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MANNHEIM PARTNER TRAJECTORIES RELATED TO PAFORS

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Abstract. In this study, we consider the concept of Mannheim partner trajectories related to the Positional Adapted Frame on Regular Surfaces (PAFORS) for the particles moving on the different regular surfaces in Euclidean 3-space. We give the relations between the PAFORS elements of these aforementioned trajectories. Also, we obtain the relations between Darboux basis vectors of these trajectories. Furthermore, some special cases of these trajectories are written.
Keywords: Mannheim partner trajectories, Positional Adapted Frame on Regular Surfaces, Darboux basis vectors.

1. Introduction

The surface theory is one of the most popular fundamental areas in differential geometry although its history is very long. The well-known moving frame Frenet-Serret frame has played an important role in the development of this theory. The steps which are performed by Frenet and Serret helped to adapt the moving frames to the curves on regular surfaces. This success was achieved by French mathematician Darboux [3]. He constructed a moving frame that is called today as Darboux frame for surface curves. Darboux frame is well-defined at every non-umbilic point

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of a surface. Therefore, it exists at every point of a regular surface curve [3, 15, 22]. Darboux frame has been used as a convenient tool for discussing many topics in the surface theory. Until today, a lot of researchers have performed many significant studies on the surface theory by means of Darboux frame. In [6, 11, 20, 26, 27], one can easily find some of these studies.

Another popular area in differential geometry is the curve theory. The concept of the special curves is an important part of this theory. In Euclidean 3-space E^3, curve pairs like Mannheim curve pairs are well-known examples of special curves. The topic of moving frames has an important place in the investigation of the local theory of these kinds of curve pairs. Developing new moving frames has always been an important effort for mathematicians. The groundbreaking discovery in this regard is the discovery of the Frenet-Serret frame, as everyone will agree. Most of the moving frames developed later include one of the basis vectors of the Frenet-Serret frame in common. Bishop frame [1], type 2-Bishop frame [29], type 3-Bishop frame [25], q-frame [5], Flc-frame [4], N-C-W frame [23], N-Bishop frame [10] can be given as examples to them. Similar to these moving frames, recently, Özen and Tosun have introduced a new moving frame on regular surfaces in Euclidean 3-space which is shortly called PAFORS by using the Darboux frame for the trajectories with non-vanishing angular momentum [17]. The authors have followed similar steps followed in the study [18] to construct this frame. The same authors also give some characterizations on asymptotic, slant helical, and geodesic trajectories with respect to PAFORS in the study [19]. Then, the idea of this new frame has been expanded to the Minkowski 3-space by Gürbüz in the study [8]. Gürbüz has taken into consideration the evolution of an electric field according to PAFORS in Minkowski 3-space in the aforementioned study.

Mannheim partner curves (according to Frenet-Serret frame) are interesting and popular special curves. The principal normal line of one of these partners matches up with the binormal line of the other partner at the corresponding points of them. Mannheim carried out the first study in 1878 on this topic [2, 13]. In the early 2000s, Mannheim partner curves were studied by Liu and Wang [12, 28]. In [12], the authors specified the necessary and sufficient conditions for a curve to possess a Mannheim partner curve in Euclidean 3-space and Minkowski 3-space. Then, Mannheim offsets of ruled surfaces were defined in [16]. On the other hand, dual Mannheim curves were discussed [7] and [21]. Another thing that can be of importance is that this topic was expanded to different frames such as Darboux frame and Bishop frame. Kazaz et al. [9] determined the Mannheim partner D-curves taking into consideration the Darboux frames of the curves on surfaces. Similar to this study, Masal and Azak investigated the Mannheim B-curves utilizing the Bishop frame [14].

In this paper, we investigate Mannheim partner trajectories related to PAFORS. Firstly, in Section 2, we mention the necessary information to understand the ensuing sections. In Section 3, Mannheim partner trajectories related to PAFORS are defined, and the relations between the PAFORS elements of these trajectories are given. Also, the relations between Darboux basis vectors of these trajectories are obtained. Moreover, some special cases of these trajectories are characterized according to PAFORS curvatures of these trajectories. Then, we give conclusions in Section 4.

2. Preliminaries

In this section, we remind some required terminology used throughout this paper.

In E^3, the standard inner product of any two vectors $\mathcal{W} = (w_1, w_2, w_3)$ and $\mathcal{X} = (x_1, x_2, x_3)$ are expressed as $\langle \mathcal{W}, \mathcal{X} \rangle = w_1x_1+w_2x_2+w_3x_3$. Based on this equality, the norm of the vector $\mathcal{W}$ is given by $\|\mathcal{W}\| = \sqrt{\langle \mathcal{W}, \mathcal{W} \rangle} = \sqrt{w_1^2 + w_2^2 + w_3^2}$. On the other hand, for a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$, if the condition $\|{}^{d\alpha}/_{ds}\| = 1 \quad \text{for all} \quad s \in I$ is satisfied, α is called a unit speed curve. In such a case, the parameter s is said to be an arc-length parameter of α. Also, if the derivative of a differentiable curve does not equal to zero everywhere along this curve, it is called a regular curve. Any regular curve always has a unit speed parameterization [24]. We must emphasize that the symbol prime $'$ will be used to show the differentiation with respect to the arc-length parameter s in the rest of this study.

The researchers generally make use of the Frenet-Serret frame to investigate many properties of regular curves. However, if these regular curves lie on regular surfaces, then using the Darboux frame offers more possibilities than the Frenet-Serret frame.

Let us suppose that a particle R moves on a regular surface M in the Euclidean 3-space along the trajectory $\alpha = \alpha(s)$ that is a unit speed curve. Thus, we can express α as $\alpha : I \subset \mathbb{R} \to M \subset \mathbb{E}^3$. The base vectors of the Darboux frame of the trajectory α are presented as $\{\mathbf{T}(s), \mathbf{Y}(s), \mathbf{U}(s)\}$ along α where $\mathbf{T}$ is called the unit tangent vector, $\mathbf{U}$ is called the unit normal vector. The remaining basis vector $\mathbf{Y}$ of the Darboux frame is found by means of the equality $\mathbf{Y} = \mathbf{U} \times \mathbf{T}$. It should be specified that the second-order derivatives of the curves, which we will consider in this article, are always non-zero (it means that $\alpha''(s)$ is zero nowhere). For Darboux frame, the derivative formulas are constructed as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{Y}'(s) \\ \mathbf{U}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & \tau_g(s) \\ -k_n(s) & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix},$$

where k_g is geodesic curvature, k_n is normal curvature and τ_g is geodesic torsion of the curve α [6,15].

Assume that the angular momentum vector of the aforesaid particle R about the origin does not vanish during the motion. In that case, PAFORS $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ is well defined along the trajectory $\alpha = \alpha(s)$. The base vectors of PAFORS are given as follows:

$$\begin{cases}\mathbf{T}(s)=\mathbf{T}(s),\\ \mathbf{G}(s)=\dfrac{\langle\alpha(s),\mathbf{U}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{Y}(s)\rangle^2+\langle\alpha(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)+\dfrac{\langle\alpha(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{Y}(s)\rangle^2+\langle\alpha(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s),\\ \mathbf{H}(s)=\dfrac{\langle-\alpha(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{Y}(s)\rangle^2+\langle\alpha(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)+\dfrac{\langle\alpha(s),\mathbf{U}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{Y}(s)\rangle^2+\langle\alpha(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s).\end{cases}$$

The relation between the Darboux frame and PAFORS exists as follows:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos\varphi(s)&-\sin\varphi(s)\\0&\sin\varphi(s)&\cos\varphi(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{Y}(s)\\ \mathbf{U}(s)\end{pmatrix}. \tag{2.1}$$

Here, $\varphi(s)$ is the angle between the vectors $\mathbf{Y}(s)$ and $\mathbf{G}(s)$ that is positively oriented from $\mathbf{Y}(s)$ to $\mathbf{G}(s)$ [17].
Furthermore, the derivative formulas of PAFORS are given as in the following [17]:

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{G}'(s)\\ \mathbf{H}'(s)\end{pmatrix}=\begin{pmatrix}0&k_1(s)&k_2(s)\\-k_1(s)&0&k_3(s)\\-k_2(s)&-k_3(s)&0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix}, \tag{2.2}$$

where

$$\begin{cases}k_1(s)=k_g(s)\cos\varphi(s)-k_n(s)\sin\varphi(s),\\ k_2(s)=k_g(s)\sin\varphi(s)+k_n(s)\cos\varphi(s),\\ k_3(s)=\tau_g(s)-\varphi'(s).\end{cases}$$

Additionally, the rotation angle $\varphi(s)$ is calculated by using the following equation [17]:

$$\varphi(s)=\begin{cases}\arctan\left(-\frac{\langle\alpha(s),\mathbf{Y}(s)\rangle}{\langle\alpha(s),\mathbf{U}(s)\rangle}\right) & if\quad \langle\alpha(s),\,\mathbf{U}(s)\rangle>0,\\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{Y}(s)\rangle}{\langle\alpha(s),\mathbf{U}(s)\rangle}\right)+\pi & if\quad \langle\alpha(s),\,\mathbf{U}(s)\rangle<0,\\ -\frac{\pi}{2} & if\quad \langle\alpha(s),\mathbf{U}(s)\rangle=0\ ,\quad \langle\alpha(s),\,\mathbf{Y}(s)\rangle>0,\\ \frac{\pi}{2} & if\quad \langle\alpha(s),\,\mathbf{U}(s)\rangle=0\ ,\quad \langle\alpha(s),\,\mathbf{Y}(s)\rangle<0.\end{cases}$$

Also, the elements of the set $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s),k_1(s),k_2(s),k_3(s)\}$ are called as PAFORS apparatuses of the trajectory $\alpha=\alpha(s)$ [17].

In order to remind the asymptotic curve and geodesic curve, we can present the following conditions [15]:

1. $k_n=0$ if and only if $\alpha=\alpha(s)$ is an asymptotic curve.
2. $k_g=0$ if and only if $\alpha=\alpha(s)$ is a geodesic curve.

Theorem 2.1. [19] *Suppose that $\alpha = \alpha(s)$ is an asymptotic curve on the regular surface M with the condition $k_g \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ if and only if $k_2 = 0$.*

Theorem 2.2. [19] *Assume that $\alpha = \alpha(s)$ is an asymptotic curve on the regular surface M with the condition $k_g \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ if and only if $k_1 = 0$.*

Theorem 2.3. [19] *Suppose that $\alpha = \alpha(s)$ is a geodesic curve on the regular surface M with the condition $k_n \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ if and only if $k_1 = 0$.*

Theorem 2.4. [19] *Assume that $\alpha = \alpha(s)$ is a geodesic curve on the regular surface M with the condition $k_n \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ if and only if $k_2 = 0$.*

For more detailed and comprehensive information about PAFORS, see [8, 17, 19].

3. Mannheim Partner Trajectories Related to PAFORS Lying on Different Regular Surfaces

In this section of this study, we introduce the Mannheim partner trajectories related to PAFORS and obtain some characterizations and geometric interpretations of them.

Definition 3.1. Let R and $\widehat{R}$ be the moving point particles on regular surfaces M and $\widehat{M}$ in Euclidean 3-space E^3. Let us show the unit speed parametrization of the trajectories of R and $\widehat{R}$ with $\alpha = \alpha(s)$ and $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$, respectively. Let $\{\mathbf{T}, \mathbf{G}, \mathbf{H}, k_1, k_2, k_3\}$ and $\left\{\widehat{\mathbf{T}}, \widehat{\mathbf{G}}, \widehat{\mathbf{H}}, \widehat{k}_1, \widehat{k}_2, \widehat{k}_3\right\}$ represent the PAFORS apparatus of the trajectories α and $\widehat{\alpha}$, respectively. If the PAFORS base vector $\mathbf{G}$ coincides with the PAFORS base vector $\widehat{\mathbf{H}}$ at the corresponding points of the trajectories α and $\widehat{\alpha}$, $\widehat{\alpha}$ is said to be a Mannheim partner trajectory of α related to PAFORS. Additionally, the pair $\{\alpha, \widehat{\alpha}\}$ is called a Mannheim pair related to PAFORS.

With the help of the definition of Mannheim pair related to PAFORS, we can give the following equation:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ 0 & 0 & 1 \\ -\sin\psi & \cos\psi & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{T}} \\ \widehat{\mathbf{G}} \\ \widehat{\mathbf{H}} \end{pmatrix}, \tag{3.1}$$

where ψ is the angle between the tangent vectors $\mathbf{T}$ and $\widehat{\mathbf{T}}$.

Theorem 3.1. *Suppose that $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ is any Mannheim pair related to PAFORS. Then, the distance between the corresponding points of α and $\widehat{\alpha}$ is constant.*

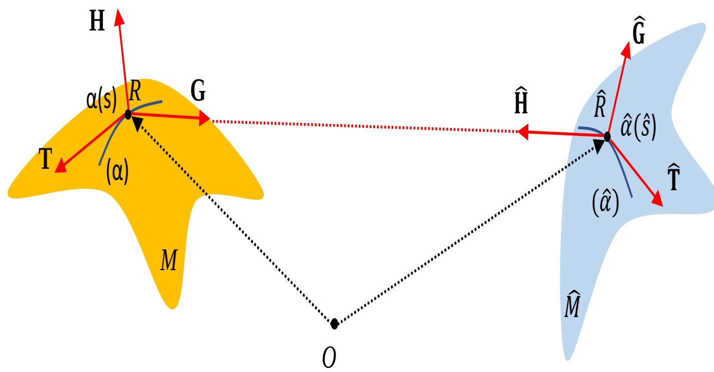

Fig. 3.1: Mannheim partner trajectories related to PAFORS

Proof. According to the definition of Mannheim trajectories related to PAFORS, the following equation can be given:

$$\alpha\left(s\right)=\widehat{\alpha}\left(\widehat{s}\right)+\eta\left(\widehat{s}\right)\widehat{\mathbf{H}}\left(\widehat{s}\right), \tag{3.2}$$

where η is a real valued smooth function of $\widehat{s}$ (cf. Figure 3.1). Differentiating the equation (3.2) with respect to $\widehat{s}$ and using the equation (2.2), we have:

$$\mathbf{T}\frac{ds}{d\widehat{s}}=\left(1-\eta\widehat{k}_2\right)\widehat{\mathbf{T}}-\eta\widehat{k}_3\widehat{\mathbf{G}}+\eta'\widehat{\mathbf{H}}. \tag{3.3}$$

Since $\mathbf{T},\widehat{\mathbf{T}}$ and $\widehat{\mathbf{G}}$ are orthogonal to $\widehat{\mathbf{H}}$, we have $\eta'=0$ with the help of the inner product. Thus, η is a non-zero constant and then we can rewrite the equation (3.3) as follows:

$$\mathbf{T}\frac{ds}{d\widehat{s}}=\left(1-\eta\widehat{k}_2\right)\widehat{\mathbf{T}}-\eta\widehat{k}_3\widehat{\mathbf{G}}. \tag{3.4}$$

Hence, the distance between the corresponding points of α and $\widehat{\alpha}$ can be written as follows:

$$d\left(\alpha\left(s\right),\widehat{\alpha}\left(\widehat{s}\right)\right)=\left\|\alpha\left(s\right)-\widehat{\alpha}\left(\widehat{s}\right)\right\|=\left\|\eta\widehat{\mathbf{H}}\right\|=\mid\eta\mid.$$

Therefore, we obtain the distance between each corresponding points of α and $\widehat{\alpha}$ as non-zero constant. □

Theorem 3.2. *Let $\{\alpha=\alpha\left(s\right),\widehat{\alpha}=\widehat{\alpha}\left(\widehat{s}\right)\}$ be any Mannheim pair related to PAFORS. In that case, the following equation is satisfied.*

$$\frac{d}{ds}(\cos\psi)=k_2\left\langle\mathbf{H},\,\widehat{\mathbf{T}}\right\rangle+\widehat{k}_1\frac{d\widehat{s}}{ds}\left\langle\mathbf{T},\widehat{\mathbf{G}}\right\rangle$$

Proof. Since ψ is the angle between the tangent vectors $\mathbf{T}$ and $\widehat{\mathbf{T}}$, we can write $\left\langle \mathbf{T}, \widehat{\mathbf{T}} \right\rangle = \|\mathbf{T}\| \left\| \widehat{\mathbf{T}} \right\| \cos\psi = \cos\psi$. If this equation is differentiated with respect to the parameter s, we obtain:

$$\begin{aligned}\frac{d}{ds}(\cos\psi) &= \frac{d}{ds}\left\langle \mathbf{T}, \widehat{\mathbf{T}} \right\rangle \\ &= \left\langle k_1\mathbf{G} + k_2\mathbf{H},\, \widehat{\mathbf{T}} \right\rangle + \left\langle \mathbf{T}, (\widehat{k_1}\widehat{\mathbf{G}} + \widehat{k_2}\widehat{\mathbf{H}})\frac{d\widehat{s}}{ds} \right\rangle \cdot\end{aligned}$$

Then, the last equation yields the desired result. $\square$

Corollary 3.1. *The angles between the tangent vectors at the corresponding points of a Mannheim pair (related to PAFORS) are generally not constant.*

Theorem 3.3. *Let* $\{\alpha = \alpha(s),\ \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair related to PAFORS. Then, the following equation is satisfied:*

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds} & -\eta\widehat{k}_3\frac{d\widehat{s}}{ds} & 0\\ 0 & 0 & 1\\ \eta\widehat{k}_3\frac{d\widehat{s}}{ds} & \left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds} & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{G}}\\ \widehat{\mathbf{H}}\end{pmatrix}\cdot \tag{3.5}$$

Proof. Let $\{\alpha, \widehat{\alpha}\}$ be a Mannheim pair related to PAFORS. With the help of the equations (3.1) and (3.4), we get:

$$\cos\psi\frac{ds}{d\widehat{s}}\widehat{\mathbf{T}} + \sin\psi\frac{ds}{d\widehat{s}}\widehat{\mathbf{G}} = \left(1-\eta\widehat{k}_2\right)\widehat{\mathbf{T}} - \eta\widehat{k}_3\widehat{\mathbf{G}}.$$

From the previous equation, we can write:

$$\begin{cases}\cos\psi = \left(1-\eta\widehat{k}_2\right)\dfrac{d\widehat{s}}{ds},\\ \sin\psi = -\eta\widehat{k}_3\dfrac{d\widehat{s}}{ds}\cdot\end{cases} \tag{3.6}$$

Substituting the equation (3.6) in the equation (3.1), we have the equation (3.5). $\square$

Corollary 3.2. *The tangent of the angle between the unit tangent vectors of the Mannheim partner trajectories (related to PAFORS)* $\alpha = \alpha(s)$ *and* $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$ *is given as follows:*

$$\tan\psi = \frac{-\eta\widehat{k}_3}{1-\eta\widehat{k}_2}\cdot \tag{3.7}$$

Corollary 3.3. *Let* $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair (related to PAFORS). In that case, the following equation is satisfied*

$$\int \cos\psi ds + \eta\int \widehat{k_2}d\widehat{s} = \widehat{s} + c_1,$$

where c_1 *shows the integration constant.*

Corollary 3.4. *Let $\{\alpha=\alpha(s), \widehat{\alpha}=\widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair (related to PAFORS). Then, the following equation is satisfied.*

$$\int \sin\psi ds+\eta\int \widehat{k}_3 d\widehat{s}=0$$

Theorem 3.4. *Let $\{\alpha=\alpha(s), \widehat{\alpha}=\widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair related to PAFORS and their Darboux frame be denoted by $\{\mathbf{T},\mathbf{Y},\mathbf{U}\}$ and $\left\{\widehat{\mathbf{T}},\widehat{\mathbf{Y}},\widehat{\mathbf{U}}\right\}$, respectively. In that case, the relations between the Darboux base vectors of this pair are given by*

$$\begin{aligned}
\widehat{\mathbf{T}}&=\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}\mathbf{T}-\eta\widehat{k}_3\sin\varphi\frac{d\widehat{s}}{ds}\mathbf{Y}-\eta\widehat{k}_3\cos\varphi\frac{d\widehat{s}}{ds}\mathbf{U},\\
\widehat{\mathbf{Y}}&=\eta\widehat{k}_3\sin\widehat{\varphi}\frac{d\widehat{s}}{ds}\mathbf{T}+\left(\cos\widehat{\varphi}\cos\varphi+\left(1-\eta\widehat{k}_2\right)\sin\widehat{\varphi}\sin\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{Y}\\
&\quad+\left(-\cos\widehat{\varphi}\sin\varphi+\left(1-\eta\widehat{k}_2\right)\sin\widehat{\varphi}\cos\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{U},\\
\widehat{\mathbf{U}}&=\eta\widehat{k}_3\cos\widehat{\varphi}\frac{d\widehat{s}}{ds}\mathbf{T}+\left(-\sin\widehat{\varphi}\cos\varphi+\left(1-\eta\widehat{k}_2\right)\cos\widehat{\varphi}\sin\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{Y}\\
&\quad+\left(\sin\widehat{\varphi}\sin\varphi+\left(1-\eta\widehat{k}_2\right)\cos\widehat{\varphi}\cos\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{U},
\end{aligned}$$

where φ is the angle between the vectors $\mathbf{U}$ and $\mathbf{H}$ and also, $\widehat{\varphi}$ is the angle between the vectors $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{H}}$.

Proof. With the help of the equation (2.1), the following equations

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos\varphi&-\sin\varphi\\0&\sin\varphi&\cos\varphi\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{Y}\\ \mathbf{U}\end{pmatrix}\tag{3.8}$$

and

$$\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{Y}}\\ \widehat{\mathbf{U}}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos\widehat{\varphi}&\sin\widehat{\varphi}\\0&-\sin\widehat{\varphi}&\cos\widehat{\varphi}\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{G}}\\ \widehat{\mathbf{H}}\end{pmatrix}\tag{3.9}$$

can be seen easily. Also, we can write the following equation according to the equation (3.5):

$$\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{G}}\\ \widehat{\mathbf{H}}\end{pmatrix}=\begin{pmatrix}\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}&0&\eta\widehat{k}_3\frac{d\widehat{s}}{ds}\\-\eta\widehat{k}_3\frac{d\widehat{s}}{ds}&0&\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}\\0&1&0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}.\tag{3.10}$$

Substituting the equation (3.10) in the equation (3.9) gives us the following:

$$\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{Y}}\\ \widehat{\mathbf{U}}\end{pmatrix}=\begin{pmatrix}\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}&0&\eta\widehat{k}_3\frac{d\widehat{s}}{ds}\\-\eta\widehat{k}_3\cos\varphi\frac{d\widehat{s}}{ds}&\sin\widehat{\varphi}&\left(1-\eta\widehat{k}_2\right)\cos\widehat{\varphi}\frac{d\widehat{s}}{ds}\\\eta\widehat{k}_3\sin\widehat{\varphi}\frac{d\widehat{s}}{ds}&\cos\widehat{\varphi}&-\left(1-\eta\widehat{k}_2\right)\sin\widehat{\varphi}\frac{d\widehat{s}}{ds}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}.\tag{3.11}$$

If the equation (3.8) is considered in the equation (3.11), the desired equations are found. □

Theorem 3.5. *Let* $\{\alpha=\alpha(s), \widehat{\alpha}=\widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair related to PAFORS. In that case, the following relations can be given:*

1. $k_1=\dfrac{\widehat{k}_2-\eta\widehat{k}_2^{\,2}-\eta\widehat{k}_3^{\,2}}{1-2\eta\widehat{k}_2+\eta^2\left(\widehat{k}_2^{\,2}+\widehat{k}_3^{\,2}\right)}$

2. $\widehat{k}_2=\dfrac{k_1-\xi k_1^2-\xi k_3^2}{1-2\xi k_1+\xi^2\left(k_1^2+k_3^2\right)}$

where ξ *is a constant satisfying* $|\xi|=|\eta|$.

Proof. 1. Assume that $\{\alpha,\widehat{\alpha}\}$ is a Mannheim pair related to PAFORS. With the help of the well-known identity $\cos^2\psi+\sin^2\psi=1$, we get:

$$\left(\frac{d\widehat{s}}{ds}\right)^2\left(\left(1-\eta\widehat{k}_2\right)^2+\eta^2\widehat{k}_3^{\,2}\right)=1$$

using the equation (3.6). Then, we can write:

$$\left(\frac{ds}{d\widehat{s}}\right)^2=1-2\eta\widehat{k}_2+\eta^2\left(\widehat{k}_2^{\,2}+\widehat{k}_3^{\,2}\right). \tag{3.12}$$

By differentiating the equation (3.4) according to the parameter $\widehat{s}$ and by using the equation (2.2), we have:

$$\begin{aligned}\frac{d^2s}{d\widehat{s}^2}\mathbf{T}+k_1\left(\frac{ds}{d\widehat{s}}\right)^2\mathbf{G}+k_2\left(\frac{ds}{d\widehat{s}}\right)^2\mathbf{H}=&\left(-\eta\left(\widehat{k}_2\right)'+\eta\widehat{k}_1\widehat{k}_3\right)\widehat{\mathbf{T}}\\&+\left(\widehat{k}_1\left(1-\eta\widehat{k}_2\right)-\eta\widehat{k}_3{}'\right)\widehat{\mathbf{G}}\\&+\left(\widehat{k}_2\left(1-\eta\widehat{k}_2\right)-\eta\widehat{k}_3^{\,2}\right)\widehat{\mathbf{H}}.\end{aligned} \tag{3.13}$$

The last equation yields:

$$k_1\left(\frac{ds}{d\widehat{s}}\right)^2=\left(1-\eta\widehat{k}_2\right)\widehat{k}_2-\eta\widehat{k}_3^{\,2}. \tag{3.14}$$

If we substitute the equation (3.12) in the equation (3.14), we get the desired result.

2. We can easily see the equality:

$$\widehat{\alpha}(\widehat{s})=\alpha(s)+\xi\mathbf{G}(s)$$

where ξ is a constant satisfying $\mid \eta \mid = \mid \xi \mid$ (cf. Figure 3.1). Derivating this equation according to the s twice, we get:

$$\widehat{\mathbf{T}}\frac{d\widehat{s}}{ds} = (1-\xi k_1)\,\mathbf{T} + \xi k_3 \mathbf{H} \tag{3.15}$$

and

$$\begin{aligned}\frac{d^2\widehat{s}}{ds^2}\widehat{\mathbf{T}} + \widehat{k}_1\left(\frac{d\widehat{s}}{ds}\right)^2\widehat{\mathbf{G}} + \widehat{k}_2\left(\frac{d\widehat{s}}{ds}\right)^2\widehat{\mathbf{H}} = &\left(-\xi k_1' - \xi k_2 k_3\right)\mathbf{T}\\ &+ \left(k_1\left(1-\xi k_1\right) - \xi k_3^2\right)\mathbf{G}\\ &+ \left(k_2\left(1-\xi k_1\right) + \xi k_3'\right)\mathbf{H}.\end{aligned} \tag{3.16}$$

By the equation (3.1), it can be seen that $\widehat{\mathbf{T}} = \cos\psi\mathbf{T} - \sin\psi\mathbf{H}$. Thus, we get:

$$\frac{d\widehat{s}}{ds}\cos\psi\mathbf{T} - \frac{d\widehat{s}}{ds}\sin\psi\mathbf{H} = (1-\xi k_1)\,\mathbf{T} + \xi k_3\mathbf{H}$$

and also $\frac{d\widehat{s}}{ds}\cos\psi = 1-\xi k_1$, $-\frac{d\widehat{s}}{ds}\sin\psi = \xi k_3$. From here we can write:

$$\left(\frac{d\widehat{s}}{ds}\right)^2 = 1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right). \tag{3.17}$$

The inner product of the vectors at the right and left sides of the equation (3.16) with the vector $\mathbf{G}$ gives us the following:

$$\widehat{k}_2\left(\frac{d\widehat{s}}{ds}\right)^2 = k_1 - \xi k_1^2 - \xi k_3^2. \tag{3.18}$$

Consequently, by using the equation (3.17), we have:

$$\widehat{k}_2 = \frac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right)}$$

and the proof is completed.

□

With the help of the Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 3.5, we can give the following corollaries.

Corollary 3.5. *Let* $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair (related to PAFORS). If* $\widehat{k}_2 = \widehat{k}_3 = 0$*, then* $k_1 = 0$.

Corollary 3.6. *Let* $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair (related to PAFORS). If* $k_1 = k_3 = 0$*, then* $\widehat{k}_2 = 0$.

Corollary 3.7. *Let $\{\alpha = \alpha(s), \ \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair related to PAFORS. Then, the followings are satisfied:*

1. *Suppose that the geodesic curvature of α never equals to zero. Then, $\alpha = \alpha(s)$ is an asymptotic curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(\mathbf{s}), \mathbf{Y}(\mathbf{s})\}$ if and only if* $\dfrac{\widehat{k}_2 - \eta\widehat{k}_2^{\,2} - \eta\widehat{k}_3^{\,2}}{1 - 2\eta\widehat{k}_2 + \eta^2\left(\widehat{k}_2^{\,2} + \widehat{k}_3^{\,2}\right)} = 0.$
2. *Assume that the geodesic curvature of $\widehat{\alpha}$ never equals to zero. Then, $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$ is an asymptotic curve whose position vector lies on the corresponding plane $Sp\{\widehat{\mathbf{T}}(\widehat{\mathbf{s}}), \widehat{\mathbf{U}}(\widehat{\mathbf{s}})\}$ if and only if* $\dfrac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right)} = 0.$

Corollary 3.8. *Let $\{\alpha = \alpha(s), \ \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair related to PAFORS. Then, the followings are satisfied:*

1. *Suppose that the normal curvature of α never equals to zero. Then, $\alpha = \alpha(s)$ is a geodesic curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(\mathbf{s}), \mathbf{U}(\mathbf{s})\}$ if and only if* $\dfrac{\widehat{k}_2 - \eta\widehat{k}_2^{\,2} - \eta\widehat{k}_3^{\,2}}{1 - 2\eta\widehat{k}_2 + \eta^2\left(\widehat{k}_2^{\,2} + \widehat{k}_3^{\,2}\right)} = 0.$
2. *Assume that the normal curvature of $\widehat{\alpha}$ never equals to zero. Then, $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$ is a geodesic curve whose position vector lies on the corresponding plane $Sp\{\widehat{\mathbf{T}}(\widehat{\mathbf{s}}), \widehat{\mathbf{Y}}(\widehat{\mathbf{s}})\}$ if and only if* $\dfrac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right)} = 0.$

4. Conclusions

The main purpose of this study is to lead the studies investigating the special classes of regular surface curves (traced out by a moving particle) by means of the new and convenient moving frame PAFORS. In accordance with this purpose, we choose the Mannheim partner curves which are well-known and preferred widely. We think this choice makes the study more remarkable.

In this study, Mannheim partner trajectories related to PAFORS are defined for the particles moving along the different regular surfaces in Euclidean 3-space. Also, the relations are given between the PAFORS elements of these aforementioned trajectories. Moreover, the relations are obtained between Darboux basis vectors of these trajectories, and some special cases of these trajectories are characterized.

We state that we plan to discuss the Bertrand partner trajectories related to PAFORS in the future study.

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