SOME FIXED POINT THEOREMS AND CYCLIC CONTRACTIONS IN DISLOCATED QUASI-METRIC SPACES

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Abstract. In this paper, we established some common fixed point theorems for types of cyclic contractions in the setting of dislocated quasi-metric spaces. Using the type of contraction defined by C-class functions in [2] and the class of continuous functions G_3 in [23] we extend, generalize and unify some results in the existing literature.

Keywords: cyclic map, cyclical contraction, dislocated quasi–metric, common fixed point

1. Introduction

There are a lot of generalizations of the concept of metric space. Some of them are the concept of dislocated metric space and dislocated quasi-metric space which were introduced by Hitzler [6, 7] and F. Zeyada [21]. These metrics have the property " nonzero self-distance" of points and play a very important role not only in topology, but also in other branches of science involving mathematics and, especially, logic programming and electronic engineering. In sequel many researchers have studied, generalized and extended the results on the existence of fixed points and common fixed points, in many directions in dislocated spaces (see e.g. [1, 6, 8, 9, 21, 22, 23, 24]).

In 2003 Kirk et al. [16] introduced cyclic contractions in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings. Since then, many authors have proved common fixed point theorems for self-mappings satisfying a different type of contractive conditions in some spaces (see results in [5, 11, 12, 13, 14, 16]).

In this paper, we introduce some notions of d_q–cyclic contractions and use the concepts of altering distance function [15], concept of C-class functions introduced by Ansari [2] and a class of continuous functions G_3 and also establish common fixed point theorems for cyclic mappings and cyclic contractions in the framework

of dislocated quasi–metric spaces. Our main theorems extend, generalize and unify most of the existing metrical fixed point results in literature.

Definition 1.1. [6, 21] Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function, called a distance function if for all $x, y, z \in X$, satisfies:

$$
\begin{aligned}
d_1 : &\quad d(x, x) = 0 \\
d_2 : &\quad d(x, y) = d(y, x) = 0 \Rightarrow x = y \\
d_3 : &\quad d(x, y) = d(y, x) \\
d_4 : &\quad d(x, y) \leq d(x, z) + d(z, y).
\end{aligned}
$$

If d satisfies the condition $d_1 - d_4$, then d is called a metric on X. If it satisfies the conditions d_1, d_2 and d_4 it is called a quasi–metric space. If d satisfies conditions d_2, d_3 and d_4 it is called a dislocated metric (or simply d–metric). If d satisfies only d_2 and d_4 then d is called a dislocated quasi–metric (or simply d_q–metric) on X.

Definition 1.2. [21] A sequence $(x_n)_{n \in \mathbb{N}}$ in a d_q–metric space (X, d_q) dislocated quasi–converges (for short, d_q –converges) to $x \in X$ if $\lim_{n \to \infty} d_q(x_n, x) = \lim_{n \to \infty} d_q(x, x_n)$ $= 0$. In this case x is called a d_q–limit of $(x_n)_{n \in \mathbb{N}}$ and we write $x_n \to x$.

Definition 1.3. [21] A sequence $(x_n)_{n \in \mathbb{N}}$ in a d_q–metric space (X, d_q) is said to be Cauchy if for every $\varepsilon > 0$, $\exists n_0 \in N$ such that $\forall m, n \geq n_0$, $d_q(x_m, x_n) < \varepsilon$ and $d_q(x_n, x_m) < \varepsilon$.

Definition 1.4. [21] A d_q–metric space (X, d_q) is complete if every Cauchy sequence is d_q–convergent in X.

Example 1.1. Let $X = [0, 1]$ and $d_q(x, y) = \max\{x, y\}$. Then the pair (X, d_q) is a dislocated metric space, but it is not a metric space.

Lemma 1.1. [21] *Every subsequence of d_q–convergent sequence to a point x_0 is d_q–convergent to x_0.*

Definition 1.5. [21] Let (X, d_q) be a d_q–metric space. A mapping $T : X \to X$ is called contraction if there exists $0 \leq \lambda < 1$ such that $d_q(Tx, Ty) \leq \lambda d_q(x, y)$ for all $x, y \in X$.

Lemma 1.2. [21] *The d_q–limit point in a d_q–metric space is unique.*

Let K and H be nonempty subsets of a metric space (X, d) . A map $T : K \cup H \to K \cup H$ is called cyclic map if $T(K) \subseteq H$ and $T(H) \subseteq K$.

Definition 1.6. [16] Let K and H be nonempty subsets of a metric space (X, d). A cyclic map $T : K \cup H \to K \cup H$ is said to be a cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in K$ and $y \in H$.

Definition 1.7. [10] Let H and K be nonempty subsets of a metric space (X, d). A cyclic map $T : K \cup H \to K \cup H$ is called a Kannan type cyclic contraction if there exists $k \in \left(0, \frac{1}{2}\right)$ such that $d(Tx, Ty) \leq k\left[d(Tx, x) + d(Ty, y)\right]$ for all $x \in K$ and $y \in H$.

In [10] Karapinar et al. has been shown that the Kannan-type cyclic contraction and cyclic contraction are independent of each other.

Definition 1.8. [10] Let K and H be nonempty subsets of a metric space (X, d). A cyclic map $T : K \cup H \to K \cup H$ is called a Chatterjee-type cyclic contraction if there exists $k \in \left(0, \frac{1}{2}\right)$ such that $d(Tx, Ty) \leq k \max\left[d(x, y), d(Tx, x), d(Ty, y)\right]$ for all $x \in K$ and $y \in H$.

Definition 1.9. [16] Let K and H be nonempty subsets of a dislocated metric space (X, d). A cyclic map $T : K \cup H \to K \cup H$ is called a d-cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in K$ and $y \in H$.

Example 1.2. [22] Let $X = [0, 1]$ and $d(x, y) = |x - y| + 3|x| + 3|y|$. Then (X, d) is a dislocated metric space, but not a metric space. Let $K = H = [0, 1]$ and define $T : K \cup H \to K \cup H$ by $Tx = \frac{1}{3}$ for $x = 1$ and $Tx = 1$ for $x \in [0, 1)$. Then T is a d-cyclic contraction in the dislocated metric space (X, d). We note that in the usual metric $d(x, y) = |x - y|$ the self map T is not cyclic contraction because for $x = \frac{11}{12}$ and $y = 1$ the cyclic contraction fails.

Hence the class of d–cyclical contractions in dislocated metric space is larger than the class of cyclical contractions in usual metric.

Definition 1.10. [22] Let K and H be nonempty subsets of a dislocated quasi–metric space (X, d_q). A cyclic map $T : K \cup H \to K \cup H$ is called a Geraghty-type d_q–cyclic contraction if there exists $\beta \in S$ such that $d_q(Tx, Ty) \leq \beta(d_q(x, y)) d_q(x, y)$ for all $x \in K$ and $y \in H$.

In 2014 the concept of C–class functions (see Definition 1.11) was introduced by Ansari [2], for example see numbers (1), (2), (9) and (15) from Example 1.3. Also see [8] and [3] .

Definition 1.11. [2] A continuous function $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called C–function if for any $s, t \in [0, \infty)$, the following conditions hold:

(1) $F(s, t) \leq s$;

(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

The letter $\mathcal{C}$ will denote the class of all C–functions.

PROBLEM: this is an open problem, we can say that for all $F \in \mathcal{C}$, $F(0, 0) = 0$.

Example 1.3. [2] The following examples show that the class $\mathcal{C}$ is nonempty.

1. $F(s,t) = s - t$.

2. $F(s,t) = ms, 0 < m < 1$.

3. $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0, \infty)$.

4. $F(s,t) = \log(t + a^s)/(1 + t)$, for some $a > 1$.

5. $F(s,t) = \ln(1 + a^s)/2$, for $a > e$. Indeed $F(s,1) = s$ implies that $s = 0$.

6. $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0, \infty)$.

7. $F(s,t) = s \log_{t+a} a$, for $a > 1$.

8. $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.

9. $F(s,t) = s\beta(s)$, where $\beta : [0, \infty) \to [0, 1)$.

10. $F(s,t) = s - \frac{t}{k+t}$.

11. $F(s,t) = s - \varphi(s)$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

12. $F(s,t) = sh(s,t)$, where $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$.

13. $F(s,t) = s - (\frac{2+t}{1+t})t$.

14. $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$.

15. $F(s,t) = \phi(s)$, where $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

16. $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0, \infty)$.

Definition 1.12. [15] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote altering distance functions as Ψ.

Definition 1.13. [2] An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0, t > 0$.

We denote ultra altering distance functions as Φ_u.

2.   Main results

Lemma 2.1. *Let (X, d_q) be a d_q- metric space and let $\{x_n\}$ be a sequence in X such that*

$$(2.1) \qquad \lim_{n \to \infty} d_q(x_{n+1}, x_n) = \lim_{n \to \infty} d_q(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}_{n \in N}$ with $n_k \geq m_k \geq k$,(positive integers) such that the following four sequences tend to ε for $k \to \infty$:

$$d_q(x_{m_k}, x_{n_k}), \quad d_q(x_{m_k}, x_{n_k+1}), \quad d_q(x_{m_k+1}, x_{n_k}), \quad d_q(x_{m_k+1}, x_{n_k+1}).$$

Proof. If suppose that $\{x_n\}$ is a sequence in X satisfying condition (2.1) which is not Cauchy then there exist $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}_{n \in N}$ such that n_k is the smallest index for which

$$(2.2) \qquad n_k \geq m_k \geq k, \quad d_q(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad d_q(x_{m_k}, x_{n_k-1}) < \varepsilon.$$

Then from triangular inequality,

$$(2.3) \qquad \begin{aligned} \varepsilon \ & \leq \ d_q(x_{m_k}, x_{n_k}) \leq d_q(x_{m_k}, x_{n_k-1}) + d_q(x_{n_k-1}, x_{n_k}) \\ & \leq \ \varepsilon + d_q(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Taking the limit as $k \to \infty$ in (2.3), using the condition (2.4) we conclude that

$$(2.4) \qquad \lim_{k \to \infty} d_q(x_{m_k}, x_{n_k}) = \varepsilon.$$

Again

$$(2.5) \qquad \begin{aligned} \varepsilon \ & \leq \ d_q(x_{m_k}, x_{n_k}) \leq d_q(x_{m_k}, x_{n_k+1}) + d_q(x_{n_k+1}, x_{n_k}) \\ & \leq \ d_q(x_{m_k}, x_{n_k}) + d_q(x_{n_k}, x_{n_k+1}) + d_q(x_{n_k+1}, x_{n_k}). \end{aligned}$$

Taking the limit as $k \to \infty$ in (2.5), using the condition (2.1) and the result (2.4) we conclude that

$$(2.6) \qquad \lim_{k \to \infty} d_q(x_{m_k}, x_{n_k+1}) = \varepsilon.$$

Further

$$\begin{aligned} \varepsilon \ & \leq \ d_q(x_{m_k}, x_{n_k}) \leq d_q(x_{m_k}, x_{m_k+1}) + d_q(x_{m_k+1}, x_{n_k}) \\ & \leq \ d_q(x_{m_k}, x_{m_k+1}) + d_q(x_{m_k+1}, x_{m_k}) + d_q(x_{m_k}, x_{n_k}). \end{aligned}$$

In the two inequalities above the taking limit as $k \to \infty$ and using the condition (2.1) and the results (2.4) we obtain:

$$(2.7) \qquad \lim_{k \to \infty} d_q(x_{m_k+1}, x_{n_k}) = \varepsilon.$$

By triangular inequality, we have

$$
\begin{aligned}
d_q(x_{m_k}, x_{n_k}) &\leq d_q(x_{m_k}, x_{m_k+1}) + d_q(x_{m_k+1}, x_{n_k+1}) + d_q(x_{n_k+1}, x_{n_k}) \\
&\leq d_q(x_{m_k}, x_{m_k+1}) + d_q(x_{m_k+1}, x_{n_k}) \\
&\quad + d_q(x_{n_k}, x_{n_k+1}) + d_q(x_{n_k+1}, x_{n_k}).
\end{aligned}
$$

In the above inequality, taking the limit as $k \to \infty$ and using the condition (2.1) and the results (2.4), (2.7) we obtain

$$
(2.8) \qquad \lim_{k \to \infty} d_q(x_{m_k+1}, x_{n_k+1}) = \varepsilon,
$$

this completes the proof. $\square$

Theorem 2.1. *Let K and H be nonempty subsets of a complete dislocated quasi–metric space (X, d_q). Let $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the condition*

$$
(2.9) \qquad
\begin{aligned}
d_q(Tx, Ty) \leq F(&\max\{d_q(x,y), d_q(x,Tx), d_q(y,Ty)\}, \\
&\varphi(\max\{d_q(x,y), d_q(x,Tx), d_q(y,Ty)\}))
\end{aligned}
$$

for all $x \in A$ and $y \in B$ and $F \in C, \varphi \in \Phi_u$. Then, T has a unique fixed point in $K \cap H$.

Proof. Taking a point $x \in K$ and using (2.9), we have

$$
\begin{aligned}
d_q(T^2 x, Tx) &= d_q(T(Tx), Tx) \\
&\leq F(\max\{d_q(Tx, x), d_q(Tx, T^2 x), d_q(x, Tx)\}, \\
&\quad\ \varphi(\max\{d_q(Tx, x), d_q(Tx, T^2 x), d_q(x, Tx)\})) \\
&\leq \max\{d_q(Tx, x), d_q(Tx, T^2 x), d_q(x, Tx)\} \\
&= \max\{d_q(Tx, x), d_q(x, Tx)\}.
\end{aligned}
$$

In the same way we have,

$$
\begin{aligned}
d_q(Tx, T^2 x,) &= d_q(Tx, T^2 x,) \\
&\leq F(\max\{d_q(x, Tx)\, d_q(x, Tx), d_q(Tx, T^2 x)\}, \\
&\quad\ \varphi(\max\{d_q(x, Tx), d_q(x, Tx), d_q(Tx, T^2 x)\})) \\
&\leq \max\{d_q(x, Tx), d_q(x, Tx), d_q(Tx, T^2 x)\} \\
&= \max\{d_q(Tx, x)\} \\
&\leq \max\{d_q(Tx, x), d_q(x, Tx)\},
\end{aligned}
$$

so,

$$
(2.10) \qquad \max\{d_q(T^2 x, Tx), d_q(Tx, T^2 x)\} \leq \max\{d_q(Tx, x), d_q(x, Tx)\}.
$$

Using (2.9) and (2.10) we get,

$$(2.11) \quad \max\left\{d_q\left(T^3x, T^2x\right), d_q\left(T^2x, T^3x\right)\right\} \le \max\left\{d_q\left(T^2x, Tx\right), d_q\left(Tx, T^2x\right)\right\}.$$

Inductively, using this process for all $n \in N$ we have

$$(2.12) \quad \begin{aligned} \max\left\{d_q\left(T^{n+1}x, T^nx\right), d_q\left(T^nx, T^{n+1}x\right)\right\} \le \max\{d_q\left(T^nx, T^{n-1}x\right), \\ d_q\left(T^{n-1}x, T^nx\right)\}. \end{aligned}$$

The sequence $\{\max\left\{d_q\left(T^{n+1}x, T^nx\right), d_q\left(T^nx, T^{n+1}x\right)\right\}\}$ is decreasing, so it tends to $r \ge 0$ now with $n \to \infty$ in the above and using (2.9) we have

$$(2.13) \quad r \le F(r, \varphi(r)).$$

So, $r = 0$ or $\varphi(r) = 0$, therefore $\lim\limits_{n\to\infty} d_q\left(T^nx, T^{n-1}x\right) = 0$ and $\lim\limits_{n\to\infty} d_q\left(T^{n-1}x, T^nx\right)$ $= 0$. Now, we shall prove that $\{x_n\}$ is a Cauchy sequence, If $\{x_n\}$ is not a Cauchy sequence, then by Lemma 2.1 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$,

$$d_q(x_{m_k}, x_{n_k}), \quad d_q(x_{m_k}, x_{n_k+1}), \quad d_q(x_{m_k+1}, x_{n_k}), \quad d_q(x_{m_k+1}, x_{n_k+1}).$$

So

$$\begin{aligned} d_q(x_{m_k+1}, x_{n_k+1}) &= d_q(Tx_{m_k}, Tx_{n_k}) \\ &\le F(\max\{d_q(x_{m_k}, x_{n_k}), d_q(x_{n_k}, Tx_{n_k}), d_q(x_{m_k}, Tx_{m_k})\}, \\ &\quad \varphi(\max\{d_q(x_{m_k}, x_{n_k}), d_q(x_{n_k}, Tx_{n_k}), d_q(x_{m_k}, Tx_{m_k})\})) \\ &= F(\max\{d_q(x_{m_k}, x_{n_k}), d_q(x_{n_k}, x_{n_k+1}), d_q(x_{m_k}, x_{m_k+1})\}, \\ (2.14) &\quad \varphi(\max\{d_q(x_{m_k}, x_{n_k}), d_q(x_{n_k}, x_{n_k+1}), d_q(x_{m_k}, x_{m_k+1})\})). \end{aligned}$$

Letting $k \to \infty$, we get

$$\varepsilon \le F(\varepsilon, \varphi(\varepsilon)).$$

So, $\varepsilon = 0$ or $\varphi(\varepsilon) = 0$, that is; $\varepsilon = 0$. Thus (T^nx) is a Cauchy sequence. Since (X, d_q) is complete, we have (T^nx) d_q –converges to some $z \in X$. We note that $(T^{2n}x)$ is a sequence in K and $(T^{2n-1}x)$ is a sequence in H in a way that both sequences tend to the same limit z.

Since K and H are closed then we have that $z \in K \cap H$. Hence $K \cap H \ne \emptyset$.

We claim that $Tz = z$.

Considering the condition (2.9) we have:

$$\begin{aligned} d_q\left(T^{2n}x, Tz\right) &= d_q\left(TT^{2n-1}x, Tz\right) \\ &\le F(\max\left\{d_q\left(T^{2n-1}x, z\right), d_q\left(T^{2n-1}x, T^{2n}x\right), d_q\left(z, Tz\right)\right\} \\ &, \quad \varphi\max\left\{d_q\left(T^{2n-1}x, z\right), d_q\left(T^{2n-1}x, T^{2n}x\right), d_q\left(z, Tz\right)\right\}). \end{aligned}$$

By $n \to \infty$ in the above inequality, we have

$$d_q(z, Tz) \leq F(d_q(z, Tz), \varphi(d_q(z, Tz))).$$

So, $d_q(z, Tz) = 0$ or $\varphi(d_q(z, Tz)) = 0$. This implies that $d_q(z, Tz) = 0$.

Similarly, considering (2.9) we have,

$$
\begin{aligned}
d_q\left(Tz, T^{2n}x\right) &= d_q\left(Tz, TT^{2n-1}x\right) \\
&\leq F(\max\left\{d_q\left(z, T^{2n-1}x\right), d_q\left(z, Tz\right), d_q\left(T^{2n-1}x, T^{2n}x\right)\right\} \\
&\quad , \varphi\max\left\{d_q\left(z, T^{2n-1}x\right), d_q\left(z, Tz\right), d_q\left(T^{2n-1}x, T^{2n}x\right)\right\}).
\end{aligned}
$$

Similarly, by $n \to \infty$ above we obtain $d_q(Tz, z) = 0$.

Hence $d_q(z, Tz) = d_q(Tz, z) = 0$, so $Tz = z$ and z is a fixed point of T.

We shall prove that z is a unique fixed point of T. Clearly, from (2.9) if u and v are the fixed points of T we have $d_q(u, u) = 0, d_q(v, v) = 0$.

Then we have,

$$
\begin{aligned}
d_q(u, v) &= d_q(Tu, Tv) \\
&\leq F\left(\max\{d_q(u, v), d_q(u, u), d_q(v, v)\}\right., \\
&\quad \varphi\left(\max\{d_q(u, v), d_q(u, u), d_q(v, v)\}\right)) \\
&= F(d_q(u, v), \varphi(d_q(u, v))),
\end{aligned}
$$

so, $d_q(u, v) = 0$ or $\varphi(d_q(u, v)) = 0$, therefore $d_q(u, v) = 0$. Similarly, we can prove that $d_q(v, u) = 0$, now from $d_q(u, v) = d_q(v, u) = 0$ we deduce that $u = v$. Hence the proof is completed. $\square$

Theorem 2.2. *Let K and H be nonempty closed subsets of a complete dislocated quasi–metric space (X, d_q) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the condition:*

$$(2.15) \qquad\qquad d_q(Tx, Ty) \leq F(d_q(x, y), \varphi(d_q(x, y)))$$

for all $x \in K$ and $y \in H$ where $F \in C, \varphi \in \Phi_u$. Then T has a unique fixed point in $K \cap H$.

Proof. If for all $a, b \in [0, \infty)$ with $a \leq b$ we have that $F(a, \varphi(a)) \leq F(b, \varphi(b))$. In this case the theorem is a corollary of Theorem 2.1, because

$$
\begin{aligned}
d_q(Tx, Ty) &\leq F(d_q(x, y), \varphi(d_q(x, y))) \\
&\leq F(\max\left\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\right\}, \\
&\quad \varphi(\max\left\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\right\})).
\end{aligned}
$$

Now assume that there is $a < b$ such that $F(a, \varphi(a)) > F(b, \varphi(b))$. In this case, fix a point $x \in A$. If $T^n x = T^{n+1}x$ for some $n \in N$, then $T^{n+1}x = T^{n+2}x$ and so

$(T^n x)$ converges to some $z \in X$. Suppose that $T^n x \neq T^{n+1} x$. Using the condition (2.15) we have,

$$d\left(T^{n+1}x, T^n x\right) \leq F(d\left(T^n x, T^{n-1}x\right), \varphi(d\left(T^n x, T^{n-1}x\right))) \leq d\left(T^n x, T^{n-1}x\right).$$

Thus the sequence $d\left(T^{n+1}x, T^n x\right)$ is decreasing and bounded from below, thus it converges to some $z \geq 0$, then from (2.15) taking the limit as $n \to \infty$, we get

$$r \leq F(r, \varphi(r)).$$

So, $r = 0$ or $\varphi(r) = 0$, therefore $\lim_{n \to \infty} d_q\left(T^n x, T^{n-1}x\right) = 0$. In a similar way we obtain $\lim_{n \to \infty} d_q\left(T^{n-1}x, T^n x\right) = 0$.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence, If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$,

$$d_q(x_{m_k}, x_{n_k}), \quad d_q(x_{m_k}, x_{n_k+1}), \quad d_q(x_{m_k+1}, x_{n_k}), \quad d_q(x_{m_k+1}, x_{n_k+1}),$$

so

(2.16) $$d_q(x_{m_k+1}, x_{n_k+1}) \leq F(d_q(x_{m_k}, x_{n_k}), \varphi(d_q(x_{m_k}, x_{n_k})))$$

then from(2.16) taking the limit as $k \to \infty$, we get

$$\varepsilon \leq F(\varepsilon, \varphi(\varepsilon)).$$

So, $\varepsilon = 0$ or $\varphi(\varepsilon) = 0$, that is; $\varepsilon = 0$, our supposition fail from this contradiction. Hence, this proves that $(T^n x)$ is a Cauchy sequence. Since (X, d_q) is complete, we have $(T^n x)$ d_q-converges to some $z \in X$. Note that $\left(T^{2n}x\right)$ is a sequence in K and $\left(T^{2n-1}x\right)$ is a sequence in H in a way that both sequences tend to the same limit $z \in K \cap H$.

Considering the condition (2.15) we have,

(2.17) $$\begin{aligned} d_q(z, Tz) &\leq d_q\left(z, T^{2n}x\right) + d_q\left(T^{2n}x, Tz\right) \\ &\leq d_q\left(z, T^{2n}x\right) + F(d_q\left(T^{2n-1}x, z\right), \varphi(d_q\left(T^{2n-1}x, z\right))). \end{aligned}$$

By $n \to \infty$ in (2.17), we have $d_q\left(Tz, z\right) = 0$ and, similarly, we have $d_q\left(z, Tz\right) = 0$ as a result $Tz = z$.

Uniqueness: Let u and v be two fixed points of T. Then

$$\begin{aligned} d_q(u, v) &= d_q(Tu, Tv) \\ &\leq F(d_q\left(u, v\right), \varphi(d_q\left(u, v\right))) \\ &\leq d_q(u, v), \end{aligned}$$

$$\begin{aligned} d_q(v, u) &= d_q(Tv, Tu) \\ &\leq F(d_q\left(v, u\right), \varphi(d_q\left(v, u\right))) \\ &\leq d_q(v, u), \end{aligned}$$

so, $d_q(u, v) = 0$ or $\varphi(d_q\left(u, v\right)) = 0$, and $d_q(v, u) = 0$ or $\varphi(d_q\left(v, u\right)) = 0$, so from those we get $d_q\left(u, v\right) = d_q\left(v, u\right) = 0$ and also, by property d_2 we have $u = v$. $\square$

Example 2.1. Let $X = [0,1]$ and $T : X \to X$ be given as $T(x) = \frac{x}{8}$. Let $K = H = [0,1]$. Define the function $d_q : X \times X \to [0,\infty)$ by $d(x,y) = \max\{x,y\}$. The function $F(s,t) = \frac{s}{1+t} : [0,\infty) \to [0,1)$ and $\varphi(t) = 1$. Considering all the cases and the general cases if $x \leq y, \forall x,y \in X$

$$
\begin{aligned}
d_q(Tx,Ty) &= d_q\left(\frac{x}{8},\frac{y}{8}\right) = \max\left\{\frac{x}{8},\frac{y}{8}\right\} \\
&= \frac{y}{8} \leq \frac{y}{1+1} = \frac{\max\{x,y\}}{1+\varphi(\max\{x,y\})} \\
&= \frac{d_q(x,y)}{1+\varphi(d_q(x,y))}
\end{aligned}
$$

Clearly, all conditions of Theorem 2.2 are satisfied and $x = 0$ is a unique fixed point of T.

We consider the set G_3 of all continuous functions [foe some examples for these functions see [23]] $g : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ with the following properties:

(i) g is non-decreasing with respect to each variable.

(ii) $g(t,t,t) \leq t$, for $t \in [0,\infty)$.

Theorem 2.3. *Let K and H be nonempty closed subsets of a complete dislocated quasi–metric space (X,d_q) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the following condition:*

(2.18)
$$
\begin{aligned}
d(Tx,Ty) \leq &F(g\left[d_q(x,y),d_q(x,Tx),d_q(y,Ty)\right], \\
&\varphi(g\left[d_q(x,y),d_q(x,Tx),d_q(y,Ty)\right]))
\end{aligned}
$$

for all $x \in K$ and $y \in H$, where $F \in C, \varphi \in \Phi_u$, $g \in G_3$. Then T has a unique fixed point in $K \cap H$.

Proof. Let x be a fixed point in X. By condition (2.18) and the properties of g we have,

$$
\begin{aligned}
d_q(T^2x,Tx) &\leq F(g\left[d_q(Tx,x),d_q(Tx,T^2x),d_q(x,Tx)\right], \\
&\quad \varphi(g\left[d_q(Tx,x),d_q(Tx,T^2x),d_q(x,Tx)\right])) \\
&\leq d_q(Tx,x),
\end{aligned}
$$

similarly, we have

$$
\begin{aligned}
d_q(T^3x,T^2x) &\leq F(g\left[d_q(T^2x,Tx),d_q(T^3x,T^2x),d_q(Tx,T^2x)\right], \\
&\quad \varphi(g\left[d_q(T^2x,Tx),d_q(T^3x,T^2x),d_q(Tx,T^2x)\right])) \\
&\leq d_q(T^2x,Tx).
\end{aligned}
$$

Generally, from the above inequalities we have,

$$
d_q(T^{n+1}x,T^nx) \leq d_q(T^nx,T^{n-1}x)
$$

for $n \in N$.

Thus the sequence $\{d_q\left(T^{n+1}x, T^n x\right)\}$ is decreasing and bounded from below, thus it converges to some $z \geq 0$, then from (2.15) taking the limit as $n \to \infty$, we get

$$r \leq F(r, \varphi(r)).$$

so, $r = 0$ or $\varphi(r) = 0$, therefore $\lim\limits_{n \to \infty} d_q\left(T^n x, T^{n-1}x\right) = 0$. In a similar way we obtain $\lim\limits_{n \to \infty} d_q\left(T^{n-1}x, T^n x\right) = 0$.

Easily, as in the above theorems, we can show that the sequence $(T^n x)$ is a Cauchy sequence in a complete dislocated quasi-metric space (X, d_q). So there exists $z \in X$ such that $(T^n x)$ dislocated quasi converges to z. Note that $\left(T^{2n}x\right)$ is a sequence in K and $\left(T^{2n-1}x\right)$ is a sequence in H in a way that both sequences tend to the same limit $z \in K \cap H$. To prove that z is a fixed point of T we reuse the contractive condition (2.18),

$$
\begin{aligned}
d_q(z, Tz) &\leq d_q(z, T^{2n}x) + d_q(T^{2n}x, Tz) \\
&\leq d_q(z, T^{2n}x) + F(g\left[d_q(T^{2n-1}x, z), d_q(T^{2n-1}x, T^{2n}x), d_q(z, Tz)\right], \\
&\quad \varphi g\left[d_q(T^{2n-1}x, z), d_q(T^{2n-1}x, T^{2n}x), d_q(z, Tz)\right]).
\end{aligned}
$$

In this inequality passing the limit as $n \to \infty$ and since g is non-decreasing and continuous we get,

$$d_q\left(z, Tz\right) \leq F(d_q\left(z, Tz\right), \varphi(d_q\left(z, Tz\right)))$$

so,

$$d_q\left(z, Tz\right) = 0 \quad \text{or} \quad \varphi(d_q\left(z, Tz\right)) = 0$$

therefore, we obtain $d_q\left(z, Tz\right) = 0$. Again, from (2.18) we get $d_q\left(Tz, z\right) = 0$. As a result $z = Tz$.

Uniqueness, Let us suppose that u and v are two fixed points of T where $Tu = u$ and $Tv = v$.

From the condition of the theorem we have,

(2.19)
$$
\begin{aligned}
d_q(Tu, Tv) \leq &F(g\left[d_q(u, v), d_q(u, Tu), d_q(v, Tv)\right], \\
&\varphi(g\left[d_q(u, v), d_q(u, Tu), d_q(v, Tv)\right])).
\end{aligned}
$$

If we replace $v = u$ in (2.19) then we obtain,

$$
\begin{aligned}
d_q(u, u) &= d_q(Tu, Tu) \\
&\leq F(g\left[d_q(u, u), d_q(u, Tu), d_q(u, Tu)\right], \\
&\quad \varphi(g\left[d_q(u, u), d_q(u, Tu), d_q(u, Tu)\right])) \\
&= F(g\left[d_q(u, u), d_q(u, u), d_q(u, u)\right], \varphi(g\left[d_q(u, u), d_q(u, u), d_q(u, u)\right])) \\
&\leq F(d_q\left(u, u\right), \varphi(d_q(u, u))).
\end{aligned}
$$

Thus $d_q(u, u) = 0$ or $\varphi(d_q(u, u)) = 0$, therefore we get $d_q(u, u) = 0$. Similarly we have that $d_q(v, v) = 0$. Therefore, using the condition (2.18) we have:

$$
\begin{aligned}
d_q(u, v) &= d_q(Tu, Tv) \\
&\leq F(g[d_q(u, v), d_q(u, u), d_q(v, v)], \varphi(g[d_q(u, v), d_q(u, u), d_q(v, v)])) \\
&\leq F(d_q(u, v), \varphi(d_q(u, v))).
\end{aligned}
$$

And also,

$$
\begin{aligned}
d_q(v, u) &= d_q(Tv, Tu) \\
&\leq F(g[d_q(v, u), d_q(v, v), d_q(u, u)], \varphi(g[d_q(v, u), d_q(v, v), d_q(u, u)])) \\
&\leq F(d_q(v, u), \varphi(d_q(v, u))).
\end{aligned}
$$

So $d_q(u, v) = 0$ or $\varphi(d_q(u, v)) = 0$, and $d_q(v, u) = 0$ or $\varphi(d_q(v, u)) = 0$, from these two we have $d_q(u, v) = d_q(v, u) = 0$, and the property d_2 implies $v = u$. Hence, the fixed point is unique. $\square$

Taking $F(s, t) = ks, 0 < k < 1$, in Theorem 2.1 we have the following remark

Remark 2.1. [22] Let K and H be nonempty subsets of a complete dislocated quasi–metric space (X, d_q). Let $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the condition

$$
(2.20) \qquad d_q(Tx, Ty) \leq k \max\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\}
$$

for all $x \in K$ and $y \in H$ and $0 \leq k < 1$. Then, T has a unique fixed point in $K \cap H$.

Taking $F(s, t) = \beta(s) s$, in Theorem 2.2 we have the following remark

Remark 2.2. [22] Let K and H be nonempty closed subsets of a dislocated quasi–metric space (X, d_q) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the Geraghty type condition:

$$
(2.21) \qquad d_q(Tx, Ty) \leq \beta(d_q(x, y)) d_q(x, y)
$$

for all $x \in K$ and $y \in H$ where $\beta \in S$. Then T has a unique fixed point in $K \cap H$.

Taking $F(s, t) = ks, 0 < k < 1$, in Theorem 2.3 we have the following remark

Remark 2.3. [22] Let K and H be nonempty closed subsets of a dislocated quasi–metric space (X, d) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the following condition:

$$
(2.22) \qquad d_q(Tx, Ty) \leq kg[d_q(x, y), d_q(x, Tx), d_q(y, Ty)]
$$

for all $x \in K$ and $y \in H$, and $0 < k < 1$, where $g \in G_3$. Then T has a unique fixed point in $K \cap H$.

Example 2.2. Let $X = [-1, 1]$ and $T : X \to X$ be given as $Tx = \frac{-x}{5}$. Let $K = [-1, 0]$ and $H = [0, 1]$. Define the function $d_q : X \times X \to [0, \infty)$ by $d_q(x, y) = |x - y| + |x|$. We note that d_q is a dislocated quasi–metric on X and the map T is cyclic on X.

If we consider from G_3 the function $g(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$ we see

$$
\begin{aligned}
d_q(x, y) &= |x - y| + |x| \\
d_q(Tx, Ty) &= \left|\frac{-x}{5} - \frac{-y}{5}\right| + \left|\frac{-x}{5}\right| = \frac{1}{5}|x - y| + \frac{1}{5}|x|.
\end{aligned}
$$

Then, clearly, we have,

$$
\begin{aligned}
d_q(Tx, Ty) &= \left|\frac{-x}{5} - \frac{-y}{5}\right| + \left|\frac{-x}{5}\right| \\
&= \frac{1}{5}(|x - y| + |x|) = \frac{1}{5}d(x, y) \\
&\leq kg[d_q(x, y), d_q(x, Tx), d_q(y, Ty)].
\end{aligned}
$$

So for the constant $\frac{1}{5} \leq k < 1$ the map T satisfies the condition (2.18) of Theorem 2.3 and 0 is a unique fixed point of T.

From the general character of Theorems 2.1, 2.2, and 2.3, and because of the class C and G_3, we can give many corollaries as follows using the functions

$$
\begin{aligned}
g(t_1, t_2, t_3) &= [\max\{t_1^p, t_2^p, t_3^p\}]^{\frac{1}{p}}, \ p > 0 \\
g(t_1, t_2, t_3) &= [\max\{t_1 t_2, t_2 t_3, t_1 t_3\}]^{\frac{1}{2}}, \ p > 0 \\
g(t_1, t_2, t_3) &= g \max\{t_1 + t_2, t_2 + t_3, t_1 + t_3\}.
\end{aligned}
$$

Corollary 2.1. *Let K and H be nonempty closed subsets of a dislocated quasi–metric space (X, d) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the following condition:*

$$
d^p(Tx, Ty) \leq k \max\{d^p(x, y), d^p(x, Tx), d^p(y, Ty)\}
$$

for all $x \in K$ and $y \in H$, and $0 \leq k < 1$. Then T has a unique fixed point in $K \cap H$.

Proof. Taking $F(s, t) = ks, 0 < k < 1$ and $g(t_1, t_2, t_3) = [\max\{t_1^p, t_2^p, t_3^p\}]^{\frac{1}{p}}, p > 0$ in Theorem 2.3. ☐

Corollary 2.2. *Let K and H be nonempty closed subsets of a dislocated quasi–metric space (X, d_q) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the following condition:*

$$
d_q{}^2(Tx, Ty) \leq k \max\{d_q(x, y) \ d_q(x, Tx), d_q(x, y) d_q(y, Ty), d_q(x, Tx) d_q(y, Ty)\}
$$

for all $x \in K$ and $y \in H$, and $0 \leq k < 1$. Then T has a unique fixed point in $K \cap H$.

Proof. Taking $F(s, t) = ks, 0 < k < 1$ and $g(t_1, t_2, t_3) = [\max\{t_1 t_2, t_2 t_3, t_1 t_3\}]^{\frac{1}{2}}$, $p > 0$ in Theorem 2.3. ☐

Corollary 2.3. *Let K and H be nonempty closed subsets of a dislocated quasi–metric space (X, d_q) and $T : K \cup H \to K \cup H$ be a cyclic mapping that satisfies the following condition:*

$$d_q(Tx, Ty) \leq k \max\{d_q(x, y) + d_q(x, Tx), d_q(x, y) \\ + d_q(y, Ty), d_q(x, Tx) + d_q(y, Ty)\}$$

for all $x \in K$ and $y \in H$, and $0 \leq k < 1$. Then T has a unique fixed point in $K \cap H$.

Proof. Taking $F(s, t) = ks, 0 < k < 1$ and $g(t_1, t_2, t_3) = \max\{t_1 + t_2, t_2 + t_3, t_1 + t_3\}$, $p > 0$ in Theorem 2.3. $\square$

Further, as common applications of fixed point theorems we provide some corollaries for cyclic maps for integral type contraction (taking $K = H = X$).

Corollary 2.4. *Let (X, d_q) be a complete dislocated quasi–metric space and $T : X \to X$ be a mapping such that for any $x, y \in X$,*

$$\int_0^{d_q(Tx, Ty)} \rho(t)\, dt \leq r \int_0^{\beta(d_q(x,y))d_q(x,y)} \rho(t)\, dt$$

where the function $\phi \in \Phi$, the constant $r \in [0, 1)$ and $\rho : [0, \infty) \to [0, \infty)$ is Lesbegue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)\, dt > 0$ for $\varepsilon > 0$. Then, T has a unique fixed point in X.

Remark 2.4. Theorem 2.3 generalizes and unifies the results for the Kannan-type cyclic contraction, Chatterjea cyclic contraction, C–cyclical contraction, Zamfirescu contraction and some existing results in dislocated–metric spaces [1, 5, 23, 24]. Statements of many theorems and results can be obtained by taking $K = H = X$.

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