 0$ then we can solve (2.15) to obtain

$$\cos u^2 = \cos u^2(u^1) = -\frac{g_1(u^1)}{g_2(u^1)}, \quad \text{(2.16)}$$

and the intersection is given by $u^2(u^1) \in I_2$ from (2.16) for those values $u^1 \in I_2$ that satisfy

$$\left|\frac{g_1(u^1)}{g_2(u^1)}\right| \leq 1.$$

2.3. The Intersections of Surfaces of Revolution

Let $RS$ and $RS^*$ be surfaces of revolution given by the parametric representations (2.1) and

$$\vec{x}^*(u^*) = (r^*(u^*) \cos u^{*2}, r^*(u^*) \sin u^{*2}, h^*(u^{*1}))$$

with domains $D = I_1 \times I_2$ and $D^* = I^*_1 \times I^*_2$. We also assume that $r(u^1) > 0$ on $I_1$ and $r^*(u^{*1}) > 0$ on $I^*_1$, and

$$|r'(u^1)| + |h'(u^1)| > 0 \text{ on } I_1 \text{ and } |r^{*'}(u^{*1})| + |h^{*'}(u^{*1})| > 0 \text{ on } I^*_1. \quad (2.17)$$
The lines of intersection of $RS$ and $RS^*$ are given by

$$\vec{x}(u^i) = \vec{x}^*(u^{*i}) \text{ for } (u^1, u^2), (u^{*1}, u^{*2}) \in D \cap D^*. \tag{2.18}$$

Squaring the equations for the first two components in (2.18), adding them and taking into account that $r(u^1), r^*(u^{*1}) > 0$, we obtain together with the third equation

$$r(u^1) = r^*(u^{*1}) \text{ and } h(u^1) = h^*(u^{*1}) \text{ for } u^1, u^{*1} \in J_1 \cap I^*_1, \tag{2.19}$$

and then $\cos u^2 = \cos u^{*2}$ and $\sin u^2 = \sin u^{*2}$ from the first two equations. Since the map $u \mapsto (\cos u, \sin u)$ is one-to-one on $(0, 2\pi)$, we obtain $u^2 = u^{*2}$ for $u^2, u^{*2} \in J_2 = I_2 \cap I^*_2$. Now, by (2.17), at least one of the functions $r, r^*, h$ and $h^*$ has a local inverse. We assume that there exists an interval $J \subset J_1$ and a function $\varphi : r^*(J) \to \mathbb{R}$ with $\varphi(r^*(u^{*1})) = u^{*1}$ for all $u^{*1} \in J$. The other cases are treated similarly. Then we obtain from the first equation in (2.19) that $u^{*1} = \varphi(r(u^1))$ and substituting this in the second equation in (2.19), we get $h(u^1) = h^*(\varphi(r(u^1)))$, hence the corresponding parts of the lines of intersection are given by the equation

$$h(u^1) - h^*(\varphi(r(u^1))) = 0 \text{ for } u^1 \in J.$$
3. Surface of Revolution with Axis Along an Arbitrary Axis

Now we consider general surfaces of revolution generated by the rotation of a planar curve about an arbitrary axis. It turns out that the solutions of the problems that arise in the graphical representation can be reduced to the methods described in detail in Subsection 2.1.

We observe that the geometry of a surface is independent of the choice of the coordinate system, in particular, the curves on a surface are determined in terms of the parameters of the surface. Thus we can use the methods in Section 2. to determine them. We make a transformation of the coordinate system, solve the visibility, contour and intersection problems in the new coordinate system exactly as in Section 2., and finally return to the original coordinate system.

Now we deal with the graphical representation of some intersections of general surfaces of revolution. First we consider some special cases.
3.1. The Intersection of Spheres

Let $S_1$ and $S_2$ be parts of spheres with centres in $C_1$ and $C_2$, radii $r_1 > 0$ and $r_2 > 0$, and domains $D_1$ and $D_2$ for their parameters. We write $\vec{c}_1$, $\vec{c}_2$ for the position vectors of $C_1$ and $C_2$, $\vec{d} = \vec{c}_2 - \vec{c}_1$, $d = ||\vec{d}||$ and $IS = S_1 \cap S_2$.

First we consider the trivial cases. If $d = 0$, then $C_1 = C_2$ and $IS = \emptyset$ for $r_1 \neq r_2$, and $IS = S_1 = S_2$ for $r_1 = r_2$.

If $d > r_1 + r_2$ then obviously $IS = \emptyset$.

Now we consider the case $0 < d < |r_1 - r_2|$. If $r_1 > r_2$ then $r_2 + d < r_1$ and the points $X \in S_2$ satisfy

$$||\overrightarrow{OX} - \overrightarrow{OC_1}|| \leq ||\overrightarrow{OX} - \overrightarrow{OC_2}|| + ||\vec{d}|| = r_2 + d < r_1,$$

that is, they are in the interior of the closed ball $B_{r_1}(C_1)$ (closed ball of radius $r$ centered at $X_0$ is $B_r(X_0) = \{X \in \mathbb{R}^3 : d(X, X_0) \leq r\}$), hence $IS = \emptyset$. If $r_2 > r_1$ then $r_1 + d < r_2$ and the points $X \in S_1$ satisfy

$$||\overrightarrow{OX} - \overrightarrow{OC_2}|| \leq ||\overrightarrow{OX} - \overrightarrow{OC_1}|| + ||\vec{d}|| = r_1 + d < r_2,$$

that is they are in the interior of the closed ball $B_{r_2}(C_2)$, hence $IS = \emptyset$. 

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**Fig. 3.2:** Tori, plane and sphere and their lines of intersections

**Fig. 3.3:** Tori and their lines of intersection
For the nontrivial case, let \(0 < |r_1 - r_2| \leq d \leq r_1 + r_2\). Then a point on \(S_1\) with position vector \(\vec{x}(u)\) in the intersection \(IS\) has to satisfy the equation

\[\|\vec{x}(u) - \vec{c}_2\|^2 = r_2^2.\]

This yields

\[
r_2^2 = \| (\vec{x}(u) - \vec{c}_1) - \vec{d}\|^2 = r_1^2 - 2(\vec{x}(u) - \vec{c}_1) \cdot \vec{d} + \|\vec{d}\|^2 = r_1^2 - 2\vec{d} \cdot \left(2(\vec{x}(u) - \vec{c}_1) - \vec{d}\right) = r_1^2 - 2\vec{d} \cdot \left(\vec{x}(u) - \frac{1}{2}(\vec{c}_1 + \vec{c}_2)\right)
\]

which is equivalent to

\[
\vec{d} \cdot \left(\vec{x}(u) - \frac{1}{2}(\vec{c}_1 + \vec{c}_2) + \frac{(r_2^2 - r_1^2)\vec{d}}{2d^2}\right) = 0.
\]

The points with position vectors \(\vec{x}(u)\) that satisfy this equation are in a plane \(PL\) orthogonal to the vector \(\vec{d}\) and through the point \(P_0\) with position vector

\[
\overrightarrow{OP_0} = \frac{1}{2d^2} \left(d^2(\vec{c}_2 + \vec{c}_1) + (r_1^2 - r_2^2)\vec{d}\right) = \vec{c}_1 + \frac{1}{2d^2} (r_1^2 + d^2 - r_2^2) \cdot \vec{d}
\]

\[
= \vec{c}_2 - \frac{1}{2d^2} (r_2^2 + d^2 - r_1^2) \cdot \vec{d}.
\]

We observe that \(|r_1 - r_2| \leq d \leq r_1 + r_2\) implies \(-d \leq r_1 - r_2 \leq d \leq r_1 + r_2\), hence \(-r_2 \leq r_1 - d \leq r_2, r_2 \leq r_1 + d, -r_1 \leq r_2 - d \leq r_1\) and \(r_1 \leq r_2 + d\), that is...
\[|r_1 - d| \leq r_2, \quad r_2 \leq r_1 + d, \quad |r_2 - d| \leq r_1 \quad \text{and} \quad r_1 \leq r_2 + d. \]

Thus we have
\[-2r_1 d = r_1^2 + d^2 - (r_1 + d)^2 \leq r_1^2 + d^2 - r_2^2 = (r_1 - d)^2 - r_2^2 + 2r_1 d\]
\[\leq r_2^2 - r_2^2 + 2r_1 d = 2r_1 d,\]
that is, \(|r_1^2 + d^2 - r_2^2| \leq 2r_1 d\).

Similarly we obtain \(|r_2^2 + d^2 - r_1^2| \leq 2r_2 d\), and consequently
\[\|\overrightarrow{OP}_0 - \vec{c}_1\| = \frac{1}{2d} |r_1^2 + d^2 - r_2^2| \leq r_1 \quad \text{and} \quad \|\overrightarrow{OP}_0 - \vec{c}_2\| = \frac{1}{2d} |r_2^2 + d^2 - r_1^2| \leq r_2\]
that is, \(P_0 \in B(C_1, r_1) \cap B(C_2, r_2)\). Thus \(IS = PL \cap S_1 = PL \cap S_2\).

**Fig. 3.5:** General spheres and their intersections

### 3.2. The Intersection of a General Surface of Revolution and a Sphere

Let \(RS\) be a general surface of revolution with a local coordinate system with origin in \(C_1\) and unit vectors \(\vec{e}_k^L\) \((k = 1, 2, 3)\) along its coordinate axes, such that \(RS\) may be given by a parametric representation
\[(3.1) \quad \vec{x}(u^i) = r(u^1) \cos u^2 \vec{e}_1^L + r(u^1) \sin u^2 \vec{e}_2^L + h(u^1) \vec{e}_3^L + \vec{c}_1 \quad \text{for} \quad (u^1, u^2) \in D_1.\]

Furthermore, let \(S_2\) be a part of a sphere given by \(D_2 \subset (-\pi/2, \pi/2) \times (0, 2\pi)\), with its centre in \(C_2\) and radius \(r_2 > 0\). Again we write \(\vec{d} = \vec{c}_2 - \vec{c}_1\). Then a point with the position vector \(\vec{x}(u^i)\) on \(RS\) in the intersection \(IS = RS \cap S_2\) has to satisfy the equation
\[\|\vec{x}(u^i) - \vec{c}_1\| = \frac{1}{2d} |r_1^2 + d^2 - r_2^2| = r_2^2.\]

If \(\vec{d}\) has the components \(d_{L,k}\) \((k = 1, 2, 3)\) with respect to the local coordinate system of \(RS\), that is, if
\[\vec{d} = d_{L,1} \vec{e}_1^L + d_{L,2} \vec{e}_2^L + d_{L,3} \vec{e}_3^L \quad \text{where} \quad d_{L,k} = \vec{d} \cdot \vec{e}_k^L \quad (k = 1, 2, 3),\]
then we must have

\[(3.2) \quad r_2^2 = r^2(u^1) + h^2(u^1) - 2 \vec{d} \cdot (\vec{x}(u^1) - \vec{c}_1) + \|\vec{d}\|^2 \]

\[= r^2(u^1) + h^2(u^1) + \|\vec{d}\|^2 - 2\left(d_L,1r(u^1)\cos u^2 + d_L,2r(u^1)\sin u^2 + d_L,3h(u^1)\right).\]

First we consider the case \(d_{L,1} = d_{L,2} = 0\) when \(C_2\) is on the axis of rotation of \(RS\). Then we must find the zeros of \(f(u^1) = r^2(u^1) + h^2(u^1) + \|\vec{d}\|^2 - r_2^2 - 2d_{L,3}h(u^1).\)

Now the intersection \(IS\) is given by the parts of the \(u^2\)-lines on \(RS\) that correspond to the zeros of \(f\) with parameters belonging to both \(D_1\) and \(D_2\).

Now we assume that \(\vec{d}_{PL} = d_{L,1}\vec{e}_L^1 + d_{L,2}\vec{e}_L^2 \neq \vec{0}\). Let \(\phi\) denote the polar angle of \(\vec{d}_{PL}\) in the plane spanned by the vectors \(\vec{e}_L^1\) and \(\vec{e}_L^2\). Then we first consider the case where \(\vec{d}_{PL} = \|\vec{d}_{PL}\|\vec{e}_L^1\). Now equation (3.2) reduces to

\[
r^2(u^1) + h^2(u^1) + \|\vec{d}\|^2 - r_2^2 - 2d_{L,3}h(u^1) - 2\|\vec{d}_{PL}\|r(u^1)\cos u^2 = 0
\]

or

\[
\cos u^2 = a(u^1) = \frac{r^2(u^1) + h^2(u^2) + \|\vec{d}\|^2 - r_2^2 - 2d_{L,3}h(u^1)}{2\|\vec{d}_{PL}\|r(u^1)}.
\]

If \(|a(u^1)| \leq 1\) then we can solve for \(u^2\) and obtain

\[
u_1^2 = u_1^2(u^1) = \arccos(a(u^1))\] and \(u_2^2 = u_2^2(u^1) = 2\pi - \arccos(a(u^1)).\)

In the general case, we have to add the angle \(\phi\) to the values \(u_1^2\) and \(u_2^2\). Now a point \(P\) is in the intersection \(IS\) if and only if its parameters satisfy \((u^1, u^2) \in D_1\) with respect to \(RS\) and \((v^1, v^2) \in D_2\) with respect to \(S_2\).

Fig. 3.6: Intersection of a catenoid and a sphere
3.3. The Intersection of a General Surface of Revolution and a Surface Given by an Equation

Let $RS$ be a general surface of revolution given by a parametric representation (3.1) with respect to its local coordinate system and $S$ be a surface that can be given by an equation

\begin{equation}
F(x^1_{L,S}, x^2_{L,S}, x^3_{L,S}) = 0,
\end{equation}

where $x^k_{L,S}$ ($k = 1, 2, 3$) denote the coordinates of the points of $S$ in the local coordinate system of $S$.

If $C_S$ with position vector $\vec{c}_2$ is the origin of the local coordinate system of $S$ and $\vec{e}^k_{L,S}$ ($k = 1, 2, 3$) denote the unit vectors along the coordinate axes of the local coordinate system of $S$ then a point $P = P(u^i)$ of $RS$ with its position vector satisfying (3.1) in the intersection of $RS$ and $S$ has to satisfy equation (3.3) with

\[ x^k_{L,S} = (\vec{x}(u^i) - \vec{c}_2) \cdot \vec{e}^k_{L,S} \text{ for } k = 1, 2, 3. \]

This involves finding the zeros of a real–valued function of two variables and drawing a curve given by an equation. The algorithms and methods needed for this task
and their implementations are described in detail in [4, 5].

Two simple examples are the intersections of a general surface of revolution with a cone with vertex in \( C_2 \) and an angle \( 2\beta \) \((\beta \in (0, \pi/2)) \) at its vertex, and with a circular cylinder of radius \( r > 0 \). Then equation (3.3) reduces to

\[
(x_{L,S}^1)^2 + (x_{L,S}^2)^2 - \tan^2 \beta (x_{L,S}^3)^2 = 0
\]

and

\[
(x_{L,S}^1)^2 + (x_{L,S}^2)^2 - r^2
\]

for a cone and a cylinder, respectively.

![Fig. 3.9: Intersections of a torus, cone and cylinder](image)

### 3.4. The Intersection of General Surfaces of Revolution

Finally we consider the intersection \( IS \) of general surfaces of revolution \( RS^{(1)} \) and \( RS^{(2)} \) given by the parametric representations

\[
\vec{x}^{(1)}(u^i) = r_1(u^1) \cos u^2 \vec{e}_{L,1}^1 + r_1(u^1) \sin u^2 \vec{e}_{L,1}^2 + h_1(u^1) \vec{e}_{L,1}^3
\]

and

\[
\vec{x}^{(2)}(v^i) = r_2(v^1) \cos v^2 \vec{e}_{L,2}^1 + r_2(v^1) \sin v^2 \vec{e}_{L,2}^2 + h_2(v^1) \vec{e}_{L,2}^3.
\]

Since the functions \( r_1, r_2, h_1 \) and \( h_2 \) satisfy the conditions in (2.17), at each \( u^i \) or \( v^i \) at least one of them has a local inverse. Here we treat the case that \( h_2 \) has a local inverse \( \psi \) in some interval \( I_{L,2}^{(2)} \). The other cases are similar. Writing \( \vec{d} = \vec{c}_2 - \vec{c}_1 \) we see that a point of intersection must satisfy

\[
r_2(v^1) \cos v^2 = (\vec{x}^{(1)}(u^i) - \vec{d}) \cdot \vec{e}_{L,2}^1,
\]

\[
r_2(v^1) \sin v^2 = (\vec{x}^{(1)}(u^i) - \vec{d}) \cdot \vec{e}_{L,2}^2 \quad \text{and} \quad \text{and}
\]

\[
h_2(v^1) = (\vec{x}^{(1)}(u^i) - \vec{d}) \cdot \vec{e}_{L,2}^3.
\]
Since
\[(3.4)\quad v^1 = v^1(u^i) = \psi \left( (\hat{x}^{(1)}(u^i) - \vec{d}) \cdot \vec{e}_{L,2}^3 \right) \quad \text{for } v^1 \in I_1^{(2)},\]
we have to find the zeros of the function \(\Phi\) with
\[
\Phi(u^1, u^2) = r_2^2(v^1(u^i)) - \left( \left( (\hat{x}^{(1)}(u^i) - \vec{d}) \cdot \vec{e}_{L,2}^1 \right) \right)^2 + \left( \left( (\hat{x}^{(1)}(u^i) - \vec{d}) \cdot \vec{e}_{L,2}^2 \right) \right)^2.
\]
Then we have to compute the values \(v_{01}^1 = v^1(u_{01}^1, u_{01}^2)\) from (3.4) that correspond to the zeros \(u_{01}^1\) and \(u_{01}^2\) of the function \(\Phi\) and finally find the corresponding values \(v_{02}^2 = v^2(u_{01}^1, u_{02}^2)\) from
\[
\cos v_{02}^2 = \frac{(\hat{x}^2(u_{01}^1) - \vec{d}) \cdot \vec{e}_{L,2}^1}{r_2(v_{01}^1)} \quad \text{and} \quad \sin v_{02}^2 = \frac{(\hat{x}^2(u_{01}^1) - \vec{d}) \cdot \vec{e}_{L,2}^2}{r_2(v_{01}^1)}.
\]

Fig. 3.10: Intersections of a catenoid and a torus

REFERENCES


