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POINTWISE 1-TYPE GAUSS MAP OF DEVELOPABLE SMARANDACHE RULED SURFACES IN $\mathbb{E}^3$ *

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Abstract. In this paper, we study the developable TN, TB, and NB-Smarandache ruled surface with a pointwise 1-type Gauss map. In particular, we obtain that every developable TN-Smarandache ruled surface has constant mean curvature, and every developable TB-Smarandache ruled surface is minimal if and only if the curve is a plane curve with non-zero curvature or a helix, and every developable NB-Smarandache ruled surface is always plane. We also provide some examples.
Keywords: Smarandache ruled surface, Gauss map, pointwise 1-type, Laplace-Beltrami operator.

1. Introduction

The fundamental theory of curves, their characterization, and the corresponding relations between the curves are very interesting and important topics in differential geometry. Bertrand curves, Mannheim curves, involute-evolute curves, etc. are some of the most famous examples of such types of curves; please see [20, 21] and references therein. Moreover, one of the most fascinating examples of such innovative curves are the Smarandache curves, which were first introduced in Minkowski space-time by authors in [22] and play an important role in Smarandache geometry.

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A Smarandache curve is a regular curve whose position vector is composed of the Frenet frame vectors of another regular curve. They are the objects of Smarandache geometry, i.e., a geometry that has at least one Smarandachely denied axiom [3]. An axiom is said to be smarandachely denied if it behaves in at least two different ways within the same space. Smarandache geometry plays a powerful role in the theory of relativity and parallel universes. Apart from the Frenet frame, many geometers studied Smarandache curves by taking different frames such as the Bishop frame, Darboux frame, etc. [2, 6, 7, 19, 22] and references therein.

In surface theory, a parametrized surface

$$X(s,v) = \gamma(s) + v\,w(s), \quad s \in I, \quad v \in \mathbb{R}, \tag{1.1}$$

is called the ruled surface generated by a one-parameter family of lines $\{\gamma(\mathrm{s}), \mathrm{w}(\mathrm{s})\}$, where $\gamma(s)$ is called a directrix of the surface $X(s,v)$ and the vector $w(s)$ defines the ruling direction [14].

Ruled surfaces have a variety of applications, including CAD/CAGD, architectural design, kinematics, wire electric discharge machining (EDM), and NC milling with a cylindrical cutter [17, 23]. In addition, ruled surfaces are widely used in mechanical industries, robotic designs, and architecture for functional and fascinating constructions.

Minimal and developable surfaces are two of the most essential types of surfaces. A surface is minimal if it has zero mean curvature. Plane, catenoid, and helicoid are examples of a minimal surface [5]. The ruled surfaces that can be transformed into a plane without any deformation or distortion with vanishing Gaussian curvature are called developable surfaces. Cylinders, cones, and tangent surfaces are examples of developable surfaces [1].

In 2021, Ourab [16] defined a Smarandache ruled surfaces whose directrix are TN, TB, and NB-Smarandache curves derived from Frenet vectors of the curve in $\mathbb{E}^3$, and studied the geometric properties of such surfaces based on the mean curvature and Gaussian curvature. Recently, in 2022, Senyurt [18] introduced some new special ruled surfaces with a directrix as the TNB-Smarandache curve and studied their geometric properties such as mean curvature and Gaussian curvature in $\mathbb{E}^3$.

On the other hand, the study of finite-type submanifolds in Euclidean spaces was initiated by Chen in the 1970s [11, 8]. Later, Chen and Piccini [9] introduced and studied submanifolds whose Gauss map G satisfies

$$\Delta\, G = \lambda\,(G + C), \tag{1.2}$$

where λ is a real number and C is a constant vector. In addition, a submanifold satisfying (1.2) is said to have a 1-type Gauss map. The study of submanifolds satisfying (1.2) were continued by many geometers [4, 8, 9].

In 2000, Kim and Yoon [15] generalized (1.2) as

$$\Delta G = f\,G, \tag{1.3}$$

where f is a smooth function. In addition, a submanifold satisfying (1.3) is said to have a pointwise 1-type Gauss map. The authors in [15] applied (1.3) to study ruled surfaces in a three-dimensional Minkowski space and classified them completely. In 2001, Choi and Kim [12] investigated ruled surfaces with (1.3) in $\mathbb{E}^3$. They proved that such surfaces are the open portions of the plane, the circular cylinder, and the minimal helicoid.

Furthermore, the generalization of (1.3) was given by Chen et al. [10] in 2005 as

$$\Delta G = f\,(G + C), \tag{1.4}$$

for a smooth function f and a constant vector C is called a pointwise 1-type Gauss map of the first kind if the vector C in (1.4) is a zero vector; otherwise, it is said to be of the second kind. In 2010, Choi et al. [13] classified ruled surfaces in a 3-dimensional Euclidean space satisfying (1.4). There are several surfaces that satisfy (1.4), including planes, cylinders, right cones, and catenoids.

Remark 1.1. The Gauss map G of a plane in $\mathbb{E}^3$ is a constant vector and $\Delta G = 0$. For $f = 0$, if we state $\Delta G = 0.G$, then M satisfies (1.4) of the first kind. If we select a non-zero smooth function f and $C = -G$, then (1.4) holds and M is of the second kind. Hence, a plane in $\mathbb{E}^3$ is a trivial surface with (1.4) of the first or second kind.

In view of the above, we study Smarandache ruled surface satisfying (1.4) in $\mathbb{E}^3$. The paper is structured as follows: In section 2, we quote some basic notations on surfaces in $\mathbb{E}^3$ that are relevant to the rest of the paper. In section 3, we obtain the condition for the developable TN-Smarandache ruled surface satisfying (1.4) in $\mathbb{E}^3$. Section 4 is devoted to the study of developable $TB-$Smarandache ruled surface with (1.4) in $\mathbb{E}^3$. In Section 5, we study the developable NB-Smarandache ruled surface satisfying (1.4) in $\mathbb{E}^3$. In Section 6, we present the conclusions of our study. In Section 7, we provide some examples.

2. Preliminary

Let $\alpha = \alpha(s)$ be a regular unit-speed curve in $\mathbb{E}^3$ where 's' measures its arc length, and the triplet $(T,\, N,\, B)$ be the Frenet frame of a curve $\alpha(s)$. Then, the Frenet-Serret formula of the curve $\alpha(s)$ is given as [14]

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where $k(s)$ and $\tau(s)$ are curvature and torsion of α, respectively.

In [2], authors defined the Smarandache curve according to the triplet (T, N, B) of the curve $\alpha = \alpha(s)$ as:

$$\beta_1(s^*(s)) = \frac{1}{\sqrt{2}}\Big(T(s) + N(s)\Big), \tag{2.2}$$

$$\beta_2(s^*(s)) = \frac{1}{\sqrt{2}}\Big(T(s) + B(s)\Big), \tag{2.3}$$

and

$$\beta_3(s^*(s)) = \frac{1}{\sqrt{2}}\Big(N(s) + B(s)\Big), \tag{2.4}$$

and β_1, β_2, and β_3 are called TN-Smarandache curves, TB-Smarandache curves, and NB-Smarandache curves, respectively.

In [16], Ouarab introduced the Smarandache ruled surfaces whose directrix are (2.2), (2.3), and (2.4) in $\mathbb{E}^3$, and defined as:

Definition 2.1. Let $\alpha = \alpha(s)$ be a regular unit-speed curve and denote $(T(s)$, $N(s)$, $B(s))$ as the Frenet frame of α in $\mathbb{E}^3$. Then

$$x(s,v) = \frac{1}{\sqrt{2}}\Big(T(s) + N(s)\Big) + vB(s), \tag{2.5}$$

$$y(s,v) = \frac{1}{\sqrt{2}}\Big(T(s) + B(s)\Big) + vN(s), \tag{2.6}$$

and

$$z(s,v) = \frac{1}{\sqrt{2}}\Big(N(s) + B(s)\Big) + vT(s), \tag{2.7}$$

are called TN-Smarandache ruled surface, TB-Smarandache ruled surface, and NB-Smarandache ruled surface, respectively.

The Laplacian operator Δ on a surface is defined by

$$\Delta = -\frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial x_i}\Big(\sqrt{g}\, g^{ij}\frac{\partial}{\partial x_j}\Big), \tag{2.8}$$

where (g_{ij}) is the first fundamental form matrix, the matrix (g^{ij}) is the inverse of (g_{ij}) and g denotes the determinant of (g_{ij}).

The Gauss map of a surface $X = X(s,v)$ in $\mathbb{E}^3$ is defined by

$$G(s,v) = \frac{X_s \times X_v}{||X_s \times X_v||}, \tag{2.9}$$

where $X_s = \frac{\partial X}{\partial s}$, and $X_v = \frac{\partial X}{\partial v}$ are partial derivatives. The relationship between the mean curvature vector $\overrightarrow{H}$ and the position vector field X of the surface $X(s,v)$ in $\mathbb{E}^3$ is

$$\Delta X = -2\,\overrightarrow{H}, \tag{2.10}$$

where

$$\overrightarrow{H} = H\,G, \tag{2.11}$$

and H denotes the mean curvature of the surface. From (2.10), we obtain the following: A surface $X(s,v)$ in $\mathbb{E}^3$ is a minimal surface if and only if [11]

$$\Delta X = 0. \tag{2.12}$$

3. Developable TN-Smarandache ruled surfaces with pointwise 1-Type Gauss map in $\mathbb{E}^3$

In this section, we study the developable TN-Smarandache ruled surface with a harmonic Gauss map and a pointwise 1-type Gauss map of the first kind. The TN-Smarandache ruled surface $x(s,v)$, as given by (2.5) is developable if and only if $\alpha(s)$ is a plane curve, i.e., $\tau = 0$ [16].

Using (2.9), the Gauss map G of the developable TN-Smarandache ruled surface $x(s,v)$ is given by

$$G(s,v) = \frac{1}{\sqrt{2}}\Big(T(s) + N(s)\Big). \tag{3.1}$$

We have

$$\begin{aligned} &g_{11} = \langle x_s, x_s\rangle = k^2, \quad g_{12} = \langle x_s, x_v\rangle = 0, \quad g_{21} = \langle x_v, x_s\rangle = 0, \\ &g_{22} = \langle x_v, x_v\rangle = 1, \quad g = k^2. \end{aligned} \tag{3.2}$$

Using (2.8) and (3.2), the Laplacian operator Δ on the developable TN-Smarandache ruled surface $x(s,v)$ is computed as:

$$\Delta = \frac{-1}{k}\Big[\frac{-k'}{k^2}\frac{\partial}{\partial s} + \frac{1}{k}\frac{\partial^2}{\partial s^2} + k\frac{\partial^2}{\partial v^2}\Big]. \tag{3.3}$$

Using (3.1) and (3.3), we obtain

$$\Delta\, G = \frac{1}{\sqrt{2}}\Big(T(s) + N(s)\Big). \tag{3.4}$$

Now, we have:

Theorem 3.1. *Every developable TN-Smarandache ruled surface $x(s,v)$ has a pointwise 1-type Gauss map of the first kind .*

Theorem 3.2. *There does not exist a developable TN-Smarandache ruled surface $x(s,v)$ with a harmonic Gauss map.*

Proof. From (3.4), we have $T(s)$ and $N(s)$, which are nonzero unit orthonormal vectors. We also know that the sum of two orthonormal vectors cannot be zero. This implies $\Delta G \neq 0$. □

Theorem 3.3. *Every developable TN-Smarandache ruled surface $x(s,v)$ satisfies the relation $\Delta\, x = G$ and has a non-zero constant mean curvature.*

Proof. Using (2.5), the partial derivatives of $x(s,v)$ are as follows:

$$\begin{aligned} &x_s = \frac{k}{\sqrt{2}}\Big(N(s) - T(s)\Big), \quad x_v = B(s), \quad x_{vv} = 0, \\ &x_{ss} = \frac{-1}{\sqrt{2}}\Big(k' + k^2\Big)\,T(s) + \frac{1}{\sqrt{2}}\Big(k' - k^2\Big)\,N(s), \end{aligned} \tag{3.5}$$

Using (2.5), (3.3), and (3.5), the Laplacian operator Δ of the developable TN-Smarandache ruled surface $x(s,v)$ is computed as

$$\Delta x = \frac{1}{\sqrt{2}}\Big(T(s) + N(s)\Big). \tag{3.6}$$

From (3.1) and (3.6), we obtain

$$\Delta\, x = G. \tag{3.7}$$

Using the relations (2.10), (2.11), and (3.7), we find that the mean curvature H of the developable TN-Smarandache ruled surface $x(s,v)$ is

$$H = -\frac{1}{2}, \tag{3.8}$$

which completes the proof of the theorem. □

4. Developable TB-Smarandache ruled surfaces with pointwise 1-type Gauss map in $\mathbb{E}^3$

In this section, we study the developable TB-Smarandache ruled surface with a harmonic Gauss map and a pointwise 1-type Gauss map of the first kind. The TB-Smarandache ruled surface as given by (2.6) is always developable [16].

Using (2.9), the Gauss map G of a developable TB-Smarandache ruled surface is

$$G = \frac{\tau T(s) + kB(s)}{\sqrt{k^2+\tau^2}}. \tag{4.1}$$

Then, we have

$$\begin{aligned} G_s &= \frac{g\,(k\,T - \tau\,B)}{\sqrt{k^2+\tau^2}}, \quad G_v = G_{sv} = G_{vv} = 0, \\ G_{ss} &= \frac{g'\,(k\,T - \tau\,B)}{\sqrt{k^2+\tau^2}} + g\left(\sqrt{k^2+\tau^2}\,N - \frac{g\,(\tau\,T + k\,B)}{\sqrt{k^2+\tau^2}}\right), \end{aligned} \tag{4.2}$$

where $g = \frac{k^2}{(k^2+\tau^2)}\left(\frac{\tau}{k}\right)'$.

Also, we have

$$\begin{aligned} g_{11} &= \langle y_s,\, y_s\rangle = \frac{(k-\tau)^2}{2} + v^2(k^2+\tau^2), \quad g_{12} = \langle y_s,\, y_v\rangle = \frac{k-\tau}{\sqrt{2}}, \\ g_{21} &= \langle y_v,\, y_s\rangle = \frac{k-\tau}{\sqrt{2}}, \quad g_{22} = \langle y_v,\, y_v\rangle = 1, \quad g = v^2(k^2+\tau^2). \end{aligned} \tag{4.3}$$

Using (2.8) and (4.3), the Laplacian operator Δ on the developable TB-Smarandache ruled surface $y(s,v)$ is found as

$$\Delta = -\psi\Big[\Big(\psi_s - \Big(\frac{k-\tau}{\sqrt{2}}\Big)\psi_v\Big)\frac{\partial}{\partial s} + a_2\frac{\partial}{\partial v} - \sqrt{2}\,\psi\,(k-\tau)\frac{\partial^2}{\partial s\partial v} + \psi\,\frac{\partial^2}{\partial s^2} + a_3\frac{\partial^2}{\partial v^2}\Big], \tag{4.4}$$

where

$$\psi = \frac{1}{\sqrt{v^2(k^2+\tau^2)}},$$
$$a_2 = \frac{4\,\psi\,v\,(k^2+\tau^2)}{2} - \frac{(k-\tau)\,\psi_s}{\sqrt{2}} - \frac{\psi\,(k'-\tau')}{\sqrt{2}} + \psi_v\Big(\frac{(k-\tau)^2}{2} + v^2\,(k^2+\tau^2)\Big),$$
$$a_3 = \frac{\psi}{2}\Big((k-\tau)^2 + 2v^2(k^2+\tau^2)\Big).$$

Using (4.1), (4.2), and (4.4), we find

$$\Delta G = -\frac{\psi^2}{\sqrt{k^2+\tau^2}}\Big[(-a_4\,k - g^2\,\tau)\,T + k^2\left(\frac{\tau}{k}\right)'N + (a_4\tau - g^2k)\,B\Big], \tag{4.5}$$

where

$$a_4 = -g' + g\left(\frac{kk'+\tau\tau'}{k^2+\tau^2} - \frac{(k-\tau)}{\sqrt{2}\,v}\right).$$

Now, we have:

Theorem 4.1. *Let $\alpha(s)$ be a unit-speed space curve with the Frenet frame (T,N,B). Then, every developable TB-Smarandache ruled surface satisfying* (1.4) *of the first kind must have a harmonic Gauss map if and only if the curve $\alpha(s)$ is a helix.*

Proof. Let $y(s,v)$ be a developable TB-Smarandache ruled surface satisfying (1.4) of the first kind, i.e., $\Delta G = f(G+C)$, and

$$C = 0. \tag{4.6}$$

From (1.4) and (4.5), the following holds on $y(s,v)$:

$$a_5 = \langle C,\, T\rangle = \frac{\psi^2(a_4\,k + g^2\tau)}{f\sqrt{k^2+\tau^2}} - \frac{\tau}{\sqrt{k^2+\tau^2}}, \tag{4.7}$$

$$a_6 = \langle C,\, N\rangle = \frac{-\psi^2}{f\sqrt{k^2+\tau^2}}\left(\frac{\tau}{k}\right)', \tag{4.8}$$

$$a_7 = \langle C,\, B\rangle = \frac{-\psi^2(a_4\,\tau - g^2\,k)}{f\sqrt{k^2+\tau^2}} - \frac{k}{\sqrt{k^2+\tau^2}}. \tag{4.9}$$

Using (4.7), (4.8) and (4.9), we can express C as

$$C = a_5 T + a_6 N + a_7 B. \tag{4.10}$$

Using (4.6) and (4.10), we obtain

$$a_5 = a_6 = a_7 = 0. \tag{4.11}$$

And $a_6 = 0$ gives

$$\left(\frac{\tau}{k}\right)' = 0. \tag{4.12}$$

Using (4.7), (4.8), (4.9) and (4.12) in (4.11), we obtain

$$a_4 = 0, \quad f = 0. \tag{4.13}$$

From (4.12) and (4.13), we obtain that the developable $TB-$Smarandache ruled surface that satisfies (1.4) of the first kind has a harmonic Gauss map with a curve $\alpha(s)$ as a helix.

Conversely, if the space curve $\alpha(s)$ is a helix, then by the property of a helix, we have

$$\left(\frac{\tau}{k}\right)' = 0. \tag{4.14}$$

Using (4.14) in (4.5), we get $\Delta G = 0$. Hence, the proof is complete. □

Theorem 4.2. *Let $\alpha(s)$ be a unit-speed plane curve with nonzero curvature and with Frenet frame (T, N, B). Then, every developable TB-Smarandache ruled surface $y(s, v)$ has a harmonic Gauss map.*

Proof. Let $\alpha(s)$ be a plane curve with $k \neq 0$. Since the curve lies on the osculating plane, we have binormal B, which is a constant.

As B is a constant, we have $\frac{dB}{ds} = 0$ which implies $\tau = |\frac{dB}{ds}| = 0$ at all points of the curve. Putting $\tau = 0$ in (4.1), we obtain

$$G = B. \tag{4.15}$$

Using (4.15) in (4.5), we obtain $\Delta G = 0$. □

Now, using theorems 4.1 and 4.2, we have

Corollary 4.1. *Let $y(s, v)$ be the developable TB-Smarandache ruled surface. Then, the following are equivalent:*

(i) the curve $\alpha(s)$ is either a helix or a plane curve

(ii) $y(s, v)$ has a harmonic Gauss map

(iii) $y(s, v)$ is a harmonic surface

(iv) $y(s, v)$ is minimal.

Proof. We have two cases:

Case 1. Let the curve $\alpha(s)$ be a helix. Then, using Theorem 4.1, we obtain (ii).

Using (2.6), we find the partial derivative of $y(s,v)$ as follows:

$$y_s = -v\,k\,T(s) + \Big(\frac{k-\tau}{\sqrt{2}}\Big)\,N(s) + v\,\tau\,B(s),\ y_v = N(s),\ y_{vv} = 0,$$

$$y_{sv} = -k\,T(s) + \tau\,B(s), \quad y_{ss} = \Big(\frac{-k^2}{\sqrt{2}} + \frac{k\,\tau}{\sqrt{2}} - v\,k'\Big)\,T(s) + \Big(\frac{k'}{\sqrt{2}} - \frac{\tau'}{\sqrt{2}} - v\,(k^2+\tau^2)\Big)\,N(s) + \Big(\frac{k\,\tau}{\sqrt{2}} - \frac{\tau^2}{\sqrt{2}} + \tau'\,v\Big)\,B(s). \tag{4.16}$$

Using (4.4) and (4.16), the Laplacian of $y(s,v)$ is

$$\Delta y = \frac{-k^2}{v(k^2+\tau^2)^2}\Big(\frac{\tau}{k}\Big)'(\tau\,T + k\,B). \tag{4.17}$$

Using (4.14) in (4.17), we obtain (iii), i.e.,

$$\Delta y = 0. \tag{4.18}$$

From (4.18), we obtain (iv).

Case 2. Let $\alpha(s)$ be a plane curve. Then, using Theorem 4.2, we obtain (ii). Using $\tau = 0$ in (4.16), we obtain

$$y_s = -v\,k\,T(s) + \frac{k}{\sqrt{2}}\,N(s), \quad y_v = N(s), \quad y_{sv} = -k\,T(s), \quad y_{vv} = 0, \quad y_{ss} = \Big(\frac{-k^2}{\sqrt{2}} - v\,k'\Big)\,T(s) + \Big(\frac{k'}{\sqrt{2}} - v\,k^2\Big)\,N(s). \tag{4.19}$$

Now, using $\tau = 0$ in (4.4), we find

$$\Delta = -\frac{1}{v\,k}\Big[\Big(\frac{-k'}{v\,k^2} + \frac{1}{\sqrt{2}\,v^2}\Big)\frac{\partial}{\partial s} + k\Big(1 - \frac{1}{2\,v^2}\Big)\frac{\partial}{\partial v} - \frac{\sqrt{2}}{v}\frac{\partial^2}{\partial s\partial v} + \frac{1}{v\,k}\frac{\partial^2}{\partial s^2} + \frac{k\,(1+2\,v^2)}{2v}\frac{\partial^2}{\partial v^2}\Big]. \tag{4.20}$$

Using (4.19) and (4.20), the Laplacian of $y(s,v)$ is

$$\Delta y = 0. \tag{4.21}$$

From (4.21), we obtain (iii). Then, from (iii), we obtain (iv).

Hence, the proof is complete. □

Theorem 4.3. *Every developable TB-Smarandache ruled surface satisfies the relation $\Delta\, y = -g\,\psi\, G$, and has mean curvature $H = \frac{g\,\psi}{2}$.*

Proof. Using (2.6) and (4.4), we obtain the Laplacian operator Δ of the developable TB-Smarandache ruled surface $y(s, v)$ as

$$\Delta\, y = -g\,\psi \frac{(\tau\, T + k\, B)}{\sqrt{k^2 + \tau^2}}. \tag{4.22}$$

Using (4.1) and (4.22), we obtain

$$\Delta\, y = -g\,\psi\, G. \tag{4.23}$$

Using the relations (2.10), (2.11), and (4.23), we find that the mean curvature of the developable TB-Smarandache ruled surface $y(s, v)$ is $H = \frac{g\,\psi}{2}$.

Thus, the proof is complete. □

5. Developable NB-Smarandache ruled surface with pointwise 1-type Gauss map in $\mathbb{E}^3$

In this section, we study the developable NB-Smarandache ruled surface with a harmonic Gauss map and a pointwise 1-type Gauss map of the first kind. The NB-Smarandache ruled surface, as given by (2.7) is developable if and only if $\alpha(s)$ is a plane curve, i.e., $\tau = 0$ [16].

Using (2.9), the Gauss map of the developable NB-Smarandache ruled surface $z(s, v)$ is computed as

$$G(s, v) = -B(s). \tag{5.1}$$

Also, we have

$$\begin{aligned} g_{11} &= \langle z_s,\, z_s\rangle = \frac{k^2}{2} + (vk)^2, \quad g_{12} = \langle z_s,\, z_v\rangle = \frac{-k}{\sqrt{2}}, \\ g_{21} &= \langle z_v,\, z_s\rangle = \frac{-k}{\sqrt{2}}, g_{22} = \langle z_v,\, z_v\rangle = 1, \quad g = (vk)^2. \end{aligned} \tag{5.2}$$

Using (2.8) and (5.2), the Laplacian operator Δ on the NB-Smarandache ruled surface $z(s, v)$ is

$$\begin{aligned} \Delta = \frac{-1}{vk}\Big[\Big(\frac{-k'}{vk^2} - \frac{1}{\sqrt{2}v^2}\Big)\frac{\partial}{\partial s} + k\Big(1 - \frac{1}{2v^2}\Big)\frac{\partial}{\partial v} + \frac{\sqrt{2}}{v}\frac{\partial^2}{\partial v \partial s} + \\ \frac{1}{vk}\frac{\partial^2}{\partial s^2} + k\Big(\frac{1}{2v} + v\Big)\frac{\partial^2}{\partial v^2}\Big]. \end{aligned} \tag{5.3}$$

Now, we have

Theorem 5.1. *Let $z(s,v)$ be the developable NB-Smarandache ruled surface. Then we have the following:*

(i) $z(s,v)$ has a harmonic Gauss map

(ii) $z(s,v)$ is a harmonic surface

(iii) $z(s,v)$ is minimal.

Proof. Using (5.1), we have

$$G_s = G_v = G_{sv} = G_{ss} = G_{vv} = 0. \tag{5.4}$$

Using (5.3) and (5.4), we find that the Laplacian operator Δ of the Gauss map of the developable $NB-$Smarandache ruled surface is

$$\Delta G = 0. \tag{5.5}$$

By taking partial derivatives of (2.7) with respect to s and v, we obtain

$$z_s = k\Big(\frac{-T(s)}{\sqrt{2}} + v\,N(s)\Big), \quad z_v = T(s), \quad z_{sv} = k\,N(s), \tag{5.6}$$

$$z_{vv} = 0, \quad z_{ss} = -\Big(\frac{k'}{\sqrt{2}} + v\,k^2\Big)\,T(s) + \Big(\frac{-k^2}{\sqrt{2}} + v\,k'\Big)\,N(s).$$

Using (5.3) and (5.6), the Laplacian operator Δ of the developable NB-Smarandache ruled surface $z(s,v)$ is

$$\Delta z = 0. \tag{5.7}$$

Using (5.5) and (5.7), the developable NB-Smarandache ruled surface is harmonic, minimal, and has a harmonic Gauss map. □

6. Conclusions

On the basis of equation (1.4), we have drawn the following conclusions about developable Smarandache ruled surface in $\mathbb{E}^3$ as follows:

1. Every developable TN-Smarandache ruled surface and NB-Smarandache ruled surface satisfy (1.4) of the first kind with $f = 1$ and $f = 0$, respectively.

2. The developable TN-Smarandache ruled surface has constant mean curvature and does not admit a harmonic Gauss map.

3. The developable NB-Smarandache ruled surface is always plane, minimal, and has a harmonic Gauss map.

4. The developable TB-Smarandache ruled surface satisfying (1.4) of the first kind has a harmonic Gauss map and is minimal if and only if the curve $\alpha(s)$ is either a plane curve with non-zero curvature or a helix.

7. Examples

Example 7.1. Let $\alpha_1(s)$ be a regular unit-speed plane curve given by

$$\alpha_1(s) = \Big(a(s)\,\cos\ln[a(s)],\, a(s)\,\sin\ln[a(s)],\, 0\Big),$$

where $a(s) = \left(\frac{s}{\sqrt{2}} + 1\right)$ and has curvature $k(s) = \frac{1}{\sqrt{2}\,a(s)}$.

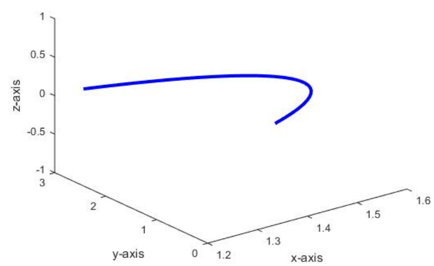

FIG. 7.1: Curve $\alpha_1(s)$ with $s \in [\pi/4,\, \pi]$.

The Frenet-Serret frame of $\alpha_1(s)$ are as follows:

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{2}}\Big(-\sin\ln[a(s)] + \cos\ln[a(s)],\, \cos\ln[a(s)] + \sin\ln[a(s)],\, 0\Big),\\ N(s) &= \frac{1}{\sqrt{2}}\Big(-\cos\ln[a(s)] - \sin\ln[a(s)],\, -\sin\ln[a(s)] + \cos\ln[a(s)],\, 0\Big),\\ B(s) &= (0,\, 0,\, 1), \end{aligned} \tag{7.1}$$

1. The TN-Smarandache ruled surface according to (7.1) is

$$x(s,v) = (-\sin\ln[a(s)],\, \cos\ln[a(s)],\, v).$$

Now, we have

$$x_s = \frac{1}{\sqrt{2}\,a(s)}\,(-\cos\ln[a(s)],\, -\sin\ln[a(s)],\, 0), \quad x_v = (0,\, 0\, 1). \tag{7.2}$$

Putting (7.2) in (2.9), the Gauss map G of $x(s,v)$ is computed as

$$G(s,\, v) = \Big(-\sin\ln[a(s)],\, \cos\ln[a(s)],\, 0\Big). \tag{7.3}$$

In addition, from (3.2), we obtain

$$g_{11} = \frac{1}{2\,a(s)^2}, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1, \quad g = \frac{1}{2\,a(s)^2}. \tag{7.4}$$

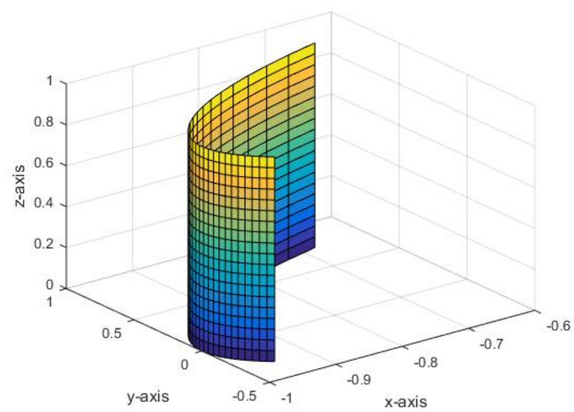

FIG. 7.2: TN-Smarandache ruled surface $x(s,v)$ with $s \in [\pi/2, 5\pi/2], v \in [0,1]$.

Using (2.8), the Laplacian operator on $x(s,\, v)$ is

$$\Delta = -\sqrt{2}\, a(s) \left[\frac{\partial}{\partial s} + \sqrt{2}\, a(s)\, \frac{\partial^2}{\partial s^2} \right]. \tag{7.5}$$

Using (7.3) and (7.5), we obtain

$$\Delta\, G = \Big(-\sin \ln[a(s)],\, \cos \ln[a(s)],\, 0 \Big). \tag{7.6}$$

From (7.3) and (7.6), we obtain

$$\Delta\, G = G. \tag{7.7}$$

Thus, $x(s,\, v)$ satisfies (1.4) of the first kind.

2. The NB-Smarandache ruled surface according to (7.1) is

$$z(s,v) = \frac{1}{\sqrt{2}} \Bigg(\cos \ln[a(s)] \left(\frac{-1}{\sqrt{2}} + v \right) - \sin \ln[a(s)] \left(\frac{1}{\sqrt{2}} + v \right),\, \sin \ln[a(s)] \left(\frac{-1}{\sqrt{2}} + v \right) \\ + \cos \ln[a(s)] \left(\frac{1}{\sqrt{2}} + v \right),\, 1 \Bigg).$$

Now, we have

$$\begin{aligned} z_s &= \frac{1}{2\, a(s)} \Bigg(-\sin \ln[a(s)] \left(\frac{-1}{\sqrt{2}} + v \right) - \cos \ln[a(s)] \left(\frac{1}{\sqrt{2}} + v \right), \\ &\qquad \cos \ln[a(s)] \left(\frac{-1}{\sqrt{2}} + v \right) - \sin \ln[a(s)] \left(\frac{1}{\sqrt{2}} + v \right),\, 0 \Bigg), \\ z_v &= \frac{1}{\sqrt{2}} (\cos \ln[a(s)] - \sin \ln[a(s)],\, \sin \ln[a(s)] + \cos \ln[a(s)],\, 0). \end{aligned} \tag{7.8}$$

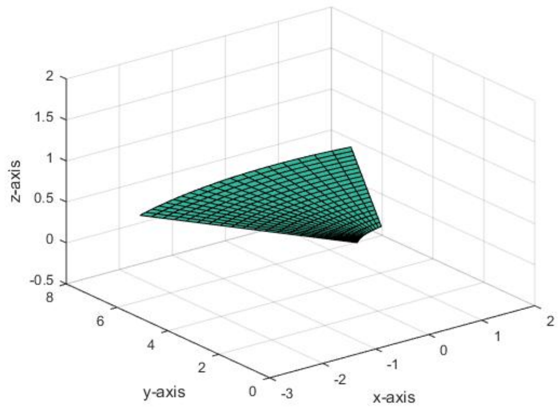

FIG. 7.3: NB-Smarandache ruled surface $x(s,v)$ with $s \in [\pi/4, \pi]$, $v \in [1, 6]$.

Using (7.8) in (2.9), we find that the Gauss map of $z(s,v)$ as

$$G = -(0,\, 0,\, 1). \tag{7.9}$$

Since G is constant, we get

$$\Delta G = 0. \tag{7.10}$$

Thus, $z(s,v)$ has a harmonic Gauss map.

3. The TB-Smarandache ruled surface according to (7.1) is given by

$$y(s,v) = \frac{1}{\sqrt{2}}\Bigg(-\sin\ln[a(s)]\bigg(\frac{1}{\sqrt{2}}+v\bigg) + \cos\ln[a(s)]\bigg(\frac{1}{\sqrt{2}}-v\bigg), \sin\ln[a(s)]\bigg(\frac{1}{\sqrt{2}}-v\bigg) + \cos\ln[a(s)]\bigg(\frac{1}{\sqrt{2}}+v\bigg),\, 1\Bigg).$$

Now, we have

$$\begin{aligned} y_s &= \frac{1}{2\,a(s)}\Bigg(-\cos\ln[a(s)]\left(\frac{1}{\sqrt{2}}+v\right) - \sin\ln[a(s)]\left(\frac{1}{\sqrt{2}}-v\right), \\ &\qquad \cos\ln[a(s)]\left(\frac{1}{\sqrt{2}}-v\right) - \sin\ln[a(s)]\left(\frac{1}{\sqrt{2}}+v\right),\, 0\Bigg), \\ y_v &= \frac{1}{\sqrt{2}}(-\sin\ln[a(s)] - \cos\ln[a(s)],\, -\sin\ln[a(s)] + \cos\ln[a(s)],\, 0). \end{aligned} \tag{7.11}$$

Using (7.11) in (2.9), the Gauss map G of $y(s,v)$ is computed as

$$G = (0,\, 0,\, 1). \tag{7.12}$$

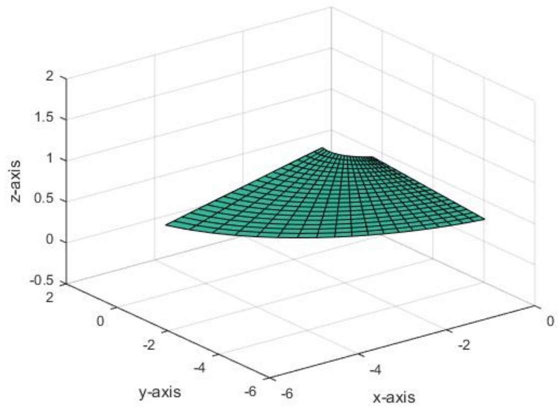

FIG. 7.4: TB-Smarandache ruled surface $x(s,v)$ with $s \in [\pi, 4\pi]$, $v \in [1,6]$.

Since G is constant, we get

$$\Delta G = 0. \tag{7.13}$$

Thus, $y(s,v)$ has a harmonic Gauss map.

Example 7.2. Let $\alpha_2(s)$ be a regular unit-speed helix given by

$$\alpha_2(s) = \frac{1}{\sqrt{2}}\left(\frac{\sin(\sqrt{2}s)}{\sqrt{2}},\, \frac{-\cos(\sqrt{2}s)}{\sqrt{2}},\, s\right).$$

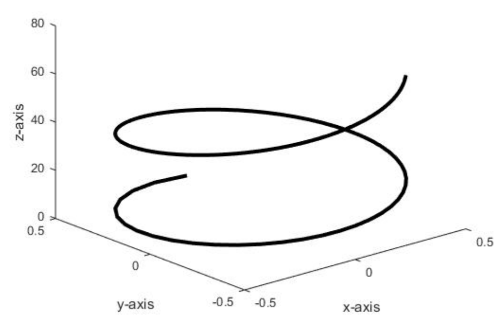

FIG. 7.5: Curve $\alpha_2(s)$ with $s \in [\sqrt{2}\,\pi, 20\,\sqrt{2}\,\pi]$.

The Frenet-Serret frame of the curve $\alpha_2(s)$ is given by

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{2}}\Big(\cos(\sqrt{2}s),\, \sin(\sqrt{2}s),\, 1\Big), \\ N(s) &= \Big(-\sin(\sqrt{2}s),\, \cos(\sqrt{2}s), 0)\Big), \\ B(s) &= \frac{1}{\sqrt{2}}\Big(-\cos(\sqrt{2}s),\, -\sin(\sqrt{2}s),\, 1\Big). \end{aligned} \tag{7.14}$$

The TB-Smarandache ruled surface $y(s,v)$ according to (7.14) is given by

$$y(s,v) = (-v\,\sin(\sqrt{2}s),\, v\,\cos(\sqrt{2}s),\, 1).$$

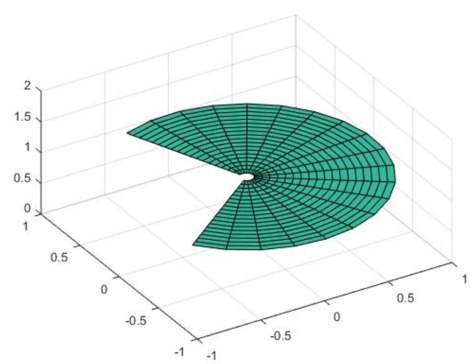

FIG. 7.6: TB-Smarandache ruled surface with $s \in [\pi, 3\pi]$, $v \in [0,1]$.

Using (2.9), its Gauss map is computed as

$$G = (0,\, 0,\, 1). \tag{7.15}$$

Since G is constant, we get

$$\Delta\, G = 0. \tag{7.16}$$

This implies that the TB-Smarandache ruled surface has a harmonic Gauss map with a curve $\alpha(s)$ helix.

Remark 7.1. We now show that a non-helical space curve does not admit (1.4) of the first kind. The non-helical space curve in $\mathbb{E}^3$ given by

$$\alpha_3(s) = \left(\frac{-2}{\sqrt{3}}\,\sin\sqrt{3}s\,\cos s + \sin s\,\cos\sqrt{3}s,\, \frac{2}{\sqrt{3}}\,\cos\sqrt{3}s\,\cos s + \sin s\,\sin\sqrt{3}s,\, -\frac{\sqrt{2}}{\sqrt{3}}\,\cos s\right),$$

with curvature $k = \sqrt{2}\,\cos s$ and torsion $\tau = \sqrt{2}\,\sin s$.

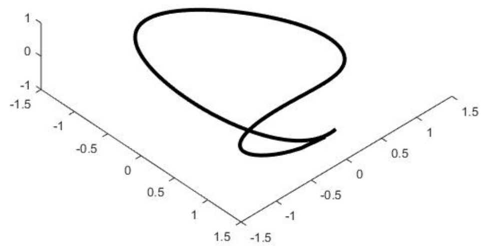

FIG. 7.7: Curve $\alpha_3(s)$ with $s \in [0, 4\pi]$.

The Frenet-Serret frame of $\alpha_3(s)$ is given by

$$
\begin{aligned}
T &= \frac{1}{\sqrt{3}}\Big(-\sqrt{3}\cos s\, \cos\sqrt{3}s - \sin s\, \sin\sqrt{3}s, -\sqrt{3}\, \sin\sqrt{3}s\, \cos s \\
&\qquad + \sin s\, \cos\sqrt{3}s,\ \sqrt{2}\, \sin s\Big), \\
N &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\, \sin\sqrt{3}s, -\sqrt{2}\, \cos\sqrt{3}s, 1\right), \qquad (7.17)\\
B &= \frac{1}{\sqrt{3}}\Big(-\sin\sqrt{3}s\, \cos s + \sqrt{3}\, \sin s\, \cos\sqrt{3}s,\ \cos s\, \cos\sqrt{3}s + \\
&\qquad \sqrt{3}\, \sin s\, \sin\sqrt{3}s, \sqrt{2}\, \cos s\Big).
\end{aligned}
$$

The TB-Smarandache ruled surface according to (7.17) is

$$
\begin{aligned}
y(s,v) = \frac{1}{\sqrt{6}}\bigg(&\sqrt{3}\, A(s)\, \cos(\sqrt{3}s) - \sin(\sqrt{3}s)\,(\sin s + \cos s - 2v), \sqrt{3}\, A(s)\, \sin(\sqrt{3}s) \\
&+ \cos(\sqrt{3}s)\,(\sin s + \cos s - 2v),\ \sqrt{2}\,(\sin s + \cos s + v)\bigg),
\end{aligned}
$$

where $A(s) = -\cos s + \sin s$.

Now, we have

$$
\begin{aligned}
y_s &= \frac{1}{\sqrt{6}}\Big(-2\, \sin(\sqrt{3}s)\, A(s) + 2\sqrt{3}\, v\, \cos(\sqrt{3}s),\ 2\, \cos(\sqrt{3}s)\, A(s) \\
&\qquad +2\sqrt{3}\, v\, \sin(\sqrt{3}s),\ -\sqrt{2}\, A(s)\Big), \qquad (7.18)\\
y_v &= \frac{1}{\sqrt{6}}\left(2\, \sin(\sqrt{3}s),\ -2\, \cos(\sqrt{3}s), \sqrt{2}\right).
\end{aligned}
$$

Using (7.18) in (2.9), we obtain

$$
G = \frac{1}{\sqrt{3}}\bigg(\sin(\sqrt{3}s),\ -\cos(\sqrt{3}s),\ -\sqrt{2}\bigg). \qquad (7.19)
$$

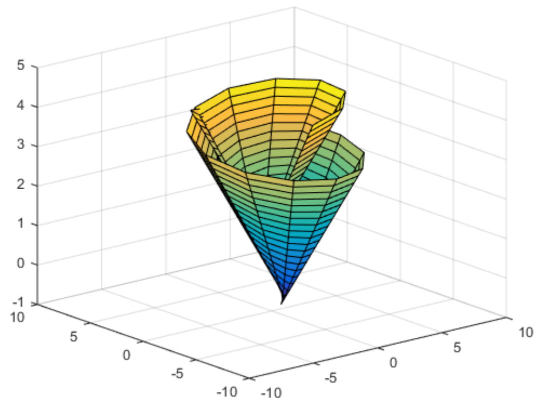

Fig. 7.8: TB-Smarandache ruled surface of Example 6.2 with $s \in [0, 2\pi]$ and $v \in [0, 2\pi]$.

From (4.3), we have

$$g_{11} = A(s)^2 + 2v^2, \quad g_{12} = -A(s), \quad g_{21} = -A(s), \quad g_{22} = 1. \tag{7.20}$$

Using (4.4), (7.19) and (7.20), we obtain

$$\Delta G = \frac{-1}{2\,v^2}\left(\frac{-A(s)\,\cos(\sqrt{3}s)}{v} - \sqrt{3}\,\sin(\sqrt{3}s),\ \frac{-A(s)\,\sin(\sqrt{3}s)}{v} + \sqrt{3}\,\cos(\sqrt{3}s),\ 0\right),$$

which can be written in the form of

$$\Delta G = \frac{1}{2\,v^2}\left[\left(\frac{A(s)\,\cos(\sqrt{3}s)}{v}, \frac{A(s)\,\sin(\sqrt{3}s)}{v}, \sqrt{6}\right) + 3\,G\right]. \tag{7.21}$$

From (7.21), we can say that the developable TB-Smarandache ruled surface $x(s, v)$ does not satisfy (1.4) since $\Delta G \neq f\,G$.

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