FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 5 (2023), 869 - 893
https://doi.org/10.22190/FUMI230824056M
Original Scientific Paper

QUASI-PARA-SASAKIAN MANIFOLD ADMITTING ZAMKOVOY CONNECTION

Sandeep K. Mishra[1], Giteshwari Pandey[2], Shravan K. Pandey[1] and Ram N. Singh[1]

[1]**Department of Mathematical Sciences, A.P.S. University,Rewa, India**
[2]**Department of Mathematics, Govt. Tulsi College, Anuppur, India**

Abstract. The purpose of the present study is to deduce some curvature properties of quasi-para-Sasakian manifold equipped with respect to Zamkovoy connection. In the present article we have studied Locally ϕ-symmetric quasi-para-Sasakian manifold, ϕ-recurrent quasi-para-Sasakian manifold, Locally projective ϕ-symmetric quasi-para-Sasakian manifold, ϕ-projectively flat quasi-para-Sasakian manifold, pseudo-quasi-conformally flat quasi-para-Sasakian manifold, ϕ-pseudo-quasi-conformally flat quasi-para-Sasakian manifold with respect to Zamkovoy connection. Also we have shown that the quasi-para-Sasakian manifold with respect to Zamkovoy connection $\bar{\nabla}$ satisfying $\bar{\tilde{V}}(\xi, U).\bar{S} = 0$, where $\bar{\tilde{V}}$ and $\bar{S}$ are the pseudo-quasi-conformal curvature tensor and Ricci tensor with respect to Zamkovoy connection respectively.
Keywords: Quasi-Para-Sasakian manifold, Projective curvature tensor, Pseudo-quasi-conformal curvature tensor, Zamkovoy connection.

1. Introduction

In [13], Kaneyuki and Konzai defined the almost paracontact structure on pseudo-Riemannian manifold M^n of dimension $(2n+1)$ and constructed the almost paracontact structure on $M^{(2n+1)} \times \mathcal{R}$.

In 2008, S. Zamkovoy associated the almost paracontact structure [26] to a pseudo-Riemannian metric of signature $(n+1, n)$ and showed that any almost para-

Received August 24, 2023. accepted October 30, 2023.
Communicated by Uday Chand De
Corresponding Author: Shravan K. Pandey, Department of Mathematical Sciences, A.P.S. University,Rewa, India | E-mail: shravan.math@gmail.com
2010 *Mathematics Subject Classification.*

contact structure admits such a pseudo-Riemannian metric which is called compatible metric. He introduced the notion of Zamkovoy connection and it showed that its torsion is the obstruction of the paracontact manifold to be para-Sasakian. He defined a Zamkovoy connection on a paracontact metric manifold which seems to be the paracontact analogue of the (generalized) Tanaka-Webster connection [24].

In [2], A. M. Blaga introduced Zamkovoy connection on para Kenmotsu manifolds. This affine connection was further studied by[1]. For an n-dimensional almost contact metric manifold M^n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g the Zamkovoy connection is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{1.1}$$

A systematic study of almost paracontact metric manifolds was given in one of the Zamkovoy's papers [26]. Z. Olszak [16] studied normal almost contact metric manifolds of dimension 3. He obtained certain necessary and sufficient conditions for an almost contact metric structure to be normal and curvature properties of such structures were studied. Normal almost paracontact metric manifolds were studied by many other authors ([10], [11]).

The notion of quasi-Sasakian manifolds was introduced by D. E. Blair [3]. A quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\Omega := g(., \phi.)$ is closed. Quasi-Sasakian manifolds can be viewed as an odd-dimensional counterpart of Kaehler structures. These manifolds have been studied by many authors ([12], [15], [23]). Although quasi-Sasakian manifolds were studied by several different authors and are considered a well-established topic in contact Riemannian geometry to the authors knowledge, there do not exists any comprehensive study about quasi-para-Sasakian manifold.

The authors in [25] introduced and studied the notion of pseudo-quasi-conformal curvature tensor $\tilde{V}$ on a Riemannian manifold of dimension $(n \geq 3)$ which includes the projective, quasi-conformal, Weyl-conformal and concircular curvature tensor as special cases. This tensor is defined as:

$$\begin{aligned}\tilde{V}(X,Y)Z =&(p+d)R(X,Y)Z + (q - \frac{d}{n-1})[S(Y,Z)X - S(X,Z)Y] \\ &+ q[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n(n-1)}\{p + 2(n-1)q\} \\ &[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.2}$$

where $X,\ Y,\ Z \in \chi(M)$ and $p,\ q,\ d$ are real constants such that $p^2 + q^2 + d^2 > 0$. In particular, if:
i) $p = q = 0,\ d = 1 \implies$ Projective curvature tensor [4],
ii) $p \neq 0,\ q \neq 0,\ d = 0 \implies$ Quasi-conformal curvature tensor [6],
iii) $p = 1,\ q = -\frac{1}{n-2},\ d = 0 \implies$ Conformal curvature tensor [5],
iv) $p = 1,\ d = q = 0 \implies$ Concircular curvature tensor [7].

In 2005, Shaikh and Jana [21] introduced and studied a tensor field, called pseudo-quasi-conformal curvature tensor $\tilde{V}$ on a Riemannian manifold of dimension greater than or equal to 3. Recently, pseudo-quasi-conformal curvature tensors has been studied by many authors in various kind such as Kundu [14], Prakasha et al. [17] and many others.

In a quasi-para-Sasakian manifold M^n of dimension $n \geq 3$, the pseudo-quasi-conformal curvature tensor $\bar{\tilde{V}}$ with respect to Zamkovoy connection $\bar{\nabla}$ is given by

$$\begin{aligned}\bar{\tilde{V}}(X,Y)Z =&(p+d)\bar{R}(X,Y)Z + (q - \frac{d}{n-1})[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] \\ &+ q[g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] - \frac{\bar{r}}{n(n-1)}\{p+2(n-1)q\} \\ &[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.3}$$

where $\bar{R}$, $\bar{S}$ *and* $\bar{Q}$ are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection $\bar{\nabla}$ respectively.

Motivated by these considerations, in this paper, we studied the projective curvature tensor and pseudo-quasi-conformal curvature tensor of quasi-para-Sasakian manifold admitting Zamkovoy connection and established some new results.

Definition 1.1. An n-dimensional quasi-para Sasakian manifold M^n is said to be η-Einstein manifold if the Ricci tensor S is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, for all $X,\ Y \in \chi(M)$, where a and b are scalars.

2. Preliminaries

An odd dimensional smooth manifold M^n (n=2m+1) has an almost paracontact structure (ϕ,ξ,η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi(\xi) = 0, \tag{2.1}$$

$$\eta \circ \phi = 0, \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

$$\phi^2 X = X - \eta(X)\xi. \tag{2.4}$$

Distribution $D : P \in M \to D_P \subseteq T_P M$; $D_P = Ker\eta$=$\{X \in T_P M : \eta(X) = 0\}$ is called paracontact distribution generated by η.

If a manifold M^n with (ϕ,ξ,η) structure admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.5}$$

then we say that M^n has an almost paracontact metric structure and g is called compatible. Any compatible metric g with the given almost paracontact structure g is necesserialy of signature $(n+1, n)$. Also if

$$\eta(X) = g(X, \xi) \tag{2.6}$$

and

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.7}$$

where,

$$d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta[X, Y]\}, \tag{2.8}$$

holds then η is paracontact form and the almost paracontact metric manifold (M, ϕ, η, ξ, g) is said to be paracontact metric manifold [9].

A paracontact metric manifold is para-Sasakian manifold [8] if and only if

$$(\nabla_X \phi)(Y) = -g(X, Y)\xi + \eta(Y)X, \tag{2.9}$$

for all vector fields X and Y.

If

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$

then the manifold (M, ϕ, η, ξ, g) is said to be a quasi-para-Sasakian manifold.

Also

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.11}$$

$$(\nabla_X \eta)(Y) = -g(X, \phi Y), \tag{2.12}$$

$$(\nabla_X \xi) = \phi X, \tag{2.13}$$

$$d\eta(X, Y) = -g(X, \phi Y), \tag{2.14}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.15}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.16}$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \tag{2.17}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.18}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X). \tag{2.19}$$

In view of equation (2.12) and (2.13), equation (1.1) gives the expression for the Zamkovoy connection on quasi-para Sasakian manifold

$$(\bar{\nabla}_X Y) = \nabla_X Y - g(X, \phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{2.20}$$

3. Curvature Tensor of Quasi-para-Sasakian Manifold with respect to Zamkovoy Connection

The curvature tensor $\bar{R}$ of Riemannian manifold M^n with respect to Zamkovoy connection $\bar{\nabla}$ is given by:

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

In the view of equation (2.20), we have

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ & - 2g(X,\phi Y)\phi Z + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ & + \{\eta(Y)X - \eta(X)Y\}\eta(Z),\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{3.3}$$

is the Riemannian curvature tensor of Levi-Civita connection ∇.

Equation (3.2) is the relation between Riemannian curvature tensor with respect to Zamkovoy connection $\bar{\nabla}$ and Levita-Civita connection ∇.
Transvection of V in equation (3.2) gives

$$\begin{aligned}\bar{R}(X,Y,Z,V) = {} & R(X,Y,Z,V) + g(X,\phi Z)g(V,\phi Y) - g(Y,\phi Z)g(V,\phi X) \\ & - 2g(X,\phi Y)g(V,\phi Z) + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(V) \\ & + \{\eta(Y)g(X,V) - \eta(X)g(V,Y)\}\eta(Z),\end{aligned} \tag{3.4}$$

where

$$\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$$

and

$$R(X,Y,Z,V) = g(R(X,Y)Z,V).$$

Putting $X = V = e_i$ $(1 \leq i \leq n)$ in equation (3.4) and taking summation ovre i, we get

$$\bar{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) + (n-3)\eta(Y)\eta(Z), \tag{3.5}$$

where $\bar{S}$ and S denotes the Ricci tensor with respect to the connection $\bar{\nabla}$ and ∇ respectively.

Again putting $Y = Z = e_i$ in equation (3.5) and taking summation over i, $(1 \leq i \leq n)$, we get

$$\bar{r} = r + 3n - 3, \tag{3.6}$$

where $\bar{r}$ and r denotes the scalar curvature with respect to the connection $\bar{\nabla}$ and ∇ respectively.
From equation (3.5), we have

$$\bar{Q}Y = QY + 2Y + (n-3)\eta(Y)\xi, \tag{3.7}$$

where $\bar{Q}$ and Q denotes the Ricci operator with respect to the connection $\bar{\nabla}$ and ∇ respectively and

$$\bar{S}(Y,\xi) = 0 = \bar{S}(\xi, Z). \tag{3.8}$$

Now from equation (3.2), we have

$$\bar{R}(X,Y)\xi = 0, \bar{R}(Y,\xi)Z) = 0, \bar{R}(\xi,Y)Z = 0. \tag{3.9}$$

Writing two more equations by the cyclic permutation of X,Y and Z in equation (3.2), we get

$$\begin{aligned} \bar{R}(Y,Z)X &= R(Y,Z)X + g(Y,\phi X)\phi Z - g(Z,\phi X)\phi Y - 2g(Y,\phi Z)\phi X \\ &+ \{g(Z,X)\eta(Y) - g(Y,X)\eta(Z)\}\xi + \{\eta(Z)Y - \eta(Y)Z\}\eta(X) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \bar{R}(Z,X)Y &= R(Z,X)Y + g(Z,\phi Y)\phi X - g(X,\phi Y)\phi Z - 2g(Z,\phi X)\phi Y \\ &+ \{g(X,Y)\eta(Z) - g(Z,Y)\eta(X)\}\xi + \{\eta(X)Z - \eta(Z)X\}\eta(Y). \end{aligned} \tag{3.11}$$

Adding equations (3.2), (3.10) and (3.11) and using Bianchi's first identity, we have

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 4g(X,\phi Z)\phi Y. \tag{3.12}$$

Thus, we can state as follows

Theorem 3.1. *A Qusi para-Sasakian manifold M^n with Zamkovoy connection satisfies the equation (3.12).*

Theorem 3.2. *The curvature tensor of a Zamkovoy connection in a quasi para Sasakian manifalds is*

1. skew-symmetric in first two slots.
2. skew-symmetric in last two slots.
3. symmetric in pair of slots.

Proof. 1. Interchanging X and Y in equation (3.4), we get

(3.13)
$$\begin{aligned} \bar{R}(Y,X,Z,V) &= R(Y,X,Z,V) + g(Y,\phi Z)g(V,\phi X) - g(X,\phi Z)g(V,\phi Y) \\ &- 2g(Y,\phi X)g(V,\phi Z) + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(V) \\ &+ \{\eta(X)g(Y,V) - \eta(Y)g(V,X)\}\eta(Z). \end{aligned}$$

Adding equations (3.4) and (3.13) with the fact that $R(X,Y,Z,V)+R(Y,X,Z,V) = 0$, we get

$$\bar{R}(X,Y,Z,V) + \bar{R}(Y,X,Z,V) = 0. \tag{3.14}$$

2. Interchanging Z and V in equation (3.4),we have

(3.15)
$$\begin{aligned}\bar{R}(X,Y,V,Z) &= R(X,Y,V,Z)+g(X,\phi V)g(Z,\phi Y)-g(Y,\phi V)g(Z,\phi X)\\ &-2g(X,\phi Y)g(Z,\phi V)+\{g(Y,V)\eta(X)-g(X,V)\eta(Y)\}\eta(Z)\\ &+\{\eta(Y)g(X,Z)-\eta(X)g(Z,Y)\}\eta(V).\end{aligned}$$

Adding equations (3.4) and (3.15) with the fact that $R(X,Y,Z,V)+R(X,Y,V,Z)=0$, we get

$$\bar{R}(X,Y,Z,V)+\bar{R}(X,Y,V,Z)=0. \tag{3.16}$$

3. Interchanging pair of slots in equation (3.4), we get

(3.17)
$$\begin{aligned}\bar{R}(Z,V,X,Y) &= R(Z,V,X,Y)+g(Z,\phi X)g(Y,\phi V)-g(V,\phi X)g(Y,\phi Z)\\ &-2g(Z,\phi V)g(Y,\phi X)+\{g(V,X)\eta(Z)-g(Z,X)\eta(V)\}\eta(Y)\\ &+\{\eta(V)g(Z,Y)-\eta(Z)g(Y,V)\}\eta(X).\end{aligned}$$

Subtracting equation (3.17) from equation (3.4) and using the fact that $R(X,Y,Z,V)-R(Z,V,X,Y)=0$, we get

$$\bar{R}(X,Y,Z,V)-\bar{R}(Z,V,X,Y)=0. \tag{3.18}$$

□

Theorem 3.3. *If the curvature tensor of a Zamkovoy connection vanishes, then M^n is of constant curvature with respect to Levi-Civita connection.*

Proof. Consider $\bar{R}(X,Y)Z=0$, then from equation (3.9), we have

$$\begin{aligned}0 &= R(X,Y)Z+g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X-2g(X,\phi Y)\phi Z\\ &+\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi+\{\eta(Y)X-\eta(X)Y\}\eta(Z).\end{aligned} \tag{3.19}$$

Transvection of ξ in equation (3.19), gives

$$\eta(R(X,Y)Z)=g(X,Z)\eta(Y)-g(Y,Z)\eta(X),$$

which gives

$$R(X,Y)Z=g(X,Z)Y-g(Y,Z)X, \tag{3.20}$$

which shows that M^n is of constant curvature with respect to Levi-Civita connection. □

Theorem 3.4. *If a quasi-para-Sasakian manifold M^n is Ricci flat with respect to the Zamkovoy connection then M^n is an η-Einstein manifold.*

Proof. Suppose that the quasi-para-Sasakian manifold is Ricci flat with respect to the Zamkovoy connection,then from equation (3.2), we have

$$S(Y,Z)=-2g(Y,Z)-(n-3)\eta(Y)\eta(Z), \tag{3.21}$$

which shows that M^n is an η-Einstein manifold. □

4. Locally ϕ-symmetric Quasi-para-Sasakian Manifold with respect to Zamkovoy Connection

Definition 4.1. A quasi-para-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds ([22], [20]).

Definition 4.2. A quasi-para-Sasakian manifolds M^n is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{4.2}$$

for arbitrary vector fields X, Y, Z, W.

Analogous to the definition of locally ϕ-symmetric quasi-para-Sasakian manifolds with respect to Levi-Civita connection, we define a locally ϕ-symmetric quasi-para-Sasakian manifold with respect to the Zamkovoy connection by

$$\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) = 0 \tag{4.3}$$

Theorem 4.1. *In a quasi-para-Sasakian manifold the Zamkovoy connection $\bar{\nabla}$ is locally ϕ-symmetric iff the Levi-Civita is so.*

Proof. In the view of equation (2.20) and (3.2), we have

$$\begin{aligned}(\bar{\nabla}_W \bar{R})(X,Y)Z =&(\nabla_W \bar{R})(X,Y)Z - g(W,\phi(\bar{R}(X,Y)Z))\xi \\ &- \eta(\bar{R}(X,Y)Z)\phi W + \eta(W)\phi(\bar{R}(X,Y)Z).\end{aligned} \tag{4.4}$$

Now differentiating equation (3.2) covariantly with respect to W, we get

$$\begin{aligned}(\nabla_W \bar{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + g(X,(\nabla_W \phi)Z)\phi Y \\ &+ g(X,\phi Z)(\nabla_W \phi)(Y) - g(Y,(\nabla_W \phi)Z)\phi X - g(Y,\phi Z)(\nabla_W \phi)X \\ &+ \{g(Y,Z)(\nabla_W \eta)(X) - g(X,Z)(\nabla_W \eta)(Y)\}\xi + \{g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\}\nabla_W \xi + \{(\nabla_W \eta)(Y)X - (\nabla_W \eta)(X)Y\}\eta(Z) \\ &+ \{\eta(Y)X - \eta(X)Y\}(\nabla_W \eta)(Z).\end{aligned} \tag{4.5}$$

Using equations (2.10), (2.12) and (2.13) in above equation, we get

$$\begin{aligned}(\nabla_W \bar{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + \{g(Y,W)\eta(Z) - g(W,Z)\eta(Y)\}\phi X \\ &+ \{g(W,Z)\eta(X) - g(X,W)\eta(Z)\}\phi Y + \{g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\}\phi W + \{g(X,\phi Z)g(Y,W) - g(X,W)g(Y,\phi Z) \\ &- g(Y,Z)g(W,\phi X) + g(X,Z)g(W,\phi Y)\}\xi + \{g(Y,\phi Z)\eta(X) \\ &- g(X,\phi Z)\eta(Y)\}W - \{g(W,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y)\}X \\ &+ \{g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)\}Y.\end{aligned} \tag{4.6}$$

Now taking the inner product of the equation (3.2) with ξ, and using equations (2.1), (2.3) and (2.19), we get

$$\eta(\bar{R}(X,Y)Z) = 0. \tag{4.7}$$

Also from equation 3.2, we have

$$\begin{aligned} g(W,\phi(\bar{R}(X,Y)Z))\xi = \{&g(X,Z)g(W,\phi Y) - g(Y,Z)g(W,\phi X) \\ &+ g(X,\phi Z)g(Y,W) - g(X,\phi Z)\eta(Y)\eta(W) - g(Y,\phi Z)g(X,W) \\ &+ g(Y,\phi Z)\eta(X)\eta(W) - 2g(X,\phi Y)g(W,Z) \\ &+ 2g(X,\phi Y)\eta(Z)\eta(W)\}\xi, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \eta(W)\phi(\bar{R}(X,Y)Z) = \{&g(X,Z) - \eta(X)\eta(Z)\}\eta(W)\phi Y - \{g(Y,Z) \\ &- \eta(Y)\eta(Z)\}\eta(W)\phi X + \{2g(X,\phi Y)\eta(Z) + g(Y,\phi Z)\eta(X) \\ &- g(X,\phi Z)\eta(Y)\}\eta(W)\xi - g(Y,\phi Z)\eta(W)X \\ &+ g(X,\phi Z)\eta(W)Y - 2g(X,\phi Y)\eta(W)Z. \end{aligned} \tag{4.9}$$

Using equations (4.6), (4.7), (4.8), (4.9) and (2.1) in equation (4.4), we get

$$\begin{aligned} \phi^2((\bar{\nabla}_W\bar{R})(X,Y)Z) = &\phi^2((\nabla_W R)(X,Y)Z) + \{g(Y,W)\eta(Z) \\ &- g(Z,W)\eta(Y)\}\phi^2(\phi X) + \{g(W,Z)\eta(X) - g(X,W)\eta(Z)\}\phi^2(\phi Y) \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2(\phi W) + \{g(Y,\phi Z)\eta(X) \\ &- g(X,\phi Z)\eta(Y)\}\phi^2 W - \{g(W,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y)\}\phi^2 X \\ &+ \{g(W,\phi Z)\eta(X) - g(W,\phi X)\eta(Z)\}\phi^2 Y + \{g(X,Z) \\ &- \eta(X)\eta(Z)\}\eta(W)\phi^2(\phi Y) - \{g(Y,Z) - \eta(Y)\eta(Z)\}\eta(W)\phi^2(\phi X) \\ &- g(Y,\phi Z)\eta(W)\phi^2(\phi X) - g(X,\phi Z)\eta(W)\phi^2(\phi Y). \end{aligned} \tag{4.10}$$

Consider X, Y, Z and W are the orthogonal to ξ, then equation (4.10) yields

$$\phi^2((\bar{\nabla}_W\bar{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.11}$$

□

5. ϕ-Recurrent Quasi-Para-Sasakian Manifold with respect to Zamkovoy Connection

Definition 5.1. An n-dimensional quasi-para-Sasakian manifold M^n is said to be ϕ-recurrent if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z. \tag{5.1}$$

If X, Y, Z, W are orthogonal to ξ then the manifold i called locally ϕ-recurrent manifold.

If the 1-form A vanishes, then the manifold is reduced to ϕ-symmetric manifold [20].

Definition 5.2. An n-dimensional quasi-para-Sasakian manifold M^n is said to be ϕ-recurrent with respect to Zamkovoy connection if there exist a non-zero 1-form such that

$$\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) = A(W)\bar{R}(X,Y)Z, \tag{5.2}$$

fpr arbitrary vectors X, Y, Z and W.

Theorem 5.1. *If a quasi-para-Sasakian manifold is ϕ-recurrent with respect to the Zamkovoy connection then the manifold is an η-Einstien manifold with respect to the Levi-Civita connection.*

Proof. Suppose M^n is ϕ-recurrent with respect to Zamkovoy connection, then in view of equation (2.4) and (5.2), we can write

$$g((\bar{\nabla}_W \bar{R})(X,Y)Z) - \eta((\bar{\nabla}_W \bar{R})(X,Y)Z)\eta(U) = A(W)g(\bar{R}(X,Y)Z,U). \tag{5.3}$$

By the virtue of equation (4.4) and (4.7) above equation reduced to

$$\begin{aligned} &g((\nabla_W \bar{R})(X,Y)Z,U) + \eta(W)g(\phi(\bar{R}(X,Y)Z),U) \\ &\qquad - \eta((\nabla_W \bar{R})(X,Y)Z)\eta(U) = A(W)g(\bar{R}(X,Y)Z,U). \end{aligned} \tag{5.4}$$

Which on using equations (4.6) and (3.2), above equation takes the form

$$\begin{aligned} &g((\nabla_W R)(X,Y)Z,U) + g(Y,W)g(\phi X,U)\eta(Z) - g(W,Z)g(\phi X,U)\eta(Y) \\ &+ g(W,Z)g(\phi Y,U)\eta(X) - g(X,W)g(\phi Y,U)\eta(Z) + g(Y,Z)g(\phi W,U)\eta(X) \\ &- g(X,Z)g(\phi W,U)\eta(Y) + g(X,Z)g(\phi W,U)\eta(Y) + g(X,\phi Z)g(Y,W)\eta(U) \\ &- g(X,W)g(Y,\phi Z)\eta(U) - g(Y,Z)g(W,\phi X)\eta(U) + g(X,Z)g(W,\phi Y)\eta(U) \\ &+ g(Y,\phi Z)g(U,W)\eta(X) - g(X,\phi Z)g(W,U)\eta(Y) - g(X,U)g(W,\phi Y)\eta(Z) \\ &- g(X,U)g(W,\phi Z)\eta(Y) + g(Y,U)g(W,\phi Z)\eta(X) + g(Y,U)g(W,\phi X)\eta(Z) \\ &+ \eta(W)g(\phi(R(X,Y)Z),U) + g(Y,U)g(X,\phi Z)\eta(W) \\ &- g(X,\phi Z)\eta(U)\eta(W)\eta(Y) - g(X,U)g(Y,\phi Z)\eta(W) \\ &+ g(Y,\phi Z)\eta(U)\eta(X)\eta(W) - 2g(X,\phi Y)g(Z,U)\eta(W) \\ &+ 2g(X,\phi Y)\eta(U)\eta(W)\eta(Z) + g(Y,Z)\eta(U)\eta(X)\eta(W) \\ &- g(X,Z)\eta(U)\eta(W)\eta(Y) + g(X,U)\eta(W)\eta(Y)\eta(Z) \\ &- g(Y,U)\eta(X)\eta(W)\eta(Z) - \eta((\nabla_W R)(X,Y)Z)\eta(U) \\ &+ g(X,\phi Z)g(Y,W)\eta(U) - g(Y,\phi Z)g(X,W)\eta(U) \\ &- g(W,\phi X)g(Y,Z)\eta(U) + g(W,\phi Y)g(X,Z)\eta(U) + g(Y,\phi Z)\eta(U)\eta(X)\eta(W) \\ &- g(X,\phi Z)\eta(U)\eta(W)\eta(Y) - g(W,\phi Y)\eta(U)\eta(X)\eta(Z) \\ &+ g(W,\phi X)\eta(Z)\eta(U)\eta(Y) = A(W)\{g(R(X,Y)Z,U) + g(X,\phi Z)g(\phi Y,U) \\ &- g(Y,\phi Z)g(\phi X,U) - g(X,\phi Y)g(\phi Z,U) + g(Y,Z)\eta(U)\eta(X) \\ &- g(X,Z)\eta(U)\eta(Y) + g(X,U)\eta(Y)\eta(Z) - g(Y,U)\eta(X)\eta(Z)\}. \end{aligned} \tag{5.5}$$

Putting $Z = \xi$ in above equation and using equations (2.1) and 2.2), we get

$$\begin{aligned} &g((\nabla_W R)(X,Y)\xi,U) + g(Y,W)g(\phi X,U) - g(\phi X,U)\eta(Y)\eta(W) \\ &+ g(\phi Y,U)\eta(X)\eta(W) - g(X,W)g(\phi Y,U) + g(W,\phi Y)\eta(X)\eta(U) \\ &- g(W,\phi Y)g(X,U) + g(W,\phi X)g(Y,U) + g(\phi(R(X,Y)\xi),U)\eta(W) \\ &+ g(X,U)\eta(Y)\eta(W) - g(Y,U)\eta(X)\eta(W) - \eta((\nabla_W R)(X,Y)\xi)\eta(U) \\ &- g(W,\phi X)\eta(Y)\eta(U) = A(W)\{g(R(X,Y)\xi,U) \\ &+ g(X,U)\eta(Y) - g(Y,U)\eta(X)\}. \end{aligned} \tag{5.6}$$

Now, putting $X = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &(\nabla_W S)(Y,\xi) - g(W,\phi Y) + (n-1)\eta(Y)\eta(W) \\ &\quad - \sum_{i=1}^{n} g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi) = A(W)S(Y,\xi) \\ &\quad + (n-1)\eta(Y)A(W). \end{aligned} \tag{5.7}$$

Let us denote the third term of left hand side of equation (5.7) by E. In this case E vanishes. Namely, we have

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g((\nabla_W R)(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &\quad - g(R(e_i,Y)\xi,\nabla_W \xi) \end{aligned} \tag{5.8}$$

at $p \in M^n$. In local co-ordinates $\nabla_W e_i = W^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_W e_i = 0$. Since R is skew-symmetric, we have

$$g(R(e_i,\nabla_W Y)\xi,\xi) = 0. \tag{5.9}$$

Using equation (5.9) in equation (5.8), we get

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g((\nabla_W R)(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \\ &\quad - g(R(e_i,Y)\xi,\nabla_W \xi). \end{aligned} \tag{5.10}$$

In view of $g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)e_i,Y) = 0$ and $(\nabla_W g) = 0$, we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) - g(R(e_i,Y)\xi,\nabla_W \xi) = 0, \tag{5.11}$$

which implies

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W \xi) - g(R(e_i,Y)\nabla_W \xi,\xi).$$

Since R is skew-symmetric, we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{5.12}$$

Using equation (5.12) in equation (5.7), we have

$$(\nabla_W S)(Y,\xi) - g(W,\phi Y) + (n-1)\eta(Y)\eta(W) = A(W)S(Y,\xi) + (n-1)\eta(Y)A(W) \tag{5.13}$$

Now, we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{5.14}$$

Using equations (2.12), (2.13) and (2.18) in above equation, we have

$$(\nabla_W S)(Y,\xi) = -S(Y,\phi W) + (n-1)g(W,\phi Y). \tag{5.15}$$

Using equation (5.15) in equation (5.13), we get

$$S(Y,\phi W) = -(n-2)g(Y,\phi W) + (n-1\eta(Y)\eta(W). \tag{5.16}$$

Now replacing W by ϕW in above equation, we get

$$S(Y,W) = -(n-2)g(Y,W) + (2n-3)\eta(Y)\eta(W). \tag{5.17}$$

□

6. Locally Projective ϕ-Symmetric Quasi-para-Sasakian Manifold with respect to Zamkovoy Connection

Definition 6.1. An n-dimensional quasi-para-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W P)(X,Y)Z) = 0, \tag{6.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where P is the projective curvature tensor defined as [18]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{6.2}$$

Equivalently,

Definition 6.2. An n-dimensional quasi-para-Sasakian manifold M^n is said to be locally projective ϕ-symmetric with respect to Zamkovoy connection if

$$\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) = 0, \tag{6.3}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where $\bar{P}$ is the projective curvature tensor with respect to Zamkovoy connection given by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \tag{6.4}$$

where $\bar{R}$ and $\bar{S}$ are the Riemannian curvature tensor and Ricci tensor with respect to Zamkovoy connection $\bar{\nabla}$.

Theorem 6.1. *An n-dimensional quasi-para-Sasakian manifold is locally projective ϕ-symmetric with respect to $\bar{\nabla}$ if and only if it is locally projective ϕ-symmetric with respect to Levi-Civita connection ∇.*

Proof. Using equation (2.20), we can write

$$(\bar{\nabla}_W \bar{P})(X,Y)Z =(\nabla_W \bar{P})(X,Y)Z - g(W,\phi(\bar{P}(X,Y)Z))\xi - \eta(\bar{P}(X,Y)Z)\phi W + \eta(W)\phi(\bar{P}(X,Y)Z). \tag{6.5}$$

Now differentiating equation (6.4) coveriently with respect to W, we get

$$(\nabla_W \bar{P})(X,Y)Z =(\nabla_W \bar{R})(X,Y)Z - \frac{1}{(n-1)}[(\nabla_W \bar{S})(Y,Z)X - (\nabla_W \bar{S})(X,Z)Y]. \tag{6.6}$$

From equation (3.5), we have

$$(\nabla_W \bar{S})(X,Y)Z = (\nabla_W S)(Y,Z) - (n-3)[g(W,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y)]. \tag{6.7}$$

Using equation (4.6) and (6.7) in equation (6.6), we get

$$\begin{aligned}(\nabla_W \bar{P})(X,Y)Z &= (\nabla_W R)(X,Y)Z + \{g(Y,W)\eta(Z) - g(W,Z)\eta(Y)\}\phi X \\ &+ \{g(W,Z)\eta(X) - g(X,W)\eta(Z)\}\phi Y + \{g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\}\phi W + \{g(X,\phi Z)g(Y,W) - g(X,W)g(Y,\phi Z) \\ &- g(Y,Z)g(W,\phi X) + g(X,Z)g(W,\phi Y)\}\xi + \{g(Y,\phi Z)\eta(X) \\ &- g(X,\phi Z)\eta(Y)\}W - \{g(W,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y)\}X \\ &+ \{g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)\}Y - \frac{1}{(n-1)}[(\nabla_W S)(Y,Z)X \\ &- (\nabla_W S)(X,Z)Y] + \frac{(n-3)}{(n-1)}[g(W,\phi Y)\eta(Z)X + g(W,\phi Z)\eta(Y)X \\ &+ g(W,\phi X)\eta(Z)Y + g(W,\phi Z)\eta(X)Y].\end{aligned} \tag{6.8}$$

Which on using equation (6.2), above equation reduced to

$$\begin{aligned}(\nabla_W \bar{P})(X,Y)Z &= (\nabla_W P)(X,Y)Z + \{g(Y,W)\eta(Z) - g(W,Z)\eta(Y)\}\phi X \\ &+ \{g(W,Z)\eta(X) - g(X,W)\eta(Z)\}\phi Y + \{g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\}\phi W + \{g(X,\phi Z)g(Y,W) - g(X,W)g(Y,\phi Z) \\ &- g(Y,Z)g(W,\phi X) + g(X,Z)g(W,\phi Y)\}\xi + \{g(Y,\phi Z)\eta(X) \\ &- g(X,\phi Z)\eta(Y)\}W - \frac{2}{(n-1)}\{g(W,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y)\}X \\ &+ \frac{2(n-2)}{(n-1)}\{g(W,\phi Z)\eta(X) - g(W,\phi X)\eta(Z)\}Y.\end{aligned} \tag{6.9}$$

Now using equations (3.2), (3.5) in equation (6.4), we get

$$
\begin{aligned}
\bar{P}(X,Y)Z &= R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z \\
&+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi + \{\eta(Y)X - \eta(X)Y\}\eta(Z) \\
&- \frac{1}{(n-1)}[S(Y,Z)X + 2g(Y,Z)X + (n-3)\eta(Y)\eta(Z)X - S(X,Z)Y \\
&- 2g(X,Z)Y - (n-3)\eta(X)\eta(Z)Y].
\end{aligned} \tag{6.10}
$$

Which gives,

$$
\begin{aligned}
\bar{P}(X,Y)Z &= P(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z \\
&+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi + \{\eta(Y)X - \eta(X)Y\}\eta(Z) \\
&- \frac{1}{(n-1)}[2g(Y,Z)X + (n-3)\eta(Y)\eta(Z)X - 2g(X,Z)Y \\
&- (n-3)\eta(X)\eta(Z)Y].
\end{aligned} \tag{6.11}
$$

Taking the inner product of equation (6.11) with ξ and using equation (2.1), we get

$$
\eta(\bar{P}(X,Y)Z) = \frac{(n-3)}{(n-1)}[g(Y,Z)\eta(X) - g(X,Z)Y]. \tag{6.12}
$$

Also, from equation (6.4), we have

$$
\begin{aligned}
\phi(\bar{P}(X,Y)Z) &= g(X,\phi Z)Y - g(Y,\phi Z)X + \{g(Y,\phi Z)\eta(X) \\
&- g(X,\phi Z)\eta(Y)\}\xi + \{\frac{2}{(n-1)}g(Y,Z) + \frac{2(n-2)}{(n-1)}\eta(Y)\eta(Z)\}\phi X \\
&- \{\frac{2}{(n-1)}g(X,Z) + \frac{2(n-2)}{(n-1)}\eta(X)\eta(Z)\}\phi Y.
\end{aligned} \tag{6.13}
$$

And

$$
\begin{aligned}
g(W,\phi(\bar{P}(X,Y)Z))\xi &= g(W,Y)g(X,\phi Z)\xi - g(X,W)g(Y,\phi Z)\xi \\
&+ g(Y,\phi Z)\eta(X)\eta(W)\xi - g(X,\phi Z)\eta(Y)\eta(W)\xi.
\end{aligned} \tag{6.14}
$$

Using equations (6.9), (6.12), (6.13) and (6.14) in equation (6.5), we get

$$
\begin{aligned}
\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) &= \phi^2((\nabla_W P)(X,Y)Z) + \{g(Y,W)\eta(Z) \\
&- g(W,Z)\eta(Y)\}\phi^2(\phi X) + \{g(W,Z)\eta(X) - g(X,W)\eta(Z)\}\phi^2(\phi Y) \\
&+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2(\phi W) + \{g(Y,\phi Z)\eta(X) \\
&- g(X,\phi Z)\eta(Y)\}\phi^2 W - \frac{2}{(n-1)}\{g(W,\phi Y)\eta(Z) \\
&+ g(W,\phi Z)\eta(Y)\}\phi^2 X + \frac{2(n-2)}{(n-1)}\{g(W,\phi Z)\eta(X) \\
&- g(W,\phi X)\eta(Z)\}\phi^2 Y - \frac{(n-3)}{(n-1)}\{g(Y,Z)\eta(X) \\
&- g(X,Z)Y\}\phi^2(\phi W) + \{g(X,\phi Z)\phi^2 Y - g(Y,\phi Z)\phi^2 X\}\eta(W) \\
&+ \{\frac{2}{(n-1)}g(Y,Z)\eta(X) + \frac{2(n-2)}{(n-1)}\eta(Y)\eta(Z)\}\eta(W)\phi^2(\phi X) \\
&- \{\frac{2}{(n-1)}g(X,Z) + \frac{2(n-2)}{(n-1)}\eta(X)\eta(Z)\}\eta(W)\phi^2(\phi Y).
\end{aligned} \tag{6.15}
$$

By assuming X, Y Z, W orthogonal to ξ, above equation reduced to

$$\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) = \phi^2((\nabla_W P)(X,Y)Z). \tag{6.16}$$

□

Theorem 6.2. *A ϕ-symmetric quasi-para-Sasakian manifold admitting the Zamkovoy connection $\bar{\nabla}$ is locally projective ϕ-symmetric with respect to Zamkovoy connection $\bar{\nabla}$ if and only if it is locally projective ϕ-symmetric with respect to Levi-Civita connection ∇.*

Proof. Using equations (6.8), (6.12), (6.13) and (6.14) in equation (6.5), we get

(6.17)
$$\begin{aligned}
\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z) + \{g(Y,W)\eta(Z) \\
&- g(W,Z)\eta(Y)\}\phi^2(\phi X) + \{g(W,Z)\eta(X) - g(X,W)\eta(Z)\}\phi^2(\phi Y) \\
&+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2(\phi W) + \{g(Y,\phi Z)\eta(X) \\
&- g(X,\phi Z)\eta(Y)\}\phi^2 W - \frac{2}{(n-1)}\{g(W,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y)\}\phi^2 X \\
&+ \frac{2(n-2)}{(n-1)}\{g(W,\phi Z)\eta(X) - g(W,\phi X)\eta(Z)\}\phi^2 Y - \frac{(n-3)}{(n-1)}\{g(Y,Z)\eta(X) \\
&- g(X,Z)Y\}\phi^2(\phi W) + \{g(X,\phi Z)\phi^2 Y - g(Y,\phi Z)\phi^2 X\}\eta(W) \\
&+ \{\frac{2}{(n-1)}g(Y,Z)\eta(X) + \frac{2(n-2)}{(n-1)}\eta(Y)\eta(Z)\}\eta(W)\phi^2(\phi X) \\
&- \{\frac{2}{(n-1)}g(X,Z) + \frac{2(n-2)}{(n-1)}\eta(X)\eta(Z)\}\eta(W)\phi^2(\phi Y) \\
&- \frac{1}{(n-1)}\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}\phi^2(\phi W).
\end{aligned}$$

Taking X, Y Z, W orthogonal to ξ, in equation (6.17), we get

$$\phi^2((\bar{\nabla}_W \bar{P})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{6.18}$$

□

7. ϕ-Projectively flat Quasi-para-Sasakian Manifold with respect to Zamkovoy Connection

Definition 7.1. An n-dimensional differentiable manifold (M^n, g) satisfying the equation

$$\phi^2(P(\phi X, \phi Y)\phi Z) = 0, \tag{7.1}$$

is called ϕ-projectively flat [19]. Analogous to the equation (7.1) we define an n-dimensional quasi-para-Sasakian manifold is said to be ϕ-projectively flat with respect to Zamkovoy connection if it satisfies

$$\phi^2(\bar{P}(\phi X, \phi Y)\phi Z) = 0, \tag{7.2}$$

where $\bar{P}$ is the projective curvature tensor of the manifold with respect to Zamkovoy connection.

Theorem 7.1. *An n-dimensional ϕ-projectively flat quasi-para-Sasakian manifold admitting Zamkovoy connection is an η-Einstein manifold with respect to Levi-Civita connection.*

Proof. Suppose M^n is ϕ-projectively flat quasi-para-Sasakian manifold with respect to Zamkovoy connection. It is easy to see that $\phi^2(\bar{P}(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(\bar{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{7.3}$$

for X, Y Z, $W \in \chi(M)$. So by the virtue of equation (6.4) ϕ-projectively flat means

$$\begin{aligned} \bar{g}(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{(n-1)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W) \\ - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi W)], \end{aligned} \tag{7.4}$$

which on using equation (3.2) and (3.5), the above equation reduced to

$$\begin{aligned} &g(R(\phi X, \phi Y)\phi Z, \phi W) + g(\phi X, Z)g(\phi W, Y) - g(\phi Y, Z)g(\phi W, X) \\ &\quad - 2g(\phi X, Y)g(\phi W, Z) = \frac{1}{(n-1)}[S(\phi Y, \phi Z)g(\phi X, \phi W) \\ &\quad + 2g(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W) \\ &\quad - 2g(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{7.5}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ are the local orthonormal basis of the vector field in M^n. Using th fact that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also local orthonormal basis. Putting $X = W = e_i$ in equation (7.5) and summing over i, we get

$$\begin{aligned} &\sum_{i=1}^{n-1}[g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + g(\phi e_i, Z)g(\phi e_i, Y) - g(\phi Y, Z)g(\phi e_i, e_i) \\ &\quad - 2g(\phi e_i, Y)g(\phi e_i, Z)] = \frac{1}{(n-1)}\sum_{i=1}^{n-1}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &\quad + 2g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\ &\quad - 2g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \end{aligned} \tag{7.6}$$

Also,

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(Y, Z) + g(Y, Z), \tag{7.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z), \tag{7.8}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1), \tag{7.9}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = -(n-1)\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = -(n-1)^2, \tag{7.10}$$

$$\sum_{i=1}^{n-1} g(e_i, \phi e_i) = 0, \tag{7.11}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi Y) g(\phi e_i, \phi Z) = S(\phi Y, \phi Z), \tag{7.12}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{7.13}$$

Now using equations (7.7), (7.8), (7.9), (7.10), (7.11), (7.12) and (7.13) in equation (7.6), we get

$$S(Y,Z) = -\frac{(n-2)(n-3}{(n-1)} g(Y,Z) - \frac{(n-2)(n-3}{(n-1)} \eta(Y)\eta(Z). \tag{7.14}$$

□

8. Pseudo-quasi-conformally flat Quasi-para-Sasakian Manifold with respect to Zamkovoy Connection$\bar{\nabla}$

An n-dimensional quasi-para-Sasakian manifold M^n is said to be pseudo-quasi-conformally flat if the pseudo-quasi-conformal curvature tensor vanishes.

In this section, we assume that $\bar{\bar{V}}(X,Y)Z = 0$, where $\bar{\bar{V}}$ denotes the pseudo-quasi-conformal curvature tensor with respect to the Zamkovoy connection $\bar{\nabla}$.

Theorem 8.1. *A pseudo-quasi-conformally flat quasi-para-Sasakian manifold M^n $(n \geq 2)$ admitting Zamkovoy connection $\bar{\nabla}$ is an $\eta-$Einstein manifold.*

Proof. Let M^n be an n-dimensional pseudo-quasi-conformally flat quasi-para-Sasakian manifold with respect to the Zamkovoy connection, *i.e.* $\bar{\bar{V}} = 0$, then from equation (1.3), we have

(8.1)
$$\begin{aligned}(p+d)\bar{R}(X,Y)Z = &-(q - \frac{d}{n-1})[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] \\ &- q[g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] - \frac{r}{n(n-1)}\{p + 2(n-1)q\}[g(Y,Z)X \\ &- g(X,Z)Y].\end{aligned}$$

Transvection of V in equation (8.1), gives

$$\begin{aligned}(p+d)\bar{R}(X,Y,Z,V) = &-(q-\frac{d}{n-1})[\bar{S}(Y,Z)g(X,V)-\bar{S}(X,Z)g(Y,V)]\\ &-q[g(Y,Z)\bar{S}(X,V)-g(X,Z)\bar{S}(Y,V)]\\ &-\frac{r}{n(n-1)}\{p+2(n-1)q\}[g(Y,Z)g(X,V)-g(X,Z)g(Y,V)].\end{aligned} \tag{8.2}$$

Let $e_i, (1 \le i \le n)$ be an orthonormal basis. Taking summation over $X = V = e_i$ $(1 \le i \le n)$ in above equation, we get

$$\bar{S}(Y,Z) = \frac{1}{(p+nq-q)}\{\frac{\bar{r}}{n}(p+2nq-q)-q(n^2+3n-3)\}g(Y,Z). \tag{8.3}$$

Using equations (3.5) and (3.6) in equation (8.3), we get.

$$S(Y,Z) = ag(Y,Z) - b\eta(Y)\eta(Z), \tag{8.4}$$

which shows that M^n is an η-Einstein manifold, where

$$a = \frac{1}{p+nq-q}\{\frac{(r+3n-3)}{n}(p+2nq-q)-q(n^2+3n-3)-2(p+nq-q)\}$$

and

$$b = (n-3).$$

□

Theorem 8.2. *A ξ-pseudo-quasi-conformally flat quasi-para-Sasakian manifold M^n $(n>2)$ admitting Zamkovoy connection $\bar{\nabla}$ is an η-Einstein manifold.*

Proof. If M^n be ξ-pseudo-quasi-conformally flat quasi-para-Sasakian manifold with respect to the Zamkovoy connection, *i.e.* $\bar{\tilde{V}}(X,Y)\xi = 0$, then from equation (1.3), we have

$$\begin{aligned}(q-\frac{d}{n-1})[\bar{S}(Y,\xi)X-\bar{S}(X,\xi)Y] &= q[\eta(Y)\bar{Q}X-\eta(X)\bar{Q}Y]\\ &-\frac{\bar{r}}{n(n-1)}\{p+2(n-1)q\}[\eta(Y)X-\eta(X)Y].\end{aligned} \tag{8.5}$$

Transvection of V in equation (8.5), we get

$$\begin{aligned}(q-\frac{d}{n-1})[\bar{S}(Y,\xi)g(X,V)-\bar{S}(X,\xi)g(Y,V)] = &\, q[\eta(Y)\bar{S}(X,V)\\ &-\eta(X)\bar{S}(Y,V)]-\frac{\bar{r}}{n(n-1)}\{p+2(n-1)q\}[\eta(Y)g(X,V)\\ &-\eta(X)g(Y,V)].\end{aligned} \tag{8.6}$$

Putting $Y = \xi$ and using equations (3.3) and (3.5) in equation (8.6), we get

$$S(X, V) = -ag(X, V) + b\eta(X)\eta(V), \tag{8.7}$$

which shows that the manifold M^n is an η-Einstein manifold, where

$$a = -\{\frac{r + 3n - 3}{n(n-1)}\}\{(2n-1)q + 3p - 2\}$$

and

$$b = \{\frac{r + 3n - 3}{n(n-1)}\{p + (2n-1)q\} - (n-3)\}.$$

□

Theorem 8.3. *On an n-dimensional quasi-para-Sasakian manifold M^n, ξ-pseudo-quasi-conformal-curvature tensor of Zamkovoy connection and Levi-Civita connection are identical provided that the vector fields on M^n are horizontal vector fields.*

Proof. From equations (1.2), (1.3), (3.2), (3.5), and (3.7), we have

$$\begin{aligned}\bar{\tilde{V}}(X, Y)Z &= \tilde{V}(X, Y)Z + (p + d)g(X, \phi Z)\phi Y - (p + d)g(X, \phi Z)\phi Y \\ &- 2(p + d)g(X, \phi Y)\phi Z + (p + d)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \\ &+ (p + d)\{\eta(Y)X - \eta(X)Y\}\eta(Z) + (q - \frac{d}{n-1})[2g(Y, Z)X \\ &+ (n-3)\eta(Y)\eta(Z)X - 2g(X, Z)Y - (n-3)\eta(X)\eta(Z)Y] \\ &+ q[2g(Y, Z)X + (n-3)g(Y, Z)\eta(X)\xi - 2g(X, Z)Y \\ &- (n-3)g(X, Z)\eta(Y)\xi] - \frac{3n-3}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)X \\ &- g(X, Z)Y].\end{aligned} \tag{8.8}$$

Substitute $Z = \xi$ in above equation (8.8), we get

$$\begin{aligned}\bar{\tilde{V}}(X, Y)\xi &= \tilde{V}(X, Y)\xi + [(p + d) + (q - \frac{d}{n-1}) \\ &+ 2q - \frac{3n-3}{n(n-1)}\{p + 2(n-1)q\}][\eta(Y)X - \eta(X)Y].\end{aligned} \tag{8.9}$$

If X and Y are horizontal vector fields then from equation (8.9), it follows that $\bar{\tilde{V}}(X, Y)\xi = \tilde{V}(X, Y)\xi$. □

9. $\phi-$Pseudo-quasi-conformally flat Quasi-para-Sasakian Manifold with respect to the Zamkovoy Connection $\bar{\nabla}$

Definition 9.1. Pseudo-quasi-conformal curvature tensor with respect to Zamkovoy connection is said to be ϕ-pseudo-quasi-conformally flat if

$$\bar{\tilde{V}}(\phi X, \phi Y, \phi Z, \phi U) = 0.$$

In this section we consider ϕ-pseudo-quasi-conformally flat quasi-para-Sasakian manifold admitting Zamkovoy connection and showed that it is an η-Einstein manifold.

Theorem 9.1. *A ϕ-pseudo-quasi-conformally flat quasi-para-Sasakian manifold M^n admitting Zamkovoy connection $\bar{\nabla}$ is an η-Einstein manifold.*

Proof. We assume that quasi-para-Sasakian manifold M^n be $\phi-$pseudo-quasi-conformally flat with respect to the Zamkovoy connection, *i.e.*

$$\bar{\tilde{V}}(\phi X, \phi Y, \phi Z, \phi U) = 0,$$

for all $X,\ Y,\ Z,\ U \in \chi(M)$. Then from equation (1.3), we have

$$\begin{aligned}(p+d)\bar{R}(\phi X, \phi Y, \phi Z, \phi U) &= -(q - \frac{d}{n-1}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi U) \\ &- \bar{S}(\phi X, \phi Z)g(\phi Y, \phi U) - q[g(\phi Y, \phi Z)\bar{S}(\phi X, \phi U) \\ &- g(\phi X, \phi Z)\bar{S}(\phi Y, \phi U)] - \frac{\bar{r}}{n(n-1)q}[g(\phi Y, \phi Z g(\phi X, \phi U) \\ &- g(\phi X, \phi Z)g(\phi Y, \phi U)].\end{aligned} \tag{9.1}$$

Now from equation (3.4), we have

$$\begin{aligned}&\bar{R}(X, Y, Z, U) = R(X, Y, Z, U) + g(X, \phi Z)g(\phi Y, U) \\ &- g(Y, \phi Z)g(\phi X, U) - 2g(X, \phi Y)g(U, \phi Z) - \eta(X)\eta(Z)g(Y, U) \\ &+ g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U) + \eta(Y)\eta(Z)g(X, U)\end{aligned} \tag{9.2}$$

Replacing X by ϕX, Y by ϕY, Z by ϕZ and U by ϕU in equation (9.2),we have

$$\begin{aligned}&\bar{R}(\phi X, \phi Y, \phi Z, \phi U) = R(\phi X, \phi Y, \phi Z, \phi U) + g(\phi X, Z)g(Y, \phi U) \\ &- g(\phi Y, Z)g(X, \phi U) - 2g(\phi X, Y)g(\phi U, Z).\end{aligned} \tag{9.3}$$

From equation (3.5), we have

$$\bar{S}(\phi Y, \phi Z) = S(\phi Y, \phi Z) + 2g(\phi Y, \phi Z). \tag{9.4}$$

Using equation (2.17) in equation (9.4), we get

$$\bar{S}(\phi Y, \phi Z) = (n-3)g(Y, Z) - (n-3)\eta(Y)\eta(Z). \tag{9.5}$$

Now using equations (9.3) and (9.5) in equation (9.1), we get

$$\begin{aligned}&(p+d)R(\phi X, \phi Y, \phi Z, \phi U) = -(p+d)g(\phi X, Z)g(Y, \phi U) \\ &+ g(\phi Y, Z)g(X, \phi U) + 2(p+d)g(\phi X, Y)g(\phi U, Z) \\ &- (q - \frac{d}{n-1})[(n-3)g(Y, Z)g(\phi X, \phi U) - (n-3)g(\phi X, \phi U)\eta(Y)\eta(Z) \\ &- (n-3)g(X, Z)g(\phi Y, \phi U) + (n-3)g(\phi Y, \phi U)\eta(X)\eta(Z) \\ &- (n-3)q[g(\phi Y, \phi Z)g(X, U) - g(\phi Y, \phi Z)\eta(X)\eta(U) \\ &- g(\phi X, \phi Z)g(Y, U) + g(\phi X, \phi Z)\eta(Y)\eta(U)] \\ &- \frac{\bar{r}}{n(n-1)}\{p + 2(n-1)q\}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)].\end{aligned} \tag{9.6}$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ be a local orthonormal basis of the tangent space at any point of the manifold M^n. Then $\{\phi e_i, \xi\}$, $(1 \leq i \leq n)$ is also a local orthonormal basis. Contracting $X = U = e_i$ $(1 \leq i \leq n-1)$ in equation (9.6), we have

$$\begin{aligned}(p+d)S(\phi Y, \phi Z) =& \{(p+d) + q(n-3)(n-2)\}g(Y,Z) \\ &+ \frac{\bar{r}}{n(n-1)}\{p + 2(n-1)q\}(n-2)\eta(Y)\eta(Z).\end{aligned} \tag{9.7}$$

Replacing Y by $\phi Y$and Z by ϕZ in equation (9.7), we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{9.8}$$

where

$$a = -\frac{1}{(p+d)}[(p+d) - (q - \frac{d}{n-1})(n-1)(n-3) + q(n-2)(n-3)$$

$$+\frac{r+3n-3}{n(n-1)}\{p + 2(n-1)q\}(n-2)]$$

and

$$b = \frac{1}{(p+d)}[\{\frac{r+3n-3}{n(n-1)}(p + 2(n-1)(n-2)q\} - (n-1)(p+d)]\eta(Y)\eta(Z)$$

. This shows that M^n is an η-Einstein manifold. □

10. Quasi-para-Sasakian manifold admitting Zamkovoy connection $\bar{\nabla}$ satisfying $\bar{\tilde{V}}(\xi, U).\bar{S}$=0

In this section, we define a relation on quasi-para-Sasakian manifold $\bar{\tilde{V}}.\bar{S} = 0$, where $\bar{\tilde{V}}$ and $\bar{S}$ are the pseudo-quasi-conformal curvature tensor and Ricci tensor with respect to the Zamkovoy connection respectively.

Theorem 10.1. *On an n-dimensional quasi-para-Sasakian manifold M^n admitting Zamkovoy connection $\bar{\nabla}$, if the condition $\bar{\tilde{V}}(\xi, U).\bar{S} = 0$ holds, then*

$$S^2(U,Y) = aS(U,Y) + bg(U,Y) + c\eta(U)\eta(Y).$$

satisfied on M^n,

where

$$a = \frac{1}{q}[\frac{r+3n-3}{n(n-1)}\{p + 2(n-1)q\} - 3q],$$

$$b = \frac{2}{q}[\frac{r+3n-3}{n(n-1)}\{p + 2(n-1)q\} - 2q]$$

and

$$c = \frac{2}{q}[\frac{r+3n-3}{n(n-1)}\{p + 2(n-1)q\} + (n^2 - 4n + 1)q].$$

Proof. Assume that a quasi-para-Sasakian manifold M^n admitting Zamkovoy connection satisfying the condition

$$(\bar{\tilde{V}}(\xi, U).\bar{S})(X, Y) = 0,$$

where $\bar{\tilde{V}}$ and $\bar{S}$ are the pseudo-quasi-conformal curvature tensor and Ricci tensor with respect to Zamkovoy connection respectively and $X,\ Y,\ U \in \chi(M)$, then we have

$$\bar{S}(\bar{\tilde{V}}(\xi, U)X, Y) + \bar{S}(X, \bar{\tilde{V}}(\xi, U)Y) = 0. \tag{10.1}$$

From equation (1.3), we have

$$\begin{aligned} \bar{\tilde{V}}(\xi, U)X =& q - \frac{d}{n-1})[\bar{S}(U, X)\xi + q[g(U, X)\bar{Q}\xi - \eta(X)\bar{Q}U] \\ & - \frac{\bar{r}}{n(n-1)}\{p + 2(n-1)q\}[g(U, X)\xi - \eta(X)U]. \end{aligned} \tag{10.2}$$

Using equation (10.2) in equation (10.1), we have

(10.3)

$$\begin{aligned} -q\eta(X)\bar{S}(\bar{Q}U, Y) + \frac{\bar{r}}{n(n-1)}\{p + 2(n-1)q\}\eta(X)\bar{S}(U, Y) - q\eta(Y)\bar{S}(\bar{Q}U, X) \\ + \frac{\bar{r}}{n(n-1)}\{p + 2(n-1)q\}\eta(Y)\bar{S}(X, U) = 0. \end{aligned}$$

Putting X=ξ in above equation, we get

$$-q\bar{S}(\bar{Q}U, Y) + \frac{\bar{r}}{n(n-1)}\{p + 2(n-1)q\}\bar{S}(U, Y) = 0. \tag{10.4}$$

Using equations (3.5), (3.7) and (3.8) in equation (10.4), we get

$$S^2(U, Y) = aS(U, Y) + bg(U, Y) + c\eta(U)\eta(Y), \tag{10.5}$$

where

$$a = \frac{1}{q}[\frac{r + 3n - 3}{n(n-1)}\{p + 2(n-1)q\} - 3q],$$

$$b = \frac{2}{q}[\frac{r + 3n - 3}{n(n-1)}\{p + 2(n-1)q\} - 2q]$$

and

$$c = \frac{2}{q}[\frac{r + 3n - 3}{n(n-1)}\{p + 2(n-1)q\} + (n^2 - 4n + 1)q].$$

□

11. Conclusions

Almost paracontact structure on pseudo Riemannian manifold of dimension (2n+1) was introduced and studied by Kaneyuki and Konzai in the year 1985. S. Zamkovoy defined a Zamkovoy connection on a paracontact metric manifold which seems to be the paracontact analogoue of the Tanaka-Webster connection. In this paper, we study various geometric properties of quasi-para-Sasakian manifold with respect to Zamkovoy connection. A unique relation between curvature tensors of Zamkovoy connection and Levi-Civita connection have been obtained. We study Locally ϕ-symmetric quasi-para-Sasakian manifold, ϕ-recurrent quasi-para-Sasakian manifold, Locally projective ϕ-symmetric quasi-para-Sasakian manifold, ϕ-projectively flat quasi-para-Sasakian manifold as well as quasi-pseudo-conformally flat quasi-para-Sasakian manifold with respect to Zamkovoy connection. Also, it is proved that a ϕ-pseudo-quasi-conformally flat quasi-para-Sasakian manifold with respect to Zamkovoy connection is an η-Eienstein manifold.

12. Acknowledgement

I would like to thank the Editor and the Reviewers for reviewing the paper carefully and providing valuable comments that improved the overall quality of the paper.

REFERENCES

1. A. Biswas and K.K. Baishya, *A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds,* Scientific Studies and Research Series Mathematics and Informatics., **29(1)**, **(2019)**, 233-246.
2. A. M. Blaga, *Canonical connections on para kenmotsu manifold,* Navi Sad. J. Math., **45(2)**, **(2015)**, 131-142.
3. D. E. Blair, *The theory of quasi-Sasakian structure,* J. Differential Geom., **1**, **(1967)**, 331-345 .
4. U.C. De and A. Sarkar, *On the Projective curvature tensor of generalized Sasakian-Space-Forms,* Questiones Math., **33**, **(2010)**, 245-252.
5. U.C. De and A. Sarkar, *Some Results on Generalized Sasakian-Space-Forms,* Thai Journal of Mathematics., **8(1)**, **(2010)**, 1-10.
6. U.C. De and K. De, *LP-Sasakian manifolds with quasi-conformal curvature tensor,* SUT Journal of Mathematics., **49(1)**, **(2013)** , 33-46.
7. U.C. De and P. Majhi, *Concircular curvature tensor on K-contact manifolds,* Acta Mathematica Academae Paedagogicae Nyrehaziensis., **29**, **(2013)**, 89-99.
8. U.C. De, Yanling Han and K. Mandal, *On para-Sasakian manifolds satisfying certain curvature conditions,* Filomat., **31(7)**, **(2017)**, 1941-1947.
9. U.C. De and Y.J. Suh, *Yamabe and quasi-Yamambe solitons in paracontact metric manifolds,* International Journal of Geometric Methods and Modern Physics., **18(12)**, **(2021)**.

10. I. Kupili Erken, *Some classes of 3-dimensional normal almost paracontact metric manifolds,* Honam Math. J., **37(4)**,**(2015)**, 457-468.

11. I. Kupili Erken., *On normal almost paracontact metric manifolds of dimension 3,* Facta. Univ. Ser. Math. Inform., **30(5)**, **(2015)**, 777-788.

12. S. Kaneyuki, *Quasi-Saskian manifolds,* Tohoku Math. J., **29**, **(1977)**, 227-233.

13. S. Kaneyuki, M. Konzai, *Para complex structure and affine symmetric space,* Tokyo J. Math., **8**, **(1985)**, 301-318.

14. S. Kundu, *On $P-$ Sasakian manifolds,* Mathe. Report., **15(65)**, **(2013)**, 221-232.

15. Z. Olszak, *Curvature properties of quasi-Saskian manifolds,* Tensor., **38**, **(1982)**, 19-28.

16. Z. Olszak, *Normal almost contact metric manifolds of dimension three,* Ann. Polon. Math., **(47)**, **(1986)**, 41-50.

17. D. G. Prakasha, T.R. Shivamurthy and K.K. Mirji, *On Pseude-quasi-conformal curvature tensor of $P-$Sasakian manifolds,* Elect. J. Math. Anal. Appl., **5(2)**, **(2017)**, 147-155.

18. R. N. Singh, S. K. Pandey and G. Pandey, *On a Type of Kenmotsu Manifold,* Bulletin of Mathematical Analysis and Applications, **4(1)**, **(2012)**, 117-132.

19. R. N. Singh, S. K. Pandey, G. Pandey and K. Tiwari *On a Semi-Symmetric Metric Connection in an (ϵ)-Kenmotsu Manifold,* Commun. Korean Math. Soc., **(29)**, **(2014)**, 331-343.

20. R. N. Singh and S. K. Pandey , *On a Quarter-Symmetric Metric Connection in an LP-Sasakian Manifold,* Thai Journal of Mathematics., **(12)**, **(2014)**, 357-371.

21. A. A. Shaikh and S.K. Jana, *A pseudo-quasi-conformal curvature tensor on Riemannian manifolds,* South East Asian J. Math.Science., **4**, **(2005)**, 15-20.

22. T. Takahashi, *Sasakian ϕ-symmetric spaces,* Tohoku Math. J. **(29)**, **(1977)**, 91-113.

23. S. Tanno, *Quasi-Saskian structure of rank 2p+1,* J. Differential Geom., **5**, **(1971)**, 317-324.

24. S. Tanno, *Variation problem on contact Riemannian manifolds,* Trans. Amer, Math Soc.,**(314)**, **(1989)**, 349-379.

25. K. Yano and S. Sawaki, *Riemannian manifolds admitting a conformal transmation group,* J. Differential. Geom., **2, (1968)**, 161-184.

26. S. Zamkovoy, *Canonical connections on paracontact manifolds,* Ann. Global And Geom., **36(1)**, **(2008)**, 37-60.