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DIFFERENTIAL OPERATORS OVER MODULES AND RINGS AS A PATH TO THE GENERALIZED DIFFERENTIAL GEOMETRY

Deepmala and Lakshmi N. Mishra

Abstract. The purpose of this paper is to give a short and understandable exposition on differential operators over modules and rings. The described methods allow for the use of algebra in differential geometry. Because this is a survey paper, the detailed proofs are omitted. On the other hand, various references are given for the interested reader. However, this paper should be understandable for quite a general audience, familiar with higher mathematics. The stress is put on the main idea, not on the computational details.
Keywords: Differential operators, differentiable manifold, commutative algebra, tensor product.

1. Introduction

The purpose of this paper is to give a short and understandable exposition on differential operators over modules and rings. The described methods allow for the use of algebra in differential geometry. As a result, the basic concepts of differential geometry can be expressed in the language of the commutative algebra.

It is a well–known result that infinitely differentiable manifold does not have to be defined and considered in terms of maps, charts and atlases, as it is done in the classical way. Instead, a suitable algebra can be studied. For example, if M is a manifold, then the algebraic approach focuses on studying the algebra of infinitely differentiable functions on M, i.e., $C^\infty(M)$. Of course, a certain methodology must be used such as, for example, inducing some topology, etc. For the classical approach, the interested reader can consult, for example, the book of Lee [18]. Being interested in physical motivations and examples, the classical position [1] is advised.

One of the interesting advantages of the algebraic approach is that if $C^\infty(M)$ algebra is considered, then the classical situation is described. However, the algebraic approach also works for more general spaces. Such spaces are, for example, the configuration space of a bar mechanism, borders of the gluing of different materials, sharp edges, etc. Indeed, there are various methods that try to deal with

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”singular spaces”, i.e., ones which contain points, in which the structure of a manifold breaks down. For example, the following [6, 9, 11, 19, 25, 26, 30, 31]. Some reviews of different concepts can be found in [3, 4, 7]. The methods presented in this paper are heavily based on books [21], [24] and [22]. Although we present the known results, we try to present them in a quite self-consistent and short way, in order to expose the idea of the generalization of differential geometry.

Because this is a survey paper, the detailed proofs are omitted. On the other hand, the readers are strongly advised to consult the given references for more thorough exposition. Yet, this paper should be understandable for quite a general audience, familiar with higher mathematics. The stress is put on the main idea, not on the computational details. For an introduction to commutative algebra the reader can consult books [2, 20].

2. Basic algebra

For the reader’s convenience basic algebraic notions are introduced [2, 20, 24]. A is called *an algebra*, if it is an additive, associative and commutative group with a multiplication such that $(a+b)\cdot c = a\cdot c + b\cdot c$ for every $a, b, c, \in A$. If it contains the unit $1 \in A$ and $1 \neq 0$, it is sometimes called *a ring*. *A field* is a commutative ring, in which non-zero elements make a multiplicative group. *An ideal* I is a subgroup of an additive group, such that if $a \in A$ and $b \in I$, then $ab \in I$. *A proper ideal* is an ideal I, such that $I \neq A$. Then, of course, $1 \notin I$. *A maximal ideal* is an ideal, which is not contained in any proper ideal. All invertible elements of A make the unique maximal ideal in a commutative ring. *A prime ideal* is an ideal I such that if $ab \in I$, then $a \in I$ or $b \in I$. If I is a prime ideal, then A/I does not have a zero divisors (i.e., elements a, b such that $ab = 0$ and $a \neq 0$ and $b \neq 0$). Moreover, if I is prime and maximal, then A/I is a field. P is called *a left A–module*, if P is an additive group with a multiplication $A \times P \to P$, such that $(ab)p = a(bp)$ for every $a, b \in A$ and $p \in P$ and $1p = p = p1$ for every $p \in P$. *A right A–module* is defined per analogy. A module over a field is called *a vector space*. An algebra which is a module over a ring $\mathbb{K}$ is called *a $\mathbb{K}$–algebra*.

Notice that every $\mathbb{K}$–algebra A can be extended to the algebra A' with a unit 1. A' is a direct sum of $\mathbb{K}$–modules $\mathbb{K}\bigoplus A$ with a multiplication $(\lambda_1, a_1)(\lambda_2, a_2) := (\lambda_1\lambda_2, \lambda_1 a_2 + \lambda_2 a_1 + a_1 a_2)$ where $\lambda_1, \lambda_2 \in \mathbb{K}$ and $a_1, a_2 \in A$. Elements of A' are denoted by $(\lambda, a) = \lambda 1 + a$, where $\lambda \in \mathbb{K}$ and $a \in A$. *A direct sum* of A–modules, $P_1 \bigoplus P_2$ is an additive group $P_1 \times P_2$ with a module structure $a(p_1 p_2) = (ap_1, ap_2)$ where $p_1 \in P_1, p_2 \in P_2, a \in A$. If $\{P_k\}_{k\in K}$ is a set of modules, then $\bigoplus P_k \ni (\dots, p_k, \dots)$ and $p_k \in \prod P_k = P_1 \times P_2 \times \dots \times P_k \times \dots$ where $p_k \neq 0$ for at most finite number of indices $k \in K$.

A tensor product $P \bigotimes Q$ of A–modules is an additive group generated by elements $p \otimes q$ where $p \in P, q \in Q$ with relations $(p+p')\otimes q = p\otimes q + p'\otimes q$, $p\otimes(q+q') = p\otimes q + p\otimes q'$, $pa\otimes q = p\otimes aq$ for $p \in P, q \in Q, a \in A$ and with an A–module structure $a(p\otimes q) = (ap)\otimes q = p\otimes(qa) = (p\otimes q)a$.

All A–linear morphisms from an A–module P to an A–module Q will be denoted by $\mathrm{Hom}_A(P, Q)$. $\mathrm{Hom}_A(P, Q)$ is an A–module itself. An A–module P is called *free*, if it has a basis, i.e., linear independent subset $I \subset P$ spanning P such that every element of P has a unique representation as a linear combination of elements from I with a finite number of non–zero coefficients from the algebra A. For example, every vector space.

P is called *projective*, if there exists a module Q such that $P \bigotimes Q$ is free. It is known that every projective module over a ring with a unique maximal ideal is free.

A composition of module morphisms $P \xrightarrow{i} Q \xrightarrow{j} T$ is called *exact in* Q, if $\ker j = \mathrm{im}\ i$. A composition of module morphisms $0 \to P \xrightarrow{i} Q \xrightarrow{j} T \to 0$ is called *a short exact sequence*, if it is exact in P, Q and T. Then, i is a monomorphism, $\ker j = \mathrm{im}\, i$ and j is an epimorphism and $T = Q/P$. It is known that, if $0 \to P \xrightarrow{i} Q \xrightarrow{j} T \to 0$ is a short exact sequence and R is an A–module, then $0 \to \mathrm{Hom}_A(T, R) \xrightarrow{j^*} \mathrm{Hom}_A(Q, R) \xrightarrow{i^*} \mathrm{Hom}_A(P, R)$ is exact in $\mathrm{Hom}_A(T, R)$ and in $\mathrm{Hom}_A(Q, R)$. Then, j^* is a monomorphism, but i^* might not be an epimorphism.

A directed set I is a pair $(I, <)$, where $<$ is a relation such that

1. $i < i\ \forall i \in I$,
2. $i < j, j < k \Rightarrow i < k$,
3. $\forall i, j \in I\ \exists k \in I$ such that $i < k$ and $j < k$.

It can happen that $i \neq j$ and $i < j$ and $j < i$ simultaneously.

A direct system is a family of modules over the given algebra, $\{P_i\}_{i\in I}$, where I is a directed set, such that for every $i, j \in I$, $i < j$ there exists a morphism $r^i_j : P_i \to P_j$ such that

1. $r^i_i = \mathrm{id}_{P_i}$,
2. $r^j_k \circ r^i_j = r^i_k$ for $i < j < k$.

A direct limit is understood as (P_∞, r^i_∞) where $r^i_\infty : P_i \to P_\infty$ and $r^i_\infty = r^j_\infty \circ r^i_j$ for all $i < j$. P_∞ consists of elements from $\bigoplus_I P_i$ modulo the relation identifying elements from P_i with their images in P_j for every $i < j$. For example, the direct sequence $P_0 \to P_1 \to \cdots \to P_i \xrightarrow{r^i_{i+1}} \dots$, where $I = \mathbb{N}$.

It is known that, if $\{P_i\}$ and $\{P_i\}$ are direct systems over the same algebra and are indexed by the same I and P_∞ and Q_∞ are their direct limits respectively, then direct limits of direct systems $\{P_i \bigoplus Q_i\}$ and $\{P_i \bigotimes Q_i\}$ are $P_\infty \bigoplus Q_\infty$ and $P_\infty \bigotimes Q_\infty$ respectively.

A projective limit is defined per analogy to a direct limit. In other words, it is understood as (P^∞, π_i^∞) where P^∞ is a module and π_i^∞ are morphisms such that $\pi_i^\infty : P^\infty \to P_i$ and $\pi_i^\infty = \pi_i^j \circ \pi_j^\infty$ for all $i < j$. They are elements from $\prod P_i$, i.e., $(\dots, p^i, \dots)$ such that $p^i \in P_i$ and $p^i = \pi_i^j(p^j)$ for every $i < j$. For example, if $\{P_i\}$ is a direct sequence of modules and Q is a module, then $\{\mathrm{Hom}(P_i, Q)\}$ make a projective system, whose projective limit is isomorphic to $\mathrm{Hom}(P_\infty, Q)$.

3. Differential operators

Let $\mathbb{K}$ be a field. Let A be a $\mathbb{K}$–algebra. Let P and Q be A–modules. Let $\mathrm{Hom}_{\mathbb{K}}(P, Q) := \{h : P \to Q \mid h - \text{homomorphism}\}$. It is a $\mathbb{K}$–module and it can be equipped with an A–module structure. It can be done with a help of *the left multiplication*, i.e., defining $(ah)(p) := ah(p)$; or with a help of *the right multiplication*, i.e., defining $(a^+h)(p) := h(ap)$, where $a \in A$ and $p \in P$.

Further, the following notation will be used. Let $\delta_a h := a^+h - ah$, where $a \in A$. In other words, $\delta_a : \mathrm{Hom}_{\mathbb{K}}(P, Q) \to \mathrm{Hom}_{\mathbb{K}}(P, Q)$. Of course, then $\delta_a(h)(p) = (a^+h)(p) - (ah)(p) = h(ap) - ah(p)$, where $p \in P$.

Definition 3.1. The element $\Delta \in \mathrm{Hom}_{\mathbb{K}}(P, Q)$ is called *a differential operator of order s on P with values in Q*, if $(\delta_{a_0} \circ \dots \circ \delta_{a_s})(\Delta) = 0$ for an arbitrary $a_0, \dots, a_s \in A$. The collection of all differential operators of order s on P with values in Q will be denoted by $\mathrm{Diff}_s(P, Q)$.

The following lemma can be easily proved.

Lemma 3.1. *The following conditions hold:*

1. $\mathrm{Diff}_s(P, Q)$ *inherits the structure of both left and right multiplications.*
2. $\mathrm{Diff}_s(P, Q) \subset \mathrm{Diff}_{s+1}(P, Q)$.
3. $\mathrm{Diff}_0(P, Q) = \mathrm{Hom}_A(P, Q)$.
4. $\delta_{ab} = a\delta_b + b\delta_a$ *and* $\delta_a \circ \delta_b = \delta_b \circ \delta_a$ *for* $\Delta \in \mathrm{Diff}_1(P, Q)$.

Proof. Of the 1: Let $\Delta \in \mathrm{Diff}_s(P, Q)$, then $(a\Delta)(p) := a\Delta(p)$. Also, $\delta_b(a\Delta)(p) = (a\Delta)(bp) - b(a\Delta)(p) = a\Delta(bp) - b(a\Delta)(p) = a(\Delta(bp) - b\Delta(p)) = a(\delta_b\Delta)(p)$. As a result, $a\Delta \in \mathrm{Diff}_s(P, Q)$. Similarly, $\delta_b(a^+\Delta)(p) = (a^+\Delta)(bp) - b(a^+\Delta)(p) = \Delta(abp) - b\Delta(ap) = (a^+(\delta_b\Delta))p$.

Of the 2: Trivial.

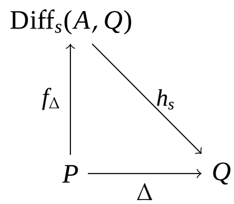
Of the 3: $\Delta \in \mathrm{Diff}_0(P, Q) \Rightarrow (\delta_a(\Delta))(p) = 0 \Rightarrow \Delta(ap) - a\Delta(p) = 0 \Rightarrow \Delta(ap) = a\Delta(p) \Rightarrow \Delta \in \mathrm{Hom}_A(P, Q)$.

Of the 4: $\Delta \in \mathrm{Diff}_1(P, Q) \Rightarrow [(\delta_a \circ \delta_b)(\Delta)](p) = 0$. But

$$[(\delta_a \circ \delta_b)(\Delta)](p) = \Delta(abp) - b\Delta(ap) - a\Delta(bp) + ab\Delta(p) \quad . \tag{3.1}$$

□

Consider the A–modules morphism $h_s : \mathrm{Diff}_s(A, Q) \to Q$, such that $h_s(\Delta) = \Delta(1)$. Then the below diagram commutes.



Above, $f_\Delta : P \to \mathrm{Diff}_s(A, Q)$ is a homomorphism, such that $(f_\Delta P)(a) = \Delta(ap)$, where $a \in A$. $\mathrm{Diff}_s(A, Q)$ should be understood as a module with the right multiplication. The mapping $\Delta \mapsto f_\Delta$ is an isomorphism, i.e., $\mathrm{Diff}_s(P, Q) = \mathrm{Hom}_A(P, \mathrm{Diff}_s(A, Q))$. In $\mathrm{Diff}_s(A, Q)$ there are both, right and left, multiplications.

Let $P = A$, $\Delta \in \mathrm{Diff}_0(A, Q)$. Then, Δ is uniquely determined by the element $\Delta(1)$. Then, there exists an isomorphism $\mathrm{Diff}_0(A, Q) = Q$, determined by $Q \ni q \mapsto \Delta_q \in \mathrm{Diff}_0(A, Q)$, where Δ_q is such that $\Delta_q(1) = q$.

Let $\Delta \in \mathrm{Diff}_1(A, Q)$. As a result of Eq. (3.1) it holds that

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ab\Delta(1) \quad , \tag{3.2}$$

where $a, b \in A$. (It has been substituted in Eq. (3.1) that $p = 1$.)

Definition 3.2. If $\Delta(1) = 0$, then the Leibniz rule holds and Δ is called *a derivation*.

Notice that an arbitrary 1–st order differential operator can be decomposed into the following sum $\Delta(a) = a\Delta(1) + (\Delta(a) - a\Delta(1))$, where the first summand belongs to $\mathrm{Diff}_0(A, Q)$ and the second is a derivation. Of course, the collection of derivations over A are an A–module. It is because if ∂ is a derivation, then $a\partial$ is also a derivation. The collection of all derivations over A with values in Q will be denoted by $\partial(A, Q)$.

Lemma 3.2. $\mathrm{Diff}_1(A, Q) = Q \bigoplus \partial(A, Q)$.

Lemma 3.3. *If $Q = A$, then ∂A is the Lie algebra over $\mathbb{K}$, i.e., $[u, u'] = u \circ u' - u' \circ u$ for arbitrary $u, u' \in \partial A$.*

Definition 3.3. $\mathrm{Diff}_s(A) := \mathrm{Diff}_s(A, A)$.

Lemma 3.4. $\mathrm{Diff}_1(A) = A \bigoplus \partial A$.

Lemma 3.5. *Let $\Delta \in \mathrm{Diff}_s(P, Q)$ and let $\nabla \in \mathrm{Diff}_r(Q, R)$. Then, it holds that*

1. $\nabla \circ \Delta \in \mathrm{Diff}_{s+r}(P, R)$.
2. $\delta_a(\nabla \circ \Delta) = \delta_a(\nabla) \circ \Delta + \nabla \circ \delta_a(\Delta)$.

For proofs of the above lemmas, see, for example, [24].

Definition 3.4. $\mathrm{Diff}(P,P) := \bigoplus_{i=1,2,3,...} \mathrm{Diff}_i(P,P)$.

$\mathrm{Diff}(P,P)$ should be understood as a direct limit in the following sense $\mathrm{End}_A P \subset \mathrm{Diff}_1(P,P) \subset \mathrm{Diff}_2(P,P) \subset \cdots \subset \mathrm{Diff}(P,P)$. $\mathrm{Diff}(P,P)$ is a $\mathbb{K}$–algebra, but not an A–algebra. It is also non–commutative.

Lemma 3.6. *If* $\Delta \in \mathrm{Diff}_l(P,Q)$, *then* $(\delta_{a_1} \circ \ldots \delta_{a_k})(\Delta) \in \mathrm{Diff}_{l-k}(P,Q)$.

For the proof of the above lemma, see, for example, [24].

Moreover, *the generalized Leibniz rule* holds, i.e., $(\delta_a \circ \delta_b)(\Delta) = \delta_a(\Delta) \circ b + a \circ \delta_b(\Delta)$.

Now, simple examples (see, [21]) of the above theory will be given. Suppose that $\mathbb{K} = \mathbb{R}$ and $A = C^\infty(M)$, where M is an infinitely differentiable manifold. Then, $\partial(A,\mathbb{R})$ are just tangent vectors in a point $x \in M$. Indeed, if $\mu_x := \{f \in C^\infty(M) \mid f(x) = 0\}$, then $C^\infty(M)/\mu_x = \mathbb{R}$.

$\partial(A,A)$ are just tangent vector fields over M. $\partial(A, A/\mu_N)$ are tangent vector fields over a submanifold $N \subset M$. $\mu_N := \{f \in C^\infty(M) \mid f(x)|_N = 0\}$.

Finally, if $\Delta \in \mathrm{Diff}_s(C^\infty(M))$ and $(x_1......x_n)$ is a local coordinate chart on $U \subset M$, then $\Delta|_U = \sum_{|\sigma|=0}^{s} \alpha_\sigma \frac{\partial^{|\sigma|}}{\partial x_\sigma}$, where $\alpha_\sigma = \alpha_\sigma(x_1....,x_n)$, $\sigma = (\sigma_1,\ldots,\sigma_n)$, $|\sigma| = \sigma_1 + \cdots + \sigma_n$ and $\partial x_\sigma = \partial x_1^{\sigma_1},\ldots,\partial x_n^{\sigma_n}$.

The algebraic definition of the differential operator (Definition 3.1) comes from Grothendieck [10]. The linkage of algebra and geometry in the above spirit is also present in the celebrated work of Swan [27]. The generalized derivations and their implications are discussed, for example, in [12, 24, 8].

4. Representations

Now, consider the tensor product $A \bigotimes_{\mathbb{K}} P$. Let $\delta^b(a \otimes p) := (ba) \otimes p - a \otimes (bp)$, where $p \in P$ and $a,b \in A$. By μ^{k+1} will be denoted a submodule of $A \bigotimes_{\mathbb{K}} P$ generated by elements $\delta^{b_0} \circ \cdots \circ \delta^{b_k}(a \otimes p)$.

Definition 4.1. $J^k(P) := (A \bigotimes_{\mathbb{K}} P)/\mu^{k+1}$ will be called *a module of k–jets.*

Elements of $J^k(P)$ will be denoted by $c \otimes_k p$. In particular, the module of 1–jets, $J^1(P)$, consists of elements $c \otimes_1 p$ modulo the relation $(\delta^a \circ \delta^b)(1 \otimes_1 p) = ab \otimes_1 p - a \otimes_1 (bp) - b \otimes_1 (ap) + 1 \otimes_1 (abp) = 0$.

$J^k(P)$ is a $\mathbb{K}$–module. However, it can be equipped with an A–module structure (with left and right multiplications), in the following way

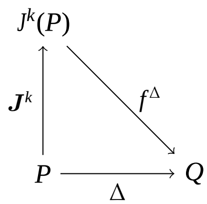
$$b(a \otimes_k p) := ba \otimes_k p \quad ,$$

$$b^+(a \otimes_k p) := a \otimes_k (bp) \quad .$$

There exists a morphism of modules $\boldsymbol{J}^k : P \to J^k(p)$, where $\boldsymbol{J}^k(p) = 1 \otimes_k p$. Notice, that P is an A–module and $J^k(p)$ is a right A–module. Now, $J^k(P)$ as a left A–module is generated by elements $\boldsymbol{J}^k(p)$, where $p \in P$.

If $r > s$, then there exists the canonical monomorphism $\mu^r \to \mu^s$. Therefore, there exists an epimorphism $\pi_i^{i+1} : J^{i+1}(P) \to J^i(P)$. In particular, $\pi_0^1 : J^1(P) \to P$, where $\pi_0^1(a \otimes_1 p) = ap$.

Lemma 4.1. *Let $\Delta \in \mathrm{Diff}_k(P, Q)$. Then, the below diagram commutes. f^Δ is a homomorphism.*



Proof. The sketch of the proof is the following. Let $\boldsymbol{J}^k(p) \in \mathrm{Diff}_k(P, J^k(p))$. For an arbitrary $f \in \mathrm{Hom}_A(A \bigotimes P, Q)$ it holds that $\delta_b(f \circ \boldsymbol{J})(p) = f(\delta^b(1 \otimes p))$, where $\boldsymbol{J} : P \to A \bigotimes P$ and $\boldsymbol{J}(p) = 1 \otimes p$. As a result, $\mathrm{Diff}_k(P, Q)$ is isomorphic to $\mathrm{Hom}_A(J^k(P), Q)$. The isomorphism is given by the mapping $\Delta \mapsto f^\Delta$. □

The immediate consequence is that in the language of the category theory $J^k(P)$ is a representation of the functor $\mathrm{Diff}_k(P, \cdot)$. This was noticed and interpreted by Vinogradov [29]. Further, on this basis a general, conceptual theory of partial differential equations was build (see, for example, [15, 17, 5]).

5. Differential operators – cont.

Let $\kappa^n := (1, 2, ..., n)$ and let $\kappa := (i_1, \ldots, i_l)$, where $l \le n$. Let $\overline{\kappa}$ denote the complement of κ in κ^n. Let $|\kappa| := l$ and let $a_\kappa := a_{i_1} \ldots a_{i_l}$. Let $\delta_{a_\kappa} := \delta_{a_{i_1}} \circ \delta_{a_{i_2}} \circ \cdots \circ \delta_{a_{i_l}}$.

It can be checked (see [21]) that

1. $\delta_{a_{\kappa^n}}(\Delta \circ \nabla) = \sum_{|\kappa| \le n} \delta_{a_\kappa}(\Delta) \circ \delta_{a_{\overline{\kappa}}}(\nabla)$,
2. $\delta_{a_{\kappa^n}}(\Delta)(b) = \sum_{|\kappa| \le n} (-1)^{|\kappa|} a_\kappa \Delta(a_{\overline{\kappa}} b)$,
3. $\Delta(a_{\kappa^n} b) = -\sum_{0 < |\kappa| \le n} (-1)^{|\kappa|} a_\kappa \Delta(a_{\overline{\kappa}} b)$.

Lemma 5.1. *If $\Delta \in \mathrm{Diff}_l A$ and $\nabla \in \mathrm{Diff}_k A$, then*

$$[\Delta, \nabla] = \Delta \circ \nabla - \nabla \circ \Delta \in \mathrm{Diff}_{k+l-1} A \quad . \tag{5.1}$$

Proof. The idea of the proof is based on the induction over $l + k$.

Suppose that $l + k = 0$. Then, $l = 0$ and $k = 0$, i.e., $\Delta \in \mathrm{Diff}_0 A = A$ and $\nabla \in \mathrm{Diff}_0 A = A$. In other words, $[\Delta, \nabla] = ab - ba = 0$.

Suppose that Eq. (5.1) holds for $l+k<n$. Then, $\delta_a(\Delta\circ\nabla-\nabla\circ\Delta)=\delta_a(\Delta)\circ\nabla+\Delta\circ\delta_a(\nabla)-\delta_a(\nabla)\circ\Delta-\nabla\circ\delta_a(\Delta)=[\delta_a(\Delta),\nabla]+[\Delta,\delta_a(\nabla)]$. Of course, $\delta_a(\Delta)$ is of $l-1$–th order, ∇ is of k–th order, Δ is of l–th order and $\delta_a(\nabla)$ is of $k-1$–th order. From the inductive step $[\Delta,\delta_a(\nabla)]$ is of the order $\leq k+l-2$. Therefore, $[\Delta,\nabla]$ is of the order $\leq l+k-1$. The proof finishes by induction. □

Definition 5.1. $S_k(A):=\mathrm{Diff}_kA/\mathrm{Diff}_{k-1}A$. Elements of $S_k(A)$ will be denoted by $\mathrm{smbl}_k\Delta$ and called *symbols*. Of course, $\Delta\in\mathrm{Diff}_kA$. $S_*(A):=\bigoplus_{i=0}^{\infty}S_i(A)$ will be called *the algebra of symbols*. The multiplication is defined in the following way $\mathrm{smbl}_l\Delta*\mathrm{smbl}_k\Delta:=\mathrm{smbl}_{k+l}(\Delta\circ\nabla)$.

Lemma 5.2. *The above definition does not depend on representing the object from S_k.*

Proof. If $\mathrm{smbl}_l\Delta=\mathrm{smbl}_l\Delta'$, then $\Delta-\Delta'\in\mathrm{Diff}_{l-1}A$ and, therefore, $(\Delta-\Delta')\circ\nabla\in\mathrm{Diff}_{l+k-1}A$. □

Lemma 5.3. *The algebra of symbols is commutative.*

Proof. It is a consequence of Eq. (5.1). □

Lemma 5.4. *The algebra of symbols is a Lie algebra.* $\{\mathrm{smbl}_l\Delta,\mathrm{smbl}_k\nabla\}:=\mathrm{smbl}_{k+l-1}[\Delta,\nabla]$.

Lemma 5.5. *$S_1(A)$ is isomorphic to $\partial(A)$. The isomorphism is given by the mapping* $\mathrm{smbl}_1\Delta\mapsto\Delta-\Delta(1)$.

For proofs of the above lemmas, see, for example, [21].

The above objects allow for reformulation in an algebraic way *a cotangent space* (see [13]).

Let $\mathrm{Spec}_{\mathbb{K}}A$ be the collection of all $\mathbb{K}$–homomorphisms from A to $\mathbb{K}$.

Definition 5.2. If $h\in\mathrm{Spec}_{\mathbb{K}}A$, then $T^*_hA:=h/h^2$ will be called *a cotangent space* and $T^*A:=\bigcup_{h\in\mathrm{Spec}_{\mathbb{K}}A}T^*_hA$ will be called *a cotangent bundle*.

Notice that in a classical case a cotangent space and a cotangent bundle can be defined in the below way.

Definition 5.3. Let M be an infinitely differentiable manifold. $T^*_xM:=\mathrm{Hom}_{\mathbb{R}}(T_xM,\mathbb{R})$ and $T^*M:=\bigcup_{x\in M}T^*_xM$.

Notice that T^*_xM is isomorphic to μ_x/μ_x^2.

Lemma 5.6. *It holds (see [13]) that*

1. *T^*M is isomorphic to* $\mathrm{Spec}_{\mathbb{R}}S_*(C^\infty(M))$,
2. *T^*A is isomorphic to* $\mathrm{Spec}_{\mathbb{K}}S_*(A)$.

Elements $h\in\mathrm{Spec}_{\mathbb{K}}A$ are in a sense generalization of the notion of a point (for physical and conceptual interpretations see, for example, [21], [23] and [8]).

6. Jets

Let $P = A$. Then, $J^k(A)$ is a commutative algebra. The multiplication (see [13]) is given by the following relations

$$a\boldsymbol{J}^k(b) * c\boldsymbol{J}^k(d) = ac\boldsymbol{J}^k(bd) \quad ,$$

$$1_{\boldsymbol{J}^k(A)} = \boldsymbol{J}^k(1_A) \quad .$$

Epimorphisms π_i^{i+1} constitute the sequence $P = J^0(P) \overset{\pi_0^1}{\leftarrow} J^1(P) \leftarrow \cdots \leftarrow J^k(P) \overset{\pi_k^{k+1}}{\leftarrow} J^{k+1}(P) \leftarrow \ldots$. This sequence is dual to the following imbeddings $\mathrm{Hom}_A(P,Q) = \mathrm{Diff}_0(P,Q) \to \mathrm{Diff}_1(P,Q) \to \cdots \to \mathrm{Diff}_k(P,Q) \to \mathrm{Diff}_{k+1}(P,Q) \to \ldots$.

As a result, it can be defined $J^\infty(P) := \varprojlim J^k(P)$, $\pi_i^\infty := J^\infty(P) \to J^i(P)$ and $\boldsymbol{J}^\infty := \varprojlim \boldsymbol{J}^k$, where $\boldsymbol{J}^\infty : P \to J^\infty(P)$.

$J^\infty(P)$ is a commutative algebra. The unit element is the following $1_{J^\infty(P)} = (1_P, \boldsymbol{J}^1(1_P), \boldsymbol{J}^2(1_P), \ldots)$.

Lemma 6.1. $\ker \pi_{i-1}^i = \langle \sum_\alpha a_\alpha(\delta^{a_\kappa}\boldsymbol{J}^k)(1)\rangle = \mu^{i-1}/\mu^i$. *In other words, elements* $\sum_\alpha a_\alpha(\delta^{a_\kappa}\boldsymbol{J}^k)(1)$ *generate* $\ker \pi_{i-1}^i$.

For the proof of the above lemma, see, for example, [24].

Definition 6.1. $C^k(P) := \ker_{k-1}^k$ and $C^*(P) := \bigoplus_{i=0}^\infty C^i(P)$. $C^*(P)$ will be called *the algebra of cosymbols*.

Lemma 6.2. *If $P = A$, then $C^*(P)$ is both a $\mathbb{K}$–algebra and an A–algebra. The multiplication is given by the following relations*

$$\theta_k \odot \theta_i := \sum_\gamma (\delta^{a_{\kappa_1}a_{\kappa_2}}\boldsymbol{J}^{k+l})(1) \in \ker \pi_{k+l-1}^{k+l} \quad ,$$

where

$$\theta_k = \sum_\alpha (\delta^{a_{\kappa_1}}\boldsymbol{J}^k)(1) \in \ker \pi_{k-1}^k \quad ,$$

$$\theta_l = \sum_\beta (\delta^{a_{\kappa_2}}\boldsymbol{J}^l)(1) \in \ker \pi_{l-1}^l \quad .$$

The proof of the above lemma is computational (see [21, 13]).

Definition 6.2. *A tangent vector* v in $h \in \mathrm{Spec}_{\mathbb{K}}(A)$ is understood as a $\mathbb{K}$–linear mapping $v : A \to \mathbb{K}$, for which the Leibniz rule holds, i.e., $v(ab) = h(a)v(b) + v(a)h(b)$ for arbitrary $a, b \in A$. The collection of all tangent vectors in h will be denoted by T_hA and called *a tangent space*. Let $TA := \bigcup_{h\in\mathrm{Spec}_{\mathbb{K}}A} T_hA$. It will be called *a tangent bundle*.

Notice that TA is a $\mathbb{K}$–module. Of course, if M is a manifold and $A = C^\infty(M)$, then the classical tangent space and the tangent bundle are obtained.

Lemma 6.3. *The following isomorphisms holds (see [21, 24, 13]):*

1. $\mathrm{Spec}_A C^*(A) = \partial(A)$,
2. $\mathrm{Spec}_A C^*(A) = \mathrm{Hom}_A(C^1(A), A)$,
3. $TA = \mathrm{Spec}_{\mathbb{K}} C^*(A)$,
4. $\mathrm{Hom}_A(C^i(A), A) = \mathrm{Diff}_i A/\mathrm{Diff}_{i-1} A$,
5. $\mathrm{Hom}_A(C^1(A), P) = \partial(A, P)$.

7. Conclusions

The concept of a differential operator over arbitrary algebra was presented along with its consequences. For example, the generalized concept of a jet was described. Besides various properties of described objects, tangent and cotangent bundles were defined in the language of algebra. It was explained that the tangent bundle TA is isomorphic to $\mathrm{Spec}_{\mathbb{K}} C^*(A)$ and that the cotangent bundle T^*A is isomorphic to $\mathrm{Spec}_{\mathbb{K}} S_*(A)$.

In addition to the already cited references, the interested reader should consult the short and concise expository article of Krasil'shchik [14]. Some extension of the discussed material is given, for example, in [28]. Finally, linear differential operators over commutative algebras are thoroughly discussed in the book [16], on which this paper also greatly relies on. This book is much more detailed and also contains a collection of exercises.

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Deepmala (Corresponding author)
SQC & OR Unit, Indian Statistical Institute,
203 B. T. Road, Kolkata, 700 108, India
Email: dmrai23@gmail.com, deepmaladm23@gmail.com

Lakshmi N. Mishra
Department of Mathematics, National Institute of Technology,
Silchar-788 010, Cachar (Assam), India
L. 1627 Awadh Puri Colony Beniganj, Phase-III,
Opposite-Industrial Training Institute (I.T.I.),
Ayodhya Main Road, Faizabad-224 001 (Uttar Pradesh), India
Email: lakshminarayanmishra04@gmail.com