CHARACTERIZATION OF BI-NULL SLANT(BNS) HELICES OF $(k, m)$-TYPE IN $R^3_1$ AND $R^5_2$

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Abstract. The present study discusses bi-null slant helices of $(k, m)$ type in $R^5_2$ and give the characterization for a curve to be certain $(k, m)$ type bi null slant helix (BNS helix). The discussion includes the proofs for the non existence cases of $(k, m)$ type bi null slant helices in $R^5_2$. Moreover certain characterizations and non existence have also been obtained for bi null slant helix to be $(k, m)$ type using modified orthogonal frame.

Keywords: k-type Slant helix, Semi Euclidean space, Bi-null curves, Frenet Formulae.

1. Introduction

In 2004, Izumiya and Takeuchi [7] introduced the notion of slant helix which is defined as a curve $\xi$ in $R^3$ where principal normal vector makes a constant angle with a fixed vector in $R^3$. Several geometers have studied slant helices [1, 8, 9] and gave characterizations for being such curves. In particular k-type slant helices have been one of the most interesting cases due to the rich geometric properties and applications in different branches of science and engineering [2, 6, 10]. Different varieties of k-type slant helices, k-type partially null and pseudo null helices etc. were further studied by Ergiin et al [6] Ahmad T et.al.[2] and E. Nesovic et.al [10] respectively.

On the other hand, in 2012, bi null cartan curves were introduced and studied by M. Sakaki [12] in $R^5_2$ with concerned distinctive Frenet frame and the related
curvatures called the Cartan curvatures. Proceeding on, in [13] and [14], some characterizations were proved for bi null Cartan curves to be \( k \)-type slant helices in semi-Euclidean spaces \( \mathbb{R}^6_3 \) and \( \mathbb{R}^5_2 \) respectively.

Later in 2020, a class of slant helices called \((k, m)\) type slant helices was considered in [3] which presented a study of \((k, m)\) type slant helices for partially null and pseudo null curves in Minkowski space \( E^4_1 \).

The aim of this paper is to give characterization for bi null curve to be \((k, m)\)-type slant helices in \( \mathbb{R}^5_2 \) using the curvature function. Moreover, characterizations of bi-null curves to be \((k, m)\) type slant helix have also been obtained in \( \mathbb{R}^3_1 \) with modified orthogonal frame.

2. Preliminaries

Assume that \( \mathbb{R}^5_2 \) is the 5-dimensional semi-Euclidean space with index 2. It is clear that if the standard co-ordinate system of \( \mathbb{R}^5_2 \) is \( \{x_1, x_2, x_3, x_4, x_5\} \), then the metric can be written as [13].

\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2
\]

The inner product on \( \mathbb{R}^5_2 \) is denoted by \( <, > \). We know that vector \( X \in \mathbb{R}^5_2 - \{0\} \) is called timelike if \( < X, X > < 0 \), spacelike if \( < X, X > > 0 \) and null (lightlike) if \( < X, X > = 0 \). If \( X = 0 \), then it will fall in the category of spacelike vectors. Also we have \( ||X|| = \sqrt{< X, X >} \). Here \( ||X|| \) denotes the norm of a vector \( X \). Two vectors \( X \) and \( Y \) are said to be orthogonal if \( < X, Y > = 0 \).

We now give a brief idea of modified orthogonal frame which in some sense generalizes Frenet frame in \( \mathbb{R}^3_1 \).

Let \( \xi \) be a general analytic curve which can be re parameterized by its arc length \( s \), where \( s \in I \) and \( I \) is a non empty open interval. Assuming that the curvature function has discrete zero points or \( k(s) \) is not identically zero, we have an orthogonal frame \( T, N, B \) defined as follows [4].

\[
(T) \quad T = \frac{d\xi}{ds}, N = \frac{dT}{ds}, B = T \times N
\]

where \( T \times N \) is the vector product of \( T \) and \( N \). The relationship between \( T, N \) and \( B \) and previous Frenet frame vectors at non zero points of \( k \) are

\[
T = t, N = kn, B = \tau b
\]

Thus from above equations we conclude that \( N = B = 0 \), when \( k = 0 \) and squares of length of \( N \) and \( B \) vary analytically in \( s \). From equation 2.1, it is easy to calculate

\[
(2.2) \quad \begin{pmatrix}
T'(s) \\
N'(s) \\
B'(s)
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
-k^2 & \frac{k'}{k} & \tau \\
0 & -\tau & \frac{k'}{k}
\end{pmatrix} \begin{pmatrix}
T \\
N \\
B
\end{pmatrix}
\]
where all the differentiation are done with respect to the arc length (s) and

$$\tau(s) = \frac{\text{det}(\xi', \xi'', \xi''')}{k^2}$$

is the torsion of $\xi$. From Frenet equation, we know that at any point, where $k^2 = 0$ is a removable singularity of $\tau$. Let $\langle, \rangle$ be the standard inner product of $E^3$, then $T, N, B$ satisfies:

$$< T, T > = 1, < N, N > = < B, B > = k^2, < T, N > = < T, B > = < N, B > = 0$$

The orthogonal frame defined in 2.2 satisfying 2.3 is called as modified orthogonal frame.

**Remark 2.1.** It can be easily seen that once we put $k=1$ in 2.3, the modified orthogonal frame coincides with Frenet frame.

**Definition 2.1.** [12] Any curve $\xi(t)$ in $R^3$ is a bi-null curve if span $\{\xi'(t), \xi''(t)\}$ is isotropic i.e $< \xi'(t), \xi'(t) >= 0, < \xi'(t), \xi''(t) >= 0$ and $< \xi''(t), \xi''(t) >= 0$, and $\{\xi'(t), \xi''(t)\}$ are linearly independent for all $t$.

We consider any bi null curve $\xi(t) \subset R^3$ with t as a parameter. Then for $\xi(t)$, there exist Frenet frame $\{T, N, B_1, B_2, B_3\}$ such that $\xi'(t) = T$ and

$$< T, T > = 1, < N, N > = < B, B > = k^2, < T, N > = < T, B > = < N, B > = 0$$

The orthogonal frame defined in 2.2 satisfying 2.3 is called as modified orthogonal frame.

**Remark 2.1.** It can be easily seen that once we put $k=1$ in 2.3, the modified orthogonal frame coincides with Frenet frame.

**Definition 2.2.** [12, 5] Any bi null curve $\xi(t)$ in $R^3$ with $\{\xi^3(t), \xi^3(t)\} \neq 0$ is a bi-null Cartan curve if $\xi'(t), \xi''(t), \xi'''(t)$ are linearly independent for all $t$.

We now quote the following theorem which guarantees the existence of a unique bi null Cartan curve with Cartan frame $[3, 11]$ for given curvatures $k_0(t)$ and $k_1(t)$.

**Theorem 2.1.** Let $k_0(t)$ and $k_1(t)$ be the differentiable functions on $(t_0 - \epsilon, t_0 + \epsilon)$ Let $p_0$ be the point in $R^3$, and $\{T, N, B_1, B_2, B_3\}$ be a pseudo-orthonormal basis of $R^3$. Then there exists a unique bi-null Cartan curve $\xi(t)$ in $R^3$ with $\xi(t_0) = p_0$, bi-null arc parameter $t$ and curvatures $k_0, k_1$, whose Cartan frame $\{T, N, B_1, B_2, B_3\}$ satisfies $T(t_0) = T, N(t_0) = N, B_1(t_0) = B_1, B_2(t_0) = B_2, B_3(t_0) = B_3$. 
3. Characterization of (k,m)-type BNS helics in $R^3_1$

First we give the definition of bi-null slant(BNS) helices of (k, m) type in $R^3_1$.

**Definition 3.1.** [3] Let $\{\Gamma_1, \Gamma_2, \Gamma_3, \}$ be the frame for a bi null curve $\xi$ in $R^3_1$. Then $\xi$ is known as a bi-null slant helix of (k, m) type, if we are able to find a fixed vector $U \neq 0 \in R^3_1$ such that $< \Gamma_k, U > = \alpha$, and $< \Gamma_m, U > = \beta$, where $\alpha, \beta$ are constants for $1 \leq k \leq 3$ and $1 \leq m \leq 3$.

We can express $U$ as $U = u_1T + u_2N + u_3B_1$, where $u_i$'s are differentiable functions of ‘t’. Here we write $\Gamma_1 = T$, $\Gamma_2 = N$, $\Gamma_3 = B_1$.

**Theorem 3.1.** (1, 2) and (1, 3) type BNS helices in $R^3_1$ with modified orthogonal frame do not exist there.

**Proof.** Let $\xi$ represents a bi null slant helix of (1, 2) type in $R^3_1$. Then by definition, for any fixed vector $U$, we have

$$< T, U > = \alpha \quad \text{and} \quad < N, U > = \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to $t$, we get

$$< T', U > = 0 \quad \text{and} \quad < N', U > = 0.$$ 

Now using equation (2.2), we get

$$< N, U > = 0$$

which contradicts our supposition. Hence there does not exist a BNS helix of (1, 2) type in $R^3_1$ with modified orthogonal frame. \(\square\)

Similarly we can show that there does not exist BNS helix of (1, 3) type in $R^3_1$ with modified orthogonal frame.

**Theorem 3.2.** $\xi$ is a bi null slant helix of (2, 3) type in $R^3_1$ with modified orthogonal frame parameterized by arclength ‘t’ with $k_0, k_1 \neq 0$ if and only if

$$\alpha^2k^2 + (\alpha^2 + \beta^2)d\left(\frac{k'}{k}\right) = 0$$

**Proof.** Let $\xi$ represents a bi null slant helix of (2, 3) type in $R^3_1$ with modified orthogonal frame.

$$< N, U > = \alpha \quad \text{and} \quad < B, U > = \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Then we can write

$$U = u_1T + \alpha N + \beta B.$$
Differentiating with respect to 't', we get
\[ u'_1 T' + u'_1 T + \alpha N' + \beta B' = 0. \]
Using equation (2.2), we get
\[ u'_1 T + u_1 N + \alpha(-k^2 T + \frac{k'}{k} N + \tau B) + \beta(-\tau N + \frac{k'}{k} B) = 0. \]
On simplification we get
\[ (3.1) \quad u'_1 - \alpha k^2 = 0, u_1 + \alpha \frac{k'}{k} = \beta \tau = 0, \alpha \tau + \beta \frac{k'}{k} = 0. \]
Solving 2nd and 3rd equation of equation 3.1, we have
\[ (3.2) \quad u_1 \alpha + \frac{k'}{k} (\alpha^2 + \beta^2) = 0. \]
Now differentiating equation 3.2 and using 1st equation of 3.1, we arrive at
\[ \alpha^2 k^2 + (\alpha^2 + \beta^2) d\left(\frac{k'}{k}\right) = 0. \]
Conversely choose \( u_1 = \beta \tau - \alpha \frac{k'}{k} \) such that
\[ U = (\beta \tau - \alpha \frac{k'}{k}) T + \alpha N + \beta B. \]
Differentiating above equation, we obtain
\[ U' = (\beta \tau - \alpha \frac{k'}{k}) T' + (\beta \tau' - \alpha \frac{kk' - k'^2}{k^2}) T + \alpha N' + \beta B'. \]
Using equation (2.2), we get
\[ U' = (\beta \tau - \alpha \frac{k'}{k}) N + (\beta \tau' - \alpha \frac{kk' - k'^2}{k^2}) T + \alpha(-k^2 T \]
\[ + \frac{k'}{k} N + \tau B) + \beta(-\tau N + \frac{k'}{k} B). \]
Finally, using equation (3.1) in above equation we get \( U' = 0 \). Hence proved. \( \square \)

As a whole, we conclude the results of this section in the form of the following table.
Table 3.1: Existence and non-existence of BNS helix in modified orthogonal frame in $R_3$

<table>
<thead>
<tr>
<th>Type of BNS helix</th>
<th>Existence/Non-existence</th>
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<tbody>
<tr>
<td>(1,2)-type</td>
<td>does not exist</td>
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</tr>
<tr>
<td>(2,3)-type</td>
<td>exists iff $\alpha^2 k^2 + (\alpha^2 + \beta^2) d(k') = 0$</td>
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</table>

4. Characterization of $(k,m)$-type BNS helices in $R^5_2$

First we give the definition of bi-null slant (BNS) helices of $(k, m)$ type in $R^5_2$.

**Definition 4.1.** [3] Let $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ be the frame for a bi-null curve $\xi$ in $R^5_2$. Then $\xi$ is known as a bi-null slant helix of $(k, m)$ type, if we are able to find a fixed vector $U \neq 0 \in R^5_2$ such that $< \Gamma_k, U >= \alpha$, and $< \Gamma_m, U >= \beta$, where $\alpha, \beta$ are constants for $1 \leq k \leq 5$ and $1 \leq m \leq 5$.

We can express $U$ as $U = u_1 T + u_2 N + u_3 B_1 + u_4 B_2 + u_5 B_3$, where $u_i$'s are differentiable functions of ‘$t$’. Here we write $\Gamma_1 = T, \Gamma_2 = N, \Gamma_3 = B_1, \Gamma_4 = B_2, \Gamma_5 = B_3$.

**Theorem 4.1.** There does not exist $(1, 2)$ type BNS helix in $R^5_2$.

**Proof.** Let $\xi$ represents a bi-null slant helix of $(1, 2)$ type in $R^5_2$. Then by definition, for any fixed vector $U$, we have

$$< T, U >= \alpha \quad \text{and} \quad < N, U >= \beta$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to $t$, we find

$$< T', U >= 0 \quad \text{and} \quad < N', U >= 0.$$

Using equation (2.4), we get

$$< N, U >= 0 \quad \text{and} \quad < B_3, U >= 0$$

which contradicts our supposition. Hence there does not exist a BNS helix of $(1, 2)$ type in $R^5_2$. \(\square\)

Similarly we can prove the non existence of $(1,5), (2,3), (2,5), (3,4), (3,5)$ and $(4,5)$ type bi null slant helices.

**Theorem 4.2.** $\xi$ is a bi null slant helix of $(1, 3)$ type in $R^5_2$ parameterized by arc length ‘$t$’ with $k_0, k_1 \neq 0$ if and only if $k_1 \neq 0$ is a constant.
Proof. Let $\xi$ represents a bi null slant helix of $(1, 3)$ type in $R_2^5$. Let $U$ a fixed vector, then by definition we have

\[
<T, U> = \alpha \quad \text{and} \quad <B_1, U> = \beta
\]

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to $t$, we get

\[
<T', U> = 0 \quad \text{and} \quad <B'_1, U> = 0.
\]

Using equation 2.4, we conclude

\[
<N, U> = 0 \quad \text{and} \quad k_1 <N, U> = 0.
\]

Differentiating first part with respect to ‘t’, we get

\[
<N', U> = 0.
\]

Again by using equation (2.4) in the above equation, we obtain

\[
<B_3, U> = 0.
\]

Differentiating the above equation with respect to t’ and using equation (2.4), we get

\[
-k_0 <N, U> - <B_2, U> = 0
\]

OR

\[
<B_2, U> = 0.
\]

Therefore we can write

\[
U = \alpha T + \beta B_1
\]

Differentiating with respect to ‘t’, we get

\[
\alpha T' + \beta B'_1 = 0.
\]

Now putting equation 2.4 in the above equation, which implies

\[
k_1 = -\frac{\alpha}{\beta} = \text{constant}.
\]

Conversely, assume that $k_1$ is a constant. For $\beta \neq 0$, choose the vector $U$ as

\[
U = -\beta k_1 T + \beta B_1.
\]

On differentiating this with respect to $t$ we get

\[
U' = 0
\]

and hence

\[
<T, U> = \text{constant} \quad \text{and} \quad <B_1, U> = \text{constant}.
\]

Hence $\xi$ is a bi null slant helix of $(1, 3)$ type in $R_2^5$. \qed
**Theorem 4.3.** ξ is a bi null slant helix of (1, 4) type in $\mathbb{R}^5_2$ if and only if $k_1 = 0$ and $\beta k'_0 - \alpha = 0$, where 't' is a parameter.

**Proof.** Let ξ represents a bi null slant helix of (1, 4) type in $\mathbb{R}^5_2$.

\[
< T, U > = \alpha \quad \text{and} \quad < B_2, U > = \beta
\]

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Then we can write

\[
U = \alpha T + u_2 N + u_3 B_1 + \beta B_2 + u_5 B_3.
\]

Differentiating with respect to 't', we get

\[
\alpha T' + u_2 N' + u_3 k_1 N + u_3' B_1 + \beta B_2' + u_5 B_3' + u_5' B_3 = 0.
\]

Using equation 2.4, we arrive at

\[
\alpha N + u_2 B_3 + u_2' N + u_3 k_1 N + u_3' B_1 + \beta(-k_1 T - B_1 + k_0 B_3) + u_5(-k_0 N - B_2) + u_5' B_3 = 0
\]

which implies that

\[
k_1 = 0, \quad \beta k'_0 - \alpha = 0.
\]

Conversely choose the vector $U$ as

\[
U = -\alpha T - k_0 \beta N + (\beta t + A) B_1 + \beta B_2.
\]

On differentiating with respect to 't' we get

\[
U' = 0
\]

which gives

\[
< T, U >= \text{constant} \quad \text{and} \quad < B_2, U >= \text{constant}.
\]

Hence ξ is a bi null slant helix of (1, 4) type in $\mathbb{R}^5_2$. \( \square \)

**Theorem 4.4.** ξ is a bi null slant helix of (2, 4) type in $\mathbb{R}^5_2$ if $\int k_1 dt + k_1 t + C = 0$, where $C$ is a constant of integration.

**Proof.** Let ξ represents a bi null slant helix of (2, 4) type in $\mathbb{R}^5_2$. Then for a fixed vector $U$, we have

\[
< T, U > = \alpha \quad \text{and} \quad < B_2, U > = \beta
\]

where $\alpha \neq 0$ and $\beta \neq 0$ are constants. Differentiating with respect to t, we get

\[
< N', U > = 0 \quad \text{and} \quad < B_2', U > = 0.
\]
Using equation 2.4, we find

\[ < B_3, U > = 0 \]

which gives

\[ U = u_1 T + \alpha N + u_3 B_1 + \beta B_2. \]

Differentiating and using equation (2.4)

\[ u_1 N + u'_1 T + \alpha B_3 + u_3 k_1 N + u'_3 B_1 + \beta (-k_1 T - B_1 + k_0 B_3) = 0. \]

Which, on simplification implies that

\[ \int k_1 dt + k_1 t + C = 0 \]

where C is constant of integration. Hence proved. \( \square \)

As a whole, the results of this section can be tabulated as follows:

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<tr>
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</tr>
<tr>
<td>(3,5)-type</td>
<td>does not exists</td>
</tr>
<tr>
<td>(4,5)-type</td>
<td>does not exists</td>
</tr>
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</table>

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REFERENCES


